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ABOUT A CERTAIN "ANOMALY" IN THE PRICING OF DEBT SECURITIES

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Abstract: The intention of the paper is the presentation of some considerations concerned with the problem of debt securities pricing without simplifying assumptions that are commonly used in practice. In the paper the deterministic and descrete time approach is used. On financial markets coupon rates are strictly connected with interest rates described by yield curve. This relation is linear but different structures of bonds can be described by taking into account particular assumptions which refer to the coefficients in this dependence. It is shown that taking into account the dependence of coupons on forward rates leads to not standard dependence of intrinsic value on spot rates. It turns out that the intrinsic value of a bond is not a decreasing function of interest rates, what in a very fundamental way changes the investment risk that accompanies that kind of bonds.

Keywords: debt securities, intrinsic value, interest rate risk, coupon bonds, yield curve

JEL classification: C02, C65, G12,

INTRODUCTION

Bonds are immensely important instruments of the capital market. Hence the question of their pricing remains vital – for instance because the intrinsic value of those instruments depends on many factors. In particular, what matters is the method of coupon payment and its structure, or bond options (the right of early redemption by the issuer, the holders' right to present bonds for redemption earlier); however, the primary factor is the spot rate structure (yield curve). It should be noted that as for the latter, what we observe is a sort of feedback: the term structure of interest rates is determined based on the prices of debt securities

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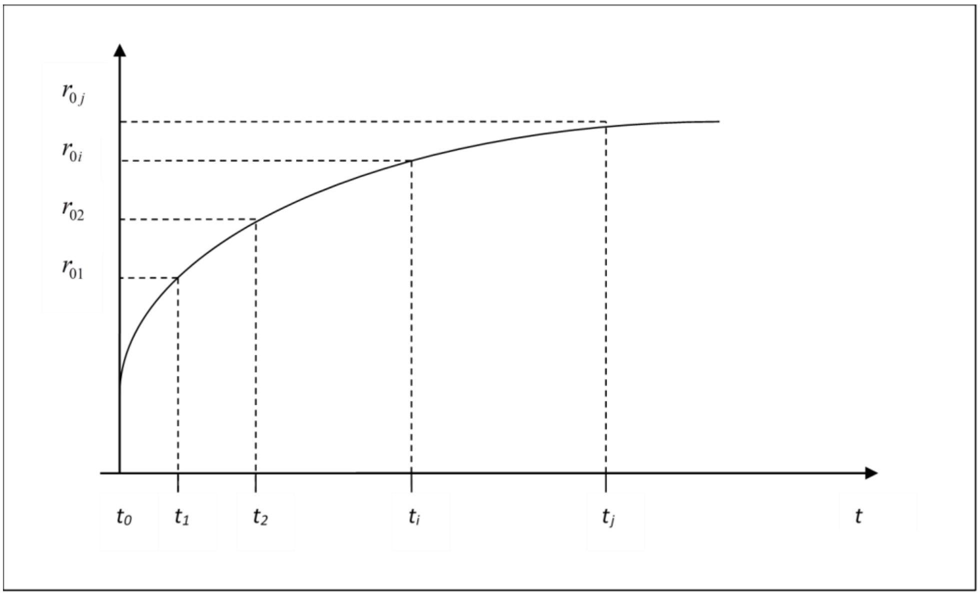
listed on the free market; on the other hand, it is the basis for pricing of such securities, which is particularly true for their new issuance. For this reason, bond pricing relies on highly advanced mathematical tools, such as non-linear approach methods [Deeba et al. 2002], partial differential equations [Zui-Cha et al. 2010], integration into trajectories known from the field of physics [Zanga et al. 2017], and Green's function [Pooe et al. 2004]. In the light of the aforementioned research, this paper relies on both elementary and deterministic methods. The underlying foundation is, of course, the notion of pricing based on the assumption that a bond intrinsic value is the sum of future financial flows discounted at present and strongly depend on market spot rates [Ritchken et al. 1966]. However, what motivates these considerations is an observation that in many cases the coupon rates depend on forward rates resulting from the yield curve. Consequently, it turns out that the intrinsic value (so the price, too) of some bonds is an increasing function of spot rates, rather than a decreasing one, as it is commonly known. This paper specifies conditions that need to be met for the "anomaly" to occur. Moreover, it touches on investment consequences and offers examples rooted in the Polish market of bonds that meet these requirements. The most introductory considerations are based on paper [Karpio 2010].

INTRINSIC BOND VALUE IN THE GENERAL CASE

A bond is a type of security, the pricing of which is based on expected interest rates. For this reason, let us start with a few comments on the term structure of interest rates (yield curve). We do not examine the curve structure methodology; instead we assume that this has been completed and the structure is given. Let us select the series of time moments (specific dates): $t_0, t_1, \dots, t_N$ with t_0 being the investment start moment, and with the $t_N = T$ moment being its end. In one case, the series may be infinite (perpetual bonds). Let us also assume that the intervals between the selected dates are not homogeneous, and we will mark them with the $\Delta t_{i-1} = t_i - t_{i-1}$ symbols. We introduce a following convention: time intervals are expressed in years, and all the rates are nominal – referenced against years. The spot rates applying to the investments starting from the current moment t_0 and ending in the future moment t_i are marked with r_{0i} symbols. The forward rates related to the investments between moments t_i and t_j ($j > i$) are marked with f_{ij}. Figure no. 1 below presents a sample and somewhat idealistic spot rate curve.

Figure 1. Example of yield curve



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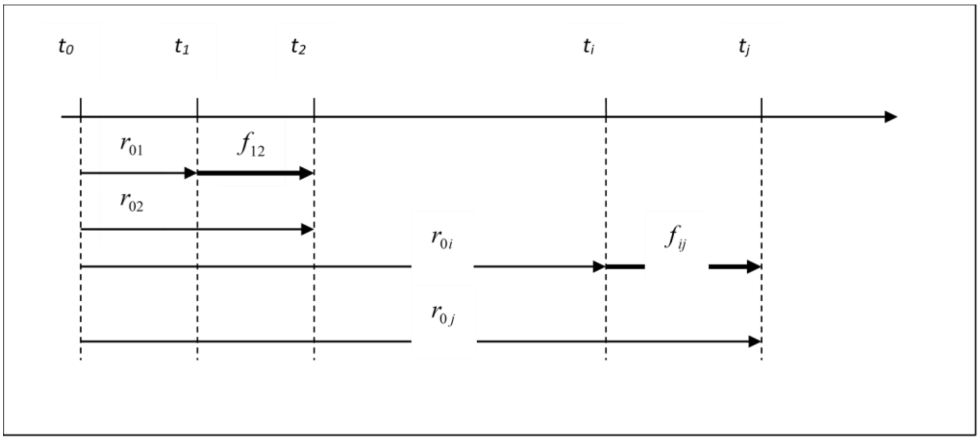
Please notice that at the starting moment t_0, the spot rate does not equal to zero. This is because the curve describes nominal rates. They are referenced against a time unit; hence their border value is different from zero. This is observable when one examines interbank rates: daily rates do not usually differ much from the weekly rates or monthly ones; in fact, occasionally they tend to be higher. Obviously, with the investment time reverting to zero, the rate of return also tends to zero, but this is an investment-based rate, and not a nominal one. It is also noteworthy that some institutions offering access to the Forex currency market provide second-based capitalisation of security deposits at nominal rates substantially different from zero.

In practical terms, the curve above is called “normal” (spot rates grow in the time function); however, other shapes of the yield curve occur: flat, reverse, humped, etc. In the first case, the nominal investment rate is a constant time function. With the reverse yield, the longer the investment period is, the lower the nominal rates are. This yield curve type can be seen, for example, when inflation is high, and the central bank takes measures to reduce it. However, investors must believe that the measures will be effective; had it not been for this belief, the yield curve would be growing.

Let us analyse investments with different implementation periods. Bank deposits with different maturities may serve as a model. In schematic terms, a set of such investments, along with corresponding spot and forward rates, is shown in Figure 2.

Figure 2. Schematic representation of spot and forward rates



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The spot rates are known, as they are observed in the form of the market prices of debt securities. The problem is the knowledge of forward rates, since they are the main factor affecting the decision whether an investment should be one-time in nature or have many stages. One of the most effective ideas followed on well-balanced financial markets – the principle of non-arbitration – justifies the following formula linking spot and forward rates:

$$\prod_{k=1}^{i}(1+r_{0i}\Delta t_{k-1})\prod_{k=i+1}^{j}\left(1+f_{ij}\Delta t_{k-1}\right)=\prod_{k=1}^{j}\left(1+r_{0j}\Delta t_{k-1}\right). \tag{1}$$

Hence, we arrive at a formula that allows one to determine forward rates (usually as an approximation):

$$\prod_{k=i+1}^{j}\left(1+f_{ij}\Delta t_{k-1}\right)=\frac{\prod_{k=1}^{j}\left(1+r_{0j}\Delta t_{k-1}\right)}{\prod_{k=1}^{i}(1+r_{0i}\Delta t_{k-1})}. \tag{2}$$

Based on this dependency, it is possible to prove a number of properties of forward rates [Karpio 2008]. For instance, with a normal (increasing) yield curve, the following relation takes place:

$$r_{0i}<r_{0j}<f_{ij}. \tag{3}$$

It has serious investment implications, since today's forward rates will be the spot rates in the future, which brings about very specific investors' actions.

In the case of the flat yield curve, forward rates are equal to spot rates. When determining the price of bonds, one-period forward rates are tremendously import. Consequently, the equation (2) makes it possible to precisely determine their value, which equals:

$$f_{i-1i} = \frac{1}{\Delta t_{i-1}}\left(\frac{\prod_{k=1}^{i}(1 + r_{0i}\Delta t_{k-1})}{\prod_{k=1}^{i-1}(1 + r_{0i-1}\Delta t_{k-1})} - 1\right). \tag{4}$$

Let us assume that we work with a debt security that in the future, in the moments: $t_1, \dots, t_N$, will pay out coupons described with streams (payouts in the annual scale) h_{i-1i} applying to periods Δt_{i-1}. The payment will be made in the moments t_i, namely in arrears. The money coupons will then be equal to: $h_{i-1i}\Delta t_{i-1}$. The pricing takes place in the current moment t_0. If P_M is the face value of the debt security, then the intrinsic value, defined as the current value of future cash flows, is equal to:

$$P_{t_0T} = \sum_{i=1}^{N}\frac{h_{i-1i}\Delta t_{i-1}}{\prod_{k=1}^{i}(1 + r_{0i}\Delta t_{k-1})} + \frac{P_M}{\prod_{k=1}^{N}(1 + r_{0N}\Delta t_{k-1})}. \tag{5}$$

Marking of the intrinsic value P_{t_0T} takes into account the pricing moment and the bond’s maturity moment. Therefore, it is a $T - t_0$ forward bond. This formulation is free from any simplifying assumptions that are often used in literature. The interpretation is based on the reinvestment of the paid-out interest, which can be easily observed when the formulation is multiplied by $\prod_{i=1}^{N}(1 + r_{0N}\Delta t_{i-1})$, and when the definition of forward rates f_{ij} is used (equation (2)). In such case, the relation (5) takes the following form:

$$P_{t_0T}\prod_{i=1}^{N}(1 + r_{0N}\Delta t_{i-1}) = \sum_{i=1}^{N} h_{i-1i}\Delta t_{i-1}\prod_{k=i+1}^{N}(1 + f_{i+1N}\Delta t_{k-1}) + P_M. \tag{6}$$

The intrinsic value P_{t_0T} is the amount, the future value of which is equal to the sum of the face value P_M and the future value of the coupon payments $h_{i-1i}\Delta t_{i-1}$, for the period from the payment moment t_i to the maturity moment T, with forward rates equal to f_{i+1N}.

In practical terms, the notion of interest rate (coupon rate) is preferred over the stream of interest. The relation between these two values is very simple, having the following form:

$$s_{i-1i} = \frac{h_{i-1i}}{P_M}. \tag{7}$$

Hence, the formulation (5) may be expressed through coupon rates and demonstrated in the form that better lends itself to discussion:

$$\frac{P_{t_0T}}{P_M} - 1 = \sum_{i=1}^{N} \frac{(s_{i-1i} - f_{i-1i})\Delta t_{i-1}}{\prod_{k=1}^{i}(1 + r_{0i}\Delta t_{k-1})} \tag{8}$$

The left side refers the intrinsic value to the nominal value; hence it can be called a normalised intrinsic value. The right side clearly depends on the relation between coupon rates and forward rates. And this fact is of fundamental significance for further considerations.

ANOMALOUS INTEREST RATE RISK

Subsequent stages of the discussion on the bond intrinsic value call for some assumptions regarding coupon rates. In practice, two types of coupon bonds are most common: with fixed and variable coupon rates. As for the latter, the coupon depends on forward rates; an analysis of the structures of various bonds used on the Polish capital market (but also in other countries) shows that it is feasible to adopt the following dependency between coupon rates and forward rates.

$$s_{i-1i} = s + pf_{i-1i}. \tag{9}$$

The different values of the coefficients s and p ($s \geq 0, p \geq 0$) lead to various types of obligations encountered on financial markets. In particular, when $p = 0$ and $s > 0$, we get bonds with fixed coupons; $p > 0$ and $s = 0$ result in bonds with the coupons proportional to forward rates; in the case of $p = 0$ and $s = 0,$ we observe zero-coupon bonds. If one attempts to interpret the aforementioned dependency, it is justifiable to conclude that the premium paid to a bondholder is composed of the part linked to the market interest rates pf_{i-1i} and the constant s that is independent of the changeable market situation reflected by the interest rate levels. Please notice that as far as the flat structure of forward rates is concerned, the coupon rate remains constant and equal to $s_{i-1i} = s + pr = const$. And so, in this case each bond (with the exception of the zero-coupon bond) is a fixed coupon bond as long as the rates remain constant.

Having applied the relation (9) to the formula (8), as well as having performed some transformations, we arrive at the following form of the normalised intrinsic value of a bond:

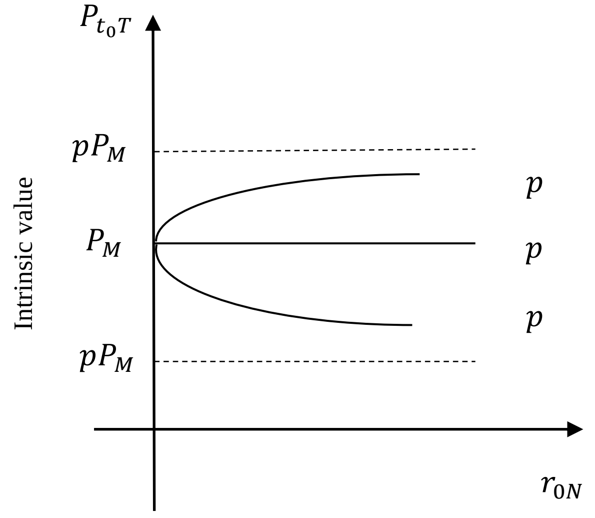
$$\frac{P_{t_0T}}{P_M} - 1 = s\sum_{i=1}^{N} \frac{\Delta t_{i-1}}{\prod_{k=1}^{i}(1 + r_{0i}\Delta t_{k-1})} + (p-1)\left(1 - \frac{1}{\prod_{k=1}^{N}(1 + r_{0N}\Delta t_{k-1})}\right). \tag{10}$$

This relation clearly makes the intrinsic value dependent on the bond structure parameters. The commonly recognised claim that the intrinsic value (price) is the decreasing function of interest rates is not quite true. Let us consider a case when $s = 0$, then in the formula (10), but other component remains unchanged. If we make an additional assumption that $p > 1$, which is often the case in practice, one can easily notice that the intrinsic value is an increasing function of spot rates. This is a direct consequence of the fact that coupon rates are functions of forward rates. The claim that the intrinsic value is a decreasing function of interest rates is based on the output formula (5) for the intrinsic value; however, what is ignored in such case is the fact that the stream of interest is, in fact, a function of forward rates.

One should also note that when $s = 0$ the intrinsic value depends only on the spot rate related to the bond maturity date r_{0N} which, among others, makes it easier to calculate the basic measure of debt securities investment feasibility – namely the yield to maturity YTM. One can also easily provide asymptotic properties of the intrinsic value when $N \to \infty$ (i.e. $T \to \infty$) or $r_{0N} \to \infty$, if only $p \neq 1$. In an exceptional case $p = 1$, we are dealing with a constant function. In terms of quality, the dependency of P_{t_0T} on spot rate r_{0N} with various scopes of the parameter p can be illustrated with figure 3. The upper curve, when $p > 1$ and the lower curve, when $p < 1$, corresponds to the increase of maturity date $T - t_0$ with the predetermined discount rate r_{0N}.

The consequence of the obtained relations is, among others, specific investment decisions, depending on the expectations towards future interest rates. For instance: if $p > 1$, then investors, awaiting an increase of interest rates, will generate demand for these bonds, expecting a price growth; this means that they will be selling bonds with the parameter $p < 1$. Moreover, longer-term bonds (the upper curve) will attract bigger interest. The investors' behaviour will be different if a drop in interest rates is expected: they will be selling their bonds with the parameter $p > 1$ and buying bonds with the parameter $p < 1$, while the shorter-term bonds will be more popular.

Figure 3. The qualitative dependence of the bond intrinsic value and interest rate when $s = 0$



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Finally, it should be noted that the described "anomalous" bonds have occurred and are present on the Polish capital market. In our reality, Treasury bonds are divided into wholesale bonds – on the primary market they are traded in tenders to the Treasury Securities Dealers, and retail bonds (savings bonds) offered to individuals. If we focus on the latter, currently, among the bonds on offer, three-year floating interest rate (TOZ) bonds can be found with a nominal value of PLN 100. Coupon rates are based on WIBOR 6M – this is the arithmetic average of the five days preceding the six-month coupon period. This rate is multiplied by the factor which in recent years is equal to 1, which in the presented considerations corresponds to the case $p = 1$. TOZ bonds have been issued since mid-1990s. For many years, the coupon has very much depended on the average yield of Treasury bills; interest was paid on a quarterly basis and the coefficient p was more than 1, when the National Bank of Poland tried to reduce inflation. However, it became less than 1, when inflation was under control. For these reasons, in 2012 issuance of Treasury bills was abandoned and the bonds were based on WIBOR 6M, while interest was paid out every six months. Since then, bonds have not been publicly traded and the coefficient p equals 1.

It was mentioned above that the coupon in subsequent interest periods is calculated on the basis of the arithmetic average of the WIBOR 6M rate from five days, the last of which is at least five days before the next interest period (all business days). The presence of the average does not undermine the previous considerations regarding the term structure of spot rates. It is determined based on prices of listed bonds, and due to the different maturities and the lack of these instruments with all possible redemption moments, the curve is generated by averaging and interpolating the actual prices observed. Thus, by definition, spot rates and the resulting forward rates are average values.

SUMMARY

The considerations presented above have been illustrated with the use of TOZ bonds, but this is not the only example. It possible to find among Treasury bonds examples of interest structures corresponding to any possible values of the parameters s and p. For instance, on the Polish market two-year zero-coupon bonds are found, i.e. OK ($s = 0$ and $p = 0$), ten-year and five-year bonds with constant coupon payments, respectively DS and PS ($s > 0$ and $p = 0$), or until recently – ten-year bonds were issued with variable coupon rate DZ ($s > 0$ and $p = 0$). Clearly, the examples above are not all the structures that can be found on the capital market. Consequently, the formula (10) presented herein covers a very large range of debt securities. Moreover, it is founded on the relation between forward and coupon rates, and the presented examples are limited to Treasury instruments. However, the considerations are directly transferred to the bonds of other issuers, business entities or local government units (municipal bonds). The only difference is the structure of the yield curve – it is determined separately for each group of issuers. The remainder of the considerations remains unchanged; it is also easy to provide examples of real bonds that are not Treasury bonds, which can be described using the formula (10) and different values of the parameter s and p.

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