


PRICING EUROPEAN OPTIONS IN THE VARIANCE GAMMA MODEL

Arkadiusz Orzechowski  <https://orcid.org/0000-0003-2872-189X>
Institute of Banking
Warsaw School of Economics, Poland
e-mail: aorzec@sgh.waw.pl

Abstract: The purpose of the article was to investigate if it is possible to speed up the process of pricing European options in the variance gamma setting. The analysis carried out for this purpose refers to the choice of the Fourier transform scheme, which allows to obtain accurately and fast the final result (theoretical value of the European option). The issues being discussed that refer to other methods of pricing options via Fourier transform are also briefly discussed.

Keywords: European options, the variance gamma model, Fourier transform

JEL classification: C02, G13

INTRODUCTION

The world's most well-known options pricing model is the one discovered by F. Black and M. Scholes [Black, Scholes 1973]. The model is based on the assumption of existence of two types of financial assets. The first one is riskless, which means that its purchase guarantees achievement of predetermined benefits. With this in mind, it can be stated that the modeling of the price path of such an instrument (in practice it is a government bond) is possible with the use of a first order ordinary differential equation. The second type of assets are stocks which prices follow geometric Brownian motion. In consequence, their prices are assumed to be unpredictable (upward and downward movements in value of the securities are described by a stochastic differential equation).

As part of the proposed theory, the following assumptions are introduced:

- short-term interest rate is known and does not change over time,
- shares' prices follow random walk,

<https://doi.org/10.22630/MIBE.2019.20.1.5>

- variance of the returns on risky assets is constant,
- conditional distribution of the market prices of risky assets is lognormal,
- stocks do not generate dividends,
- only European options are subject to analysis,
- stocks are perfectly divisible,
- short sale of both risky and risk-free assets is not restricted,
- money can be borrowed and invested at the short-term risk-free interest rate.

In order to better explain changes in the value of options, it is often assumed that the process responsible for movements of the underlying asset's prices is discontinuous. In the literature [Tankov, Voltchkova 2009], extension of the diffusion process by a jump component is explained in many ways. According to the approach of F. Black and M. Scholes [Black, Scholes 1973], the probability of achieving large profits or incurring significant losses in short time intervals is much lower than it results from the analysis of empirical data. In consequence, on the basis of historical observations, it can be concluded that out-of-the-money contracts which are close to expiration remain underestimated. Moreover, the diffusion model of option pricing is based on the concept of perfect hedging. In reality, however, perfect hedging is implausible, and one of the reasons may be the occurrence of jumps in the quotations of financial instruments. Among existing arguments supporting the inclusion of discontinuities into the models of option pricing are empirical observations confirming the occurrence of such phenomena on various segments of the financial market.

Over time, jump-diffusion models were transformed into pure jump models in which diffusion component disappears and only the one referring to jumps remains. In consequence, nowadays, all models of option pricing with discontinuities can be divided into two groups: (1) finite activity models in which continuous changes in prices of the underlying assets are occasionally disturbed by jumps [Merton 1976, Kou 2002] and (2) infinite activity models in which only discrete changes (in values of the underlying asset) in a finite time interval appear [Barndorff et al. 1991, Carr et al. 2002, Eberlein et al. 1998, Madan et al. 1998, Madan et al. 1991]. Such classification is partly confirmed by W. Schoutens [2003].

The aim of the article is to show that properly chosen Fourier transform scheme can increase efficiency of the European option pricing in the variance gamma model. The article is organized as follows. In the first section definitions of the characteristic function and the Fourier transform are given. Moreover, characteristic functions of two pure jump models are presented. In the second section, the application of the variance gamma model to the valuation of the European options is briefly discussed. In the third section, efficiency of the valuation of the contracts in the variance gamma framework is analyzed. Finally, in fourth section, the article is summarized and major conclusions are drawn.

THE FOURIER TRANSFORM AND CHARACTERISTIC FUNCTION

If $f(x)$ is a piecewise continuous real-valued function defined over the domain of real numbers which satisfies the following condition, then the Fourier transform of $f(x)$ is defined by:

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} e^{\mathbb{I}\xi x} f(x) dx, \quad (1)$$

where: \mathbb{I} is the imaginary part of the complex number and $\xi \in \mathbb{R}$.

If X is a random variable having the density function $g(x)$, then the characteristic function of X is defines by:

$$\phi_X(\xi) = \int_{-\infty}^{\infty} e^{\mathbb{I}\xi x} g(x) dx, \quad (2)$$

where the notation is consistent with previously introduced.

If G_t is a gamma process with parameters $\alpha = \beta = \frac{1}{\nu}$, where ν is the volatility of the time change, then X_t is the variance gamma process with infinite number of jumps in the finite time interval, i.e.:

$$X_t = \theta G_t + \sigma B_{G_t}, \quad (3)$$

where θ is a skewness parameter, σ is a variance, B_{G_t} is a subordinated Brownian motion and the remaining notation is consistent with previously introduced.

The distribution of the process is negatively and positively skewed when $\theta < 0$ and $\theta > 0$ respectively. Moreover, it is infinitely divisible and has stationary and independent increments.

The characteristic function of X is given by:

$$\phi_X(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{1}{\nu}} \quad (4)$$

and the characteristic function of X_t is expressed by the formula:

$$\phi_{X_t}(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{t}{\nu}}, \quad (5)$$

where the notation is consistent with previously introduced.

If X_t is a normal inverse Gaussian process with parametrs $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$, then it can be treated as an inverse Gaussian subordinated Brownian motion with a drift. If I_t is an inverse Gaussian process with parameters $\alpha = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, then X_t can be expressed by the following equation

$$X_t = \beta\delta^2 I_t + \delta B_{I_t}. \quad (6)$$

The characteristic function of X is given by:

$$\phi_X(\xi) = e^{-\delta(\sqrt{\alpha^2 - (\beta + \mathbb{I}\xi)^2} - \sqrt{\alpha^2 - \beta^2})} \quad (7)$$

and the characteristic function of X_t is expressed by the formula:

$$\phi_{X_t}(\xi) = e^{-\delta t(\sqrt{\alpha^2 - (\beta + \mathbb{I}\xi)^2} - \sqrt{\alpha^2 - \beta^2})}, \quad (8)$$

where the notation is consistent with previously introduced.

Although normal inverse Gaussian processes can be applied to pricing option they are left for later analysis. In the remaining part of the article only variance gamma processes are of special interest.

PRICING EUROPEAN OPTIONS VIA FOURIER TRANSFORM

If the general form of the characteristic function of a variable identified with log-price of the underlying asset is known, then the price of the European option can be easily determined. As there are many approaches to the calculation of the Fourier and inverse Fourier transforms final formulas for the theoretical price of the European option can take different forms. Assuming that T is the moment of option's expiration and t is the moment of option's pricing characteristic function of the natural logarithm of the spot price of the underlying asset is given by:

$$\phi(\xi) = \left(1 - \mathbb{I}\theta v\xi + \frac{1}{2}\sigma^2 v\xi^2\right)^{-\frac{(T-t)}{v}} e^{\mathbb{I}\xi\left(s_t + (T-t)\left(r + \frac{1}{v}\ln\left(1 - \theta v - \frac{1}{2}\sigma^2 v\right)\right)\right)}, \quad (9)$$

where: $s_t = \ln(S_t)$ and the remaining notation is consistent with previously introduced.

Assuming k is the natural logarithm the exercise price K , S_0 is the price of the underlying asset at time $t = 0$ and r is the risk-free rate of return, the formulas for the price of the European call in the methods of P. Carr and D. Madan [Carr, Madan 1999], G. Bakshi and D. Madan [Bakshi, Madan 2000], M. Attari [Attari 2004], and A. Orzechowski [Orzechowski 2018] are the following:

1. P. Carr and D. Madan [1999] for $\alpha = 1$ (the method is referred to as VG-CM):

$$C(S_0, 0) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{e^{-rT} \phi(\xi - (\alpha+1)\mathbb{I})}{\alpha^2 + \alpha - \xi^2 + \mathbb{I}(2\alpha+1)\xi}\right) d\xi. \quad (10)$$

2. P. Carr and D. Madan [1999] for $\alpha = 1$ (the method is referred to as VG-CMTV):

$$C(S_0, 0) = \frac{1}{\sinh(\alpha k)} \frac{1}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{\zeta(\xi - \mathbb{I}\alpha) + \zeta(\xi + \mathbb{I}\alpha)}{2}\right) d\xi, \quad (11)$$

where: $\zeta(\xi) = e^{-rT} \left(\frac{1}{1 + \mathbb{I}\xi} - \frac{\phi(-\mathbb{I})}{\mathbb{I}\xi} + \frac{\phi(\xi - \mathbb{I})}{\mathbb{I}\xi(1 + \mathbb{I}\xi)}\right)$.

3. G. Bakshi and D. Madan [2000] (the method is referred to as VG-BM):

$$C(S_0, 0) = \frac{1}{2}(S_0 - Ke^{-rT}) + \frac{S_0}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k} \phi(\xi - \mathbb{I})}{\mathbb{I}\xi \phi(-\mathbb{I})}\right) d\xi + \\ - Ke^{-rT} \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k} \phi(\xi)}{\mathbb{I}\xi}\right) d\xi. \quad (12)$$

4. M. Attari [2004] (the method is referred to as VG-A):

$$C(S_0, 0) = S_0 \left(1 + \frac{e^l}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}(\xi + \mathbb{I})} \phi_1(\xi)\right) d\xi\right) + \\ - e^{-rT} K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}\xi} \phi_1(\xi)\right) d\xi\right), \quad (13)$$

where: $\phi_1(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{t}{\nu}} e^{\mathbb{I}\xi\left(t\left(\frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)\right)\right)}$, and $l = \frac{K}{S_t e^{rT}}$.

5. D. S. Bates [2006] (the method is referred to as VG-B):

$$C(S_0, 0) = S_0 - e^{-rT}K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k}}{\mathbb{I}\xi(1-\mathbb{I}\xi)}\right) \phi(\xi) d\xi\right). \tag{14}$$

6. A. Lewis [2001] and A. Lipton [2002] (the method is referred to as VG-LL):

$$C(S_0, 0) = S_0 - \frac{\sqrt{S_0 K} e^{-\frac{rT}{2}}}{\pi} \int_0^\infty \Re\left(\phi\left(\xi - \frac{\mathbb{I}}{2}\right) \frac{e^{\mathbb{I}u(s_0 - k + rT)}}{\xi^2 + \frac{1}{4}}\right) d\xi. \tag{15}$$

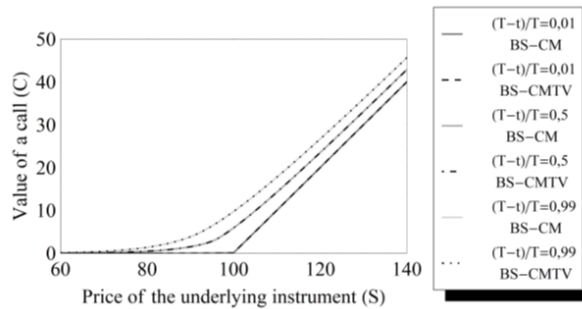
7. A. Orzechowski [2018] (the method is referred to as VG-Au):

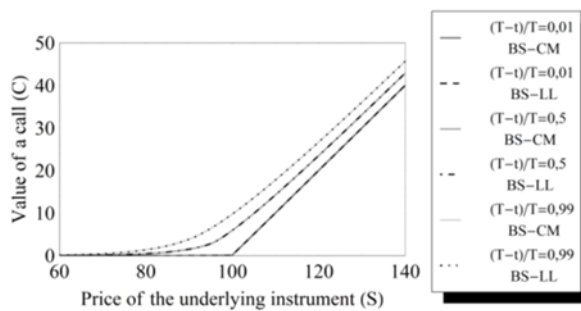
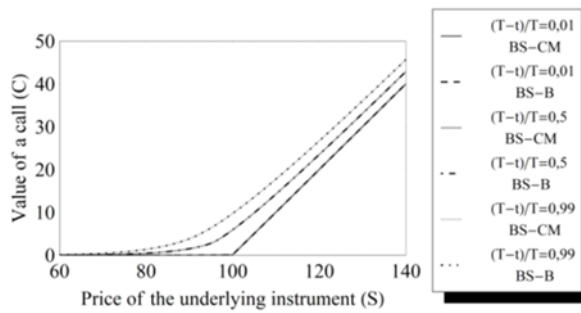
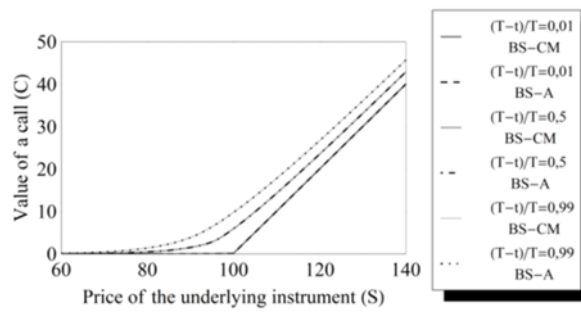
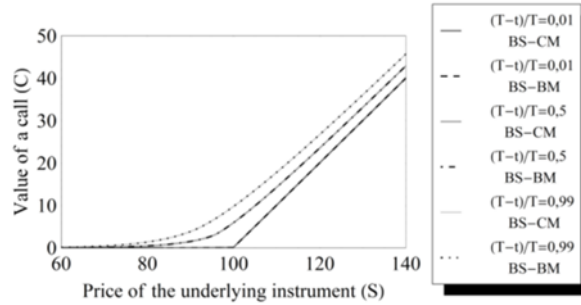
$$C(S_0, 0) = \frac{1}{2}S_0 - e^{-rT} \frac{1}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{\phi(\xi - \mathbb{I})}{\mathbb{I}\xi(\mathbb{I}\xi + 1)}\right) d\xi. \tag{16}$$

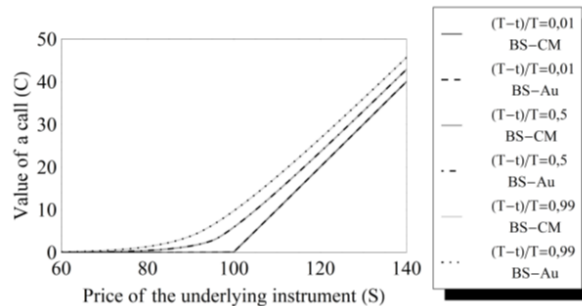
As no analytical methods of the European options' pricing were included in the article computational speed and accuracy are investigated with respect to the VG-CM method.

Computational accuracy is analyzed by comparing deviations of the theoretical prices of the European calls in the VG-CMTV, VG-BM, VG-A, VG-B, VG-LL, VG-Au methods from the theoretical values of the contracts in the VG-CMTV method. In every case it is assumed that: $K = 100$, $r = 5\%$, $\sigma = 20\%$, $\theta = -0.06$, $\nu = 1.44$ and $S \in [60, 140]$. Obtained results are presented in Figure 1.

Figure 1. Computational accuracy in the variance gamma model assuming that: $\theta = -0.06$, $\nu = 1.44$ in VG-CMTV, VG-BM, VG-A, VG-B, VG-LL, VG-Au methods comparing to VG-CMTV method





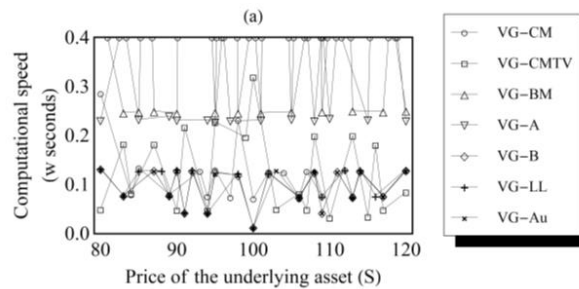


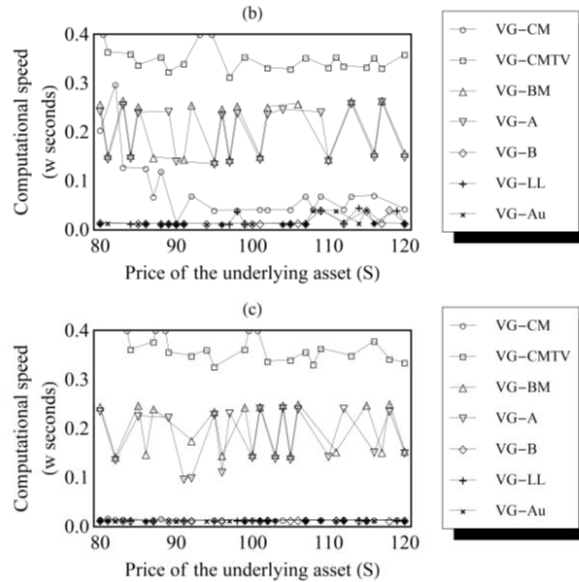
Source: own elaboration

From Figure 1 it can be easily concluded that all methods of pricing European options based on the Fourier transform are relatively well convergent to the VG-CM method. It means that all approaches to the valuation of options are comparable in terms of computational accuracy.

In order to investigate computational speed of the contracts' valuation, appropriate codes are developed in Mathematica 10.2. The package being used is run on a computer with Intel i5-4210U CPU @ 1.70 GHz processor with RAM memory of 6 GB. Each time, before the codes are started, cache memory is deleted. It is done in order to force the written blocks of commands to be restarted by the computer. The results of the tests carried out are expressed in the graphic form - see Figure 2.

Figure 2. Computational accuracy in the variance gamma model assuming that: $\theta = -0.06$, $\nu = 1.44$ for: (a) $\frac{T-t}{T} = 0.01$, (b) $\frac{T-t}{T} = 0.5$ i (c) $\frac{T-t}{T} = 0.99$





Source: own elaboration

Obtained results are ambiguous for the contracts which are close to expiration. In other cases, however, it can be easily concluded that the slowest methods of pricing European options in the variance gamma setting are: VG-CMTV, VG-BM and VG-A. A little faster is the VG-CM method and the fastest methods of pricing European options are: VG-B, VG-LL and VG-Au. In their cases parameter ξ in the denominator of the subintegral function is squared and there is only one characteristic function of the variable S_t in the subintegral function of the final formula for the price of the European call.

SUMMARY

In the finance industry more and more important is the speed of valuation of all instruments listed on the stock exchanges. This happens because developing computer technologies strongly affect the time of obtaining and processing market information. In consequence, investment strategies that are implemented by investors to a greater extent are focused on searching and immediately discounting every relevant piece of news appearing on the market. Sometimes the strategies are implemented to the high-frequency or algorithmic trading.

In the article, it was shown that the right choice of the Fourier transform scheme is needed to price the European options in variance gamma model. The scheme of the Fourier transform plays a key role in the effectiveness of the valuation of the analyzed contracts. Moreover, it was proved that the approach proposed by the author of the article, i.e. VG-Au, belongs to the group of methods

that are equally accurate as VG-CM, VG-CMTV, VG-BM, VG-A, VG-B, and VG-LL methods. At the same time, the method allows to speed up calculations comparing to some of the existing approaches to the valuation of the European options.

REFERENCES

- Attari M. (2004) Option Pricing Using Fourier Transform: A Numerically Efficient Simplification. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=520042, [dostep: 1.07.2018].
- Bakshi G., Madan D. (2000) Spanning and Derivative-Security Valuation. *Journal of Financial Economics*, 2(55), 205-238.
- Barndorff-Nielsen O. E. (1991) Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling. *Scandinavian Journal of Statistics*, 24(1), 1-13.
- Bates D. S. (2006) Maximum Likelihood Estimation of Latent Affine Processes. *Review of Financial Studies*, 19(3), 909-965.
- Black F., Scholes M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3), 637-654.
- Carr P., Geman H., Madan D. B., Yor M. (2002) The Fine Structure of Asset Returns: An Empirical Investigation. *Journal of Business*, 75(2), 305-332.
- Carr P., Madan D. B. (1999) Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, 2(4), 61-73.
- Eberlein E., Keller U., Prause K. (1998) New Insights into Smile, Mispricing and Value at Risk: The Hyperbolic Model. *Journal of Business*, 71(3), 371-405.
- Kou S. G. (2002) Jump-Diffusion Model for Option Pricing. *Management Science*, 48(8), 1086-1101.
- Lewis A. (2001) A Simple Option Formula for General Jump-Diffusion and Other Exponential Levy Processes. *SSRN Electronic Journal*, 1-25.
- Lipton A. (2002) The Vol Smile Problem. http://www.math.ku.dk/~rolf/Lipton_VolSmileProblem.pdf [access date: 8.12.2017].
- Madan D. B., Carr P., Chang E. (1998) The Variance Gamma Process and Option Pricing Model. *European Finance Review*, 2(1), 79-105.
- Madan D. B., Milne F. (1991) Option Pricing with VG Martingale Components. *Mathematical Finance*, 1(4), 39-55.
- Merton R. C. (1976) Option Pricing When Underlying Stock Returns Are Discontinuous. *Journal of Financial Economics*, 3(1-2), 125-144.
- Orzechowski A. (2018) Pricing Correlation Options: from the P. Carr and D. Madan Approach to the New Method Based on the Fourier Transform. *Economics and Business Review*, 4(1), 16-28.
- Schoutens W. (2003) *Lévy Processes in Finance: Pricing Financial Derivatives*. John Wiley & Sons, Chichester.