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ASYMMETRIC SQUARE ROOT OPTIONS – CAN WE PRICE THEM VIA THE FOURIER TRANSFORM?

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Abstract: The aim of this article is to investigate computational speed and convergence of asymmetric square root options' pricing in the F. Black and M. Scholes setting. The methodology of the conducted research is based on the comparison of pricing efficiency of the contracts with the use of BS, FT-BM and FT-B methods (including two different numerical schemes). Based on obtained results, it can be concluded that the BS method is better than methods based on the Fourier transform. However, it can be used only in the F. Black and M. Scholes setting. When analyzing other models, e.g. stochastic volatility models, one should use model based on the Fourier transform. Among those described in the article, the most efficient is FT-B with Clenshaw-Curtis numerical rule.

Keywords: asymmetric square root options, Fourier transform, Black-Scholes model

JEL classification: C02, G13

INTRODUCTION

Asymmetric square root options are exotic derivatives which payoff functions depend on the path followed by the difference between the square roots of the underlying asset's price and the exercise price at the moment of the contract's expiration (for the call option). Such instruments allow investors to implement conservative speculative or hedging strategies at a low cost - asymmetric square root options are cheaper than corresponding plain vanilla contracts or power options (assuming that the power to which the payoff function is raised is greater than one).

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There are many methods of calculating theoretical values of this kind of derivatives. All of them can be divided into three main groups:

1. methods based on partial integro-differential equations and numerical schemes dedicated to these equations [Cont, Voltchkova 2005],
2. Monte Carlo methods and other simulation techniques [Brandimarte 2014],
3. methods based on stochastic differential equations.

Taking into account the above mentioned classification, it can be easily seen that the further part of the considerations to the largest extent concerns the third group of methods. Only in their case, apart from the methods presented in the article, approaches referring to the discrete and the fast Fourier transforms [Carr, Madan 1999] as well as the Fourier series [Fang, Oosterlee 2008] can be distinguished.

The aim of this article is to investigate computational speed and accuracy of the valuation of asymmetric square root options in the F. Black and M. Scholes setting [Black, Scholes 1973]. As a part of the conducted research analytical formula for the theoretical values of the analyzed derivatives is determined. Then the Fourier transform theory is applied to the valuation of asymmetric square root options. Finally, the role of selected numerical schemes in the pricing of the analyzed contracts is discussed. In the summary, main conclusions are drawn and the best method of the valuation of asymmetric square root options is chosen.

VALUATION OF ASYMMETRIC SQUARE ROOT OPTIONS

In order to achieve the objective defined in the introduction to this article, simulation experiments were performed. In consequence, the best method of the valuation of the analyzed derivatives, out of the martingale approach [Black, Scholes 1973] and two approaches based on the Fourier transform, was selected. It is worth noting that the determination of the theoretical values of the analyzed derivatives via the Fourier transform is limited only to methodologies proposed by [Bakshi, Madan 2000] and [Bates 2006]. It can not be overlooked that square root options, as well as many other exotic options [Orzechowski 2018a, Orzechowski 2018b, Orzechowski 2018c, Orzechowski 2018d, Orzechowski 2017], according to the best knowledge of the author, are not traded on any of the exchanges. It causes great difficulty in getting access to empirical data. As a result, it is not possible to investigate how close to each other are theoretical and empirical prices of the contracts.

Martingale approach

The first method of the valuation of square root options is identified with the F. Black and M. Scholes model [Black, Scholes 1973]. The approach presented in the article, however, is more similar to the martingale method, which assumes that the option is worth as much as the expected value (calculated with respect to a certain martingale measure) of the profits generated by the contract discounted to the moment of valuation. Taking into account payoff function of asymmetric square root options, it can be stated that the following formula is correct:

$$C(S_0, 0) = e^{-rT} E^{\mathbb{Q}} \left(\left(\sqrt{S_T} - \sqrt{K}, 0 \right)^+ | \Omega_0 \right), \tag{1}$$

where: S_0 and S_T are the market values of the underlying asset at the time of writing and the expiration of the option, K is the exercise price, r is the risk-free interest rate, e^{-rT} is the discount factor, $E^{\mathbb{Q}}$ is the operator of the expected value with respect to a certain martingale measure $\mathbb{Q}$, and Ω_0 is the price history of the underlying instrument until the moment of the valuation.

Formula (1) can be transformed to the following form:

$$C(S_0, 0) = E^{\mathbb{Q}} \left(e^{-rT} \sqrt{S_T} \mathbb{1}_{\{S_T > K\}} \right) - e^{-rT} \sqrt{K} E^{\mathbb{Q}} \left(\mathbb{1}_{\{S_T > K\}} \right), \tag{2}$$

where: $\mathbb{1}_{\{S_T > K\}}$ is the indicator function taking value of 1 when $S_T > K$ and 0 otherwise, and the remaining notation is the same as previously introduced.

Changing the martingale measure of $E^{\mathbb{Q}} \left(e^{-rT} \sqrt{S_T} \mathbb{1}_{\{S_T > K\}} \right)$ and applying the Itô lemma, first for the square root of the price of the underlying asset, then for the natural logarithm of S_T allows to obtain the theoretical price of asymmetric square root options (the method is referred to as BS):

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2} rT - \frac{1}{8} \sigma^2 T} \mathcal{N}(d_1) - \sqrt{K} e^{-rT} \mathcal{N}(d_2), \tag{3}$$

where: σ is the standard deviation of the rates of return of the underlying asset, and $\mathcal{N}(.)$ is the cummulative distribution function of the standardized normal distribution with parameters d_1 and d_2 defined by the following formulas:

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + rT}{\sigma\sqrt{T}}, \tag{4}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \tag{5}$$

Based on equations (3) - (5) it is easy to determine payoff functions of the European square root options (BSSR) and compare them with the payoff functions of the plain vanilla European contracts (BSPV) - see Figures 1 - 3. Figures 1 - 3 are prepared assuming that the prices of the underlying asset are between 20 to 110, standard deviation of the rates of return of the underlying asset is equal to 29%, risk-free rate amounts to 4 %, exercise price of the option is equal to 60, and the relative time to expiration takes the following values: 0.9, 0.5 and 0.01.

Figure 1. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.9$ in the F. Black and M. Scholes model

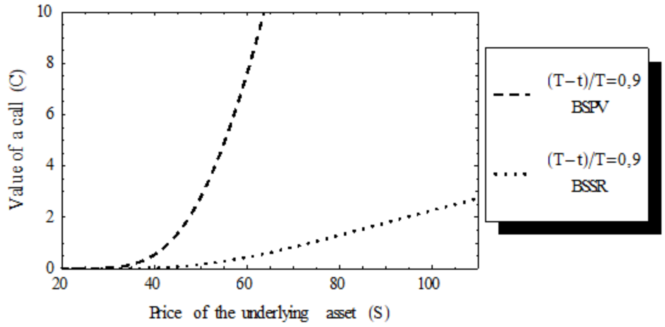

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Figure 2. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.5$ in the F. Black and M. Scholes model

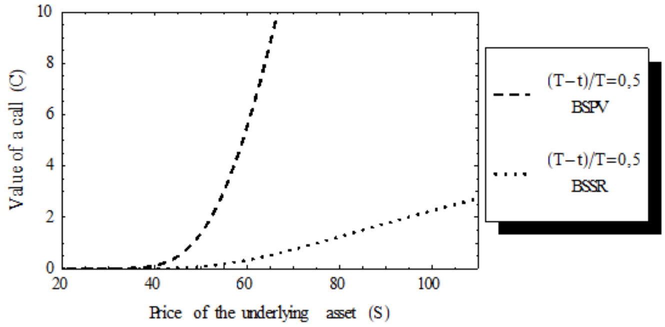

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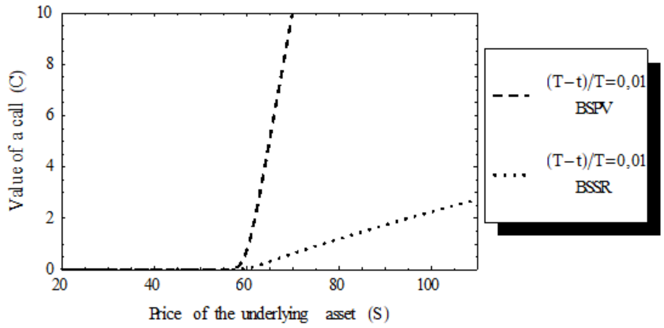
Figure 3. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.01$ in the F. Black and M. Scholes model



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Approach based on the Fourier transform

Another method of the valuation of options is based on the Fourier transform. This approach is applicable to jump-diffusion models by [Metron 1976, Kou 2002], pure jump models, such as variance-gamma model [Madan et al. 1998], NIG model [Rydberg 1997], CGMY model [Carr et al. 2002] and stochastic volatility models [Heston 1997, Stein et al. 1991].

One of the earliest methods of the valuation of options via the Fourier transform is the one developed by [Bakshi, Madan 2000]. The method (referred to as FT-BM) consists of several steps. At the beginning, logarithmic transformations of the underlying asset's price and the exercise price are performed, i.e. $s_T = lnS_T$, $k = lnK$. This allows to write the formula for the price of asymmetric square root options in the following way:

$$C(S_0, 0) = e^{-rT} \int_k^{\infty} \left(e^{\frac{1}{2}s_T} - e^{\frac{1}{2}k} \right) \mathbb{Q}(s_T | \Omega_0) ds_T \,, \tag{6}$$

where: $\mathbb{Q}(s_T|\Omega_0)$ is the probability density function of the variable s_T with filtration Ω_0, and the remaining notation is consistent with previously introduced.

Formula (6) is divided into two parts and for each of them Fourier transforms are determined. Obtained results are presented in formulas (7) and (8):

$$\chi_T^1(\xi) = \frac{e^{-rT} E\left(\sqrt{S_T}\right) \phi\left(\xi - \frac{1}{2}\mathbb{I}\right)}{i\xi \phi\left(-\frac{1}{2}\mathbb{I}\right)} \,, \tag{7}$$

$$\chi_T^2(\xi) = \sqrt{K} e^{-rT} \frac{\phi(\xi)}{\mathbb{I}\xi} \,, \tag{8}$$

where: $E(.)$ is the operator of expected value, $\phi(\xi)$ is the characteristic function of the variable s_T, $\mathbb{I}$ is the imaginary part of the complex number, and the remaining notation is consistent with previously introduced.

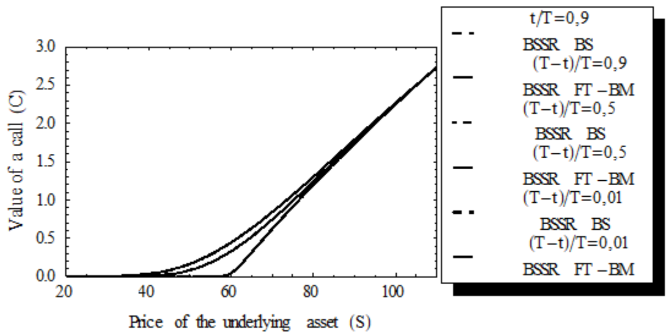
Finally, for the purpose of the valuation of analyzed derivatives, inverse Fourier transforms are calculated. As a consequence, the following equation is obtained:

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2}rT + \frac{1}{8}\sigma^2 T} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re \left[\frac{e^{-\mathbb{I}\xi k} \phi\left(\xi - \frac{1}{2}\mathbb{I}\right)}{\mathbb{I}\xi \phi\left(-\frac{1}{2}\mathbb{I}\right)} \right] d\xi \right) + \\ -\sqrt{K} e^{-rT} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re \left[\frac{e^{-\mathbb{I}\xi k} \phi(\xi)}{\mathbb{I}\xi} \right] d\xi \right), \tag{9}$$

where: $\Re[.]$ is the real part of the subintegral function, and the remaining notation is consistent with previously introduced.

It is worth noting that formula (9) can be used to price square root options almost exactly in the same way as in the Black-Scholes model (possible differences result from the fact that the integrals in formula (9) are calculated numerically). This is confirmed by Figure 4.

Figure 4. Payoff functions of European asymmetric square root call options (BBSR) for $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.9$, $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.5$ and $(\mathrm{T}-\mathrm{t})/\mathrm{T}=0.01$ determined with the use of BS and FT-BM methods



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It can be easily seen that determination of the theoretical values of square root options with the use of the FT-BM method is not effecient. From the construction of the formula (9) it can be concluded that the FT-BM method requires greater calculation effort than in the BS method. For this reason alternative concept allowing for faster and more accurate valuation of analyzed options should be considered.

From the set of available methods particular attention should be paid to the proposal of [Bates 2006]. This method is based on the transformation of formula (6) to the following form:

$$C(S_0,0)=\sqrt{S_0}e^{-\frac{1}{2}rT-\frac{1}{8}\sigma^2T}-e^{-rT}\int_{\infty}^{k}e^{\frac{1}{2}s_T}\mathbb{Q}(s_T|\Omega_0)ds_T+$$
$$-e^{-rT}e^{\frac{1}{2}k}\int_{k}^{\infty}\mathbb{Q}(s_T|\Omega_0)ds_T, \qquad (10)$$

where proposed notation is consistent with previously introduced.

For the second and third terms on the right-hand side of equation (10), the inverse Fourier transforms are determined. After appropriate simplifications it can be concluded that:

$$\chi'^{1}_{T}(\xi)=e^{-rT}\frac{1}{2\pi}e^{\frac{1}{2}k}\int_{-\infty}^{\infty}\frac{e^{-\mathbb{I}\xi k}}{\frac{1}{2}-\mathbb{I}\xi}\phi(\xi)d\xi, \qquad (11)$$

$$\chi'^{2}_{T}(\xi)=e^{-rT}\frac{1}{2\pi}e^{\frac{1}{2}k}\int_{-\infty}^{\infty}\frac{e^{-\mathbb{I}\xi k}}{\mathbb{I}\xi}\phi(\xi)d\xi, \qquad (12)$$

where proposed notation is consistent with previously introduced.

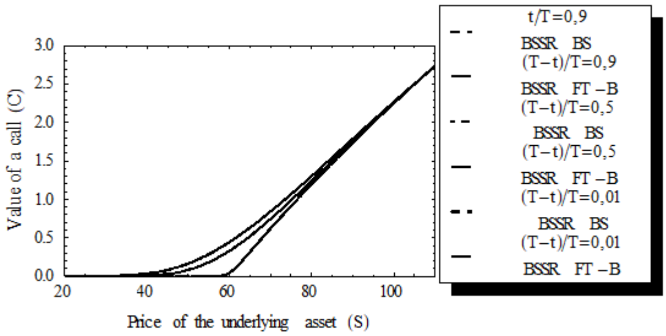
Finally, the theoretical value of asymmetric square root options is calculated as follows (the method is referred to as FT-B):

$$C(S_0,0)=\sqrt{S_0}e^{-\frac{1}{2}rT+\frac{1}{8}\sigma^2T}-\frac{1}{2}e^{-rT}\sqrt{K}+e^{-rT}\sqrt{K}\frac{1}{2\pi}\int_{0}^{\infty}\Re\left[\frac{e^{-\mathbb{I}\xi k}\phi(\xi)}{\mathbb{I}\xi\left(\mathbb{I}\xi-\frac{1}{2}\right)}\right]d\xi, \qquad (13)$$

where proposed notation is consistent with previously introduced.

Similarity of the valuation of asymmetric square root options in the BS and the FT-B methods is confirmed by Figure 5.

Figure 5. Payoff functions of European asymmetric square root call options (BBSR) for $(T-t)/T = 0.9$, $(T-t)/T = 0.5$ and $(T-t)/T = 0.01$ determined with the use of BS and FT-B methods



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It is relatively easy to see that the FT-B method is better than the FT-BM in at least two aspects, i.e. computational speed and accuracy. This is due to the fact that in formula (13), in contrast to formula (9), there is only one characteristic function and in the denominator of the subintegral function ξ is squared. It increases speed of the valuation of asymmetric square root options and improves computational accuracy of obtained results (it should be remembered that the inverse Fourier transforms in formulas (9) and (13) are calculated numerically).

Having formulas (9) and (13) one can proceed to examine how much time is required to obtain the final result of the calculations. It is assumed that the input data needed to price asymmetric square root options is identical as in the case of Figure 1-3. An important element of this part of the research is the selection of numerical schemes used for the calculation of integrals in formulas (9) and (13). For the needs of simulation experiments, the following numerical methods were selected: trapezoidal and Clenshaw-Curtis rules.

All calculations were performed in the Mathematica 8.0 run on a computer with Intel i5-4210U CPU @ 1.70 GHz processor and RAM memory of 6 GB. It is worth noting that in the proposed procedure cache memory is deleted every time before the option pricing process begins. Obtained results are presented in Tables 1-3.

Table 1. Computational speed of asymmetric square root options in seconds
$((T-t)/T=0.9)$

	OTM (S_T=50)	ATM (S_T=60)	ITM (S_T=70)
BS	0	0	0
FT-BM (trapezoidal rule)	0 .031	0 .031	0 .032
FT-BM (Clenshaw-Curtis rule)	0 .015	0 .015	0 .015
FT-B (trapezoidal rule)	0 .016	0 .015	0 .015
FT-B (Clenshaw-Curtis rule)	0 .015	0 .015	0 .015

Source: own preparation

Table 2. Computational speed of asymmetric square root options in seconds
$((T-t)/T=0.5)$

	OTM (S_T=50)	ATM (S_T=60)	ITM (S_T=70)
BS	0	0	0
FT-BM (trapezoidal rule)	0 .031	0 .031	0 .031
FT-BM (Clenshaw-Curtis rule)	0 .015	0 .016	0 .016
FT-B (trapezoidal rule))	0 .016	0 .016	0 .016
FT-B (Clenshaw-Curtis rule)	0 .016	0 .015	0 .015

Source: own preparation

Table 3. Computational speed of asymmetric square root options in seconds
$((T-t)/T=0.01)$

	OTM (S_T=50)	ATM (S_T=60)	ITM (S_T=70)
BS	0	0	0
FT-BM (trapezoidal rule)	0 .031	0 .031	0 .031
FT-BM (Clenshaw-Curtis rule)	0 .015	0 .016	0 .016
FT-B (trapezoidal rule)	0 .015	0 .016	0 .016
FT-B (Clenshaw-Curtis rule)	0 .015	0 .015	0 .015

Source: own preparation

Based on Tables 1-3 it can be concluded that the BS method allows for the fastest pricing of asymmetric square root options (irrespective of the moneyness of the contracts). Such statement, however, does not mean that the BS method is the best method of pricing analyzed contracts. Introducing assumptions concerning existence of discontinuities or stochastic volatility of the standard deviation of returns of the underlying assets makes it practically impossible to use discussed model. In consequence, there is a need to refer to alternative methods, including those based on the Fourier transform. In the case of methods in which characteristic functions are used there appear two criteria on the basis of which the best approach should be chosen:

- the number of inverse Fourier transforms (the less the better),
- the power to which ξ in the denominator of the subintegral function is raised (the higher, the faster the convergence of the method).

Moreover, the numerical scheme used to approximate theoretical values of the contracts in proposed formulas is also important. Among the methods based on the Fourier transform, the FT-B method with the Clenshaw-Curtis rule seems to be the best, regardless of the moneyness of the option and the period remaining to expiration. The worst method is the FT-BM with a trapezoidal rule.

SUMMARY

Asymmetric square root options can be priced in many ways. The choice of the appropriate method of pricing analyzed contracts depends to a large extent on the assumptions regarding the process of generating prices of underlying assets.

This article analyzes computational speed and accuracy of derivatives in the Black-Scholes setting. The research carried out indicates that the BS method is the most effecient. From the group of methods based on the Fourier transform the FT-B method with the Clenshaw-Curtis rule should be chosen. It is important to note that the drawn conclusions are true when the variance of return on underlying assets is constant and no jumps in the prices of the underlying instruments appear. At the same time, it should be remembered that the choice of the optimal method of determining the Fourier transform and the inverse Fourier transform, as well as the selection of the best numerical scheme, depends to a large extent on the assumptions under which the valuation of derivatives is carried out.

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