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SPATIAL GRAPHIC INTERPRETATION OF THE FOSTER-HART FORMULA

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Abstract: This article deals with the problem of spatial interpretation of graphical Foster-Hart formulas. The proposed approach allows the assessment of investments with specific expected payouts. This approach may also be, in a certain sense, considered as generalization in relation to the evaluation, as the author has shown how to interpret certain investment cases. It is also important that in a similar way, one can also evaluate all portfolios, which consist not only of financial instruments, but also other investment assets. The paper presents the idea of the Foster-Hart measurement on the basis of the analysis of a hypothetical action, and all simulation tests were carried out in MATLAB programming environment.

Keywords: measure of riskiness, investment, risk, portfolio management, financial model

INTRODUCTION

In the conventional models used within the portfolio management adopted in many different principles, which limit their practical utility by making them tools, whose effectiveness in real conditions is not always high. In literature, one pointed to a number of disadvantages of these models, among which dominates the lack of provisions for possible bankruptcy of the investors, both individual and institutional [Halicki 2016]. It is commonly known that the most well-known and widely used model for the Markowitz [Markowitz 1952] does not satisfy such a condition. Moreover, it assumes that the risk of a single asset forming part of the portfolio should be measured by the standard deviation. These two assumptions

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themselves reduce the usefulness of this model, both in practical and theoretical dimension, because the standard deviation is not an appropriate measure for the risk measurement (not monotone), and the possibility of bankruptcy of the investor is a decisive factor for the preference for a particular asset investment. Their palette, available on the capital market, may contain such instruments that offer on one hand attractive rates of return, with a high risk, however on the other one, increase the likelihood of bankruptcy of the investor. The problem which involves the selection of investment assets and their management, is therefore to measure the level of risk. If it is measured in the wrong way, as is the case with the Markowitz model, the construction of portfolios, taking into account the criterion of the level, will not always be correct from the perspective of diversification.

The effectiveness of portfolio management is assessed through the prism of management skills and methods they use. These skills include identifying attractive assets, and mitigate the risk. Therefore, measuring its level plays a significant role in the management of portfolios. In contrast, the investment models are the support that this measure allows. The result is a trend to develop new risk measures, improving the efficiency of portfolio management. One of them is a measure of the Hart-Foster [Foster & Hart 2009], which the literature does not pay too much attention to. This measure, being monotone, objective and universal [Foster & Hart 2009] also satisfies the condition of considering the possibility of investor's bankruptcy. Presented characteristics suggest that it could raise an interest among investors.

It should also be noted that the measure of the risk of Foster-Hart takes the form of a formula. In modern literature, the articles more often test its basic features [Chudziak & Halicki 2016; Hellmann & Riedel 2015], however, the spatial graphical interpretation of this formula was not carried out,. Such an interpretation may assist investors in the selection of investment assets in terms of the risk reduction. In this light it may seem desirable, and therefore makes an objective of the work. The reason for taking this subject up, is also the fact that the spatial interpretation of graphical formula of the Foster-Hart has also many other advantages. One of them is that they may assist investors in assessing the immediate graphical investments with specific expected payouts. In the present study, we used literature devoted to the Foster-Hart measure, as well as hypothetical data, whose analyses were performed in MATLAB software environment.

THE FOSTER-HART MEASURE AND ITS INTERPRETATION

The measure of Foster-Hart, presented in 2009 [Foster & Hart 2009], is an alternative to other well-known measures, which include among others: measure of risk based on the Markowitz Portfolio Theory, VaR (Value at Risk) and coherent risk measures. This is due to its main features, namely the ability to identify very risky investments which could lead to bankruptcy of the investor. As it is well known, other measures do not have this property. The risk measure of the Foster-Hart for a single investment takes the form of the following formula:

$$E\left[log\left(1+\frac{1}{R(g)}g\right)\right]=0\ , \qquad (1)$$

wherein *E[X]* is the expected value of a random variable *X* (the probability value that specifies the expected result of a random experiment), g means income from investments, which can be expected with a certain probability at the end of the investment period, and *R(g)* is a measure of the investment risk and the critical value of the investor's property. The *R(g)* value is calculated not only to determine the level of the risk but also for comparison with the current level of the investor's wealth. In a discrete recording of the random variable *X*, receiving the *value* x_1, x_2, ... x_n with probabilities of, respectively p_1, p_2, ... p_n, the expected value is reduced to the form:

$$E[X]=\sum_{i=1}^{n} x_i\, p_i, \qquad (2a)$$

It should be added that the validity of the equation (1) for the selection of investment occurs only when the following conditions are met:

$$\sum_{i=1}^{n} p_i=1\ ,\ \ \sum_{i=1}^{n} p_i g_i>0, \text{ and} \qquad (2b)$$

$$g_i<0\ \ then\ \ g_j>0\ , \qquad (2c)$$

wherein:

$$i\neq j \text{ and also,} \qquad (2d)$$

$$i,j=2,3,... \qquad (2e)$$

In the following discussion we will use the equivalent of a formula (1) which, for n components is as follows:

$$\left(1+\frac{g_1}{R(g)}\right)^{p_1}\cdot\left(1+\frac{g_2}{R(g)}\right)^{p_2}\cdot\ ...\ \cdot\ \left(1+\frac{g_n}{R(g)}\right)^{p_n}=1. \qquad (3)$$

The idea of the Foster-Hart measure is worth presenting by using the analysis of the hypothetical action for a specific performance, ie. the purchase price and the selling prices of the hypothetical accepted probabilities (Table 1).

Table 1. Hypothetical data for the investor who wants to purchase 1 share of interpretation of the application of the Foster-Hart formula to determine *R(g)*

The purchase price at the beginning of the investment period	1,000 USD			
The hypothetical sales price with probabilities of 25%	USD 685	USD 784	USD 1,279	USD 1,378
The corresponding investment income	USD -315	USD -216	USD 279	USD 378

Source: own study

This approach allows to determine the value of *R(g)* for which the investor is neutral towards the investment. The values of these parameters mean that the presented formula, after surgery logarithmic equation (3), takes the following form:

$$\frac{1}{4}log\left(1+\frac{-315}{R(g)}\right)+\frac{1}{4}log\left(1+\frac{-216}{R(g)}\right)+\frac{1}{4}log\left(1+\frac{279}{R(g)}\right)+\frac{1}{4}log\left(1+\frac{378}{R(g)}\right)=0. \quad (4)$$

With the solution of equation (4) we calculate the value of R(g) of 1,426.65. This means that the investor may purchase shares for USD 1,000 if in addition to this amount, he has a net worth of not less than USD 1,426.65. Therefore, the investor should have at least USD 2,426.65. This will allow him to avoid bankruptcy.

In practice, stock exchange investors may be interested in it even on the grounds that it disregards the preferences, pointing to instruments that would be too risky for them. It should be mentioned that according to a recent study, it has been shown that the optimal portfolios built with its use have high performance [Anand et al. 2016]. This means that the graphic interpretation, unprecedented in literature, may be regarded as interesting, especially if it is used in the process of evaluating various investment assets, including financial instruments, as well as for the construction of portfolios.

GRAPHIC INTERPRETATION OF THE FOSTER-HART FORMULA AND ITS ANALYSIS

An in-depth analysis of the Foster-Hart formula points out that its graphical interpretation requires appropriate mathematical transformations in order to achieve the desired relationship of parameters. For example, the relationship between g_1 and g_2 in the formula (1) or (3) for permanent p_1, p_2 and *R(g)* is determined as follows:

$$g_2 = R\cdot\left[1-\left(1+\frac{g_1}{R(g)}\right)^{\frac{p_1}{p_2}}\right]/(1+\frac{g_1}{R(g)})^{\frac{p_1}{p_2}}\text{, and} \quad (5)$$

$$p_1 + p_2 = 1. \quad (6)$$

where g_1 is the amount of income generated by an investment in the future of probability of p_1, and g_2 is the amount of income generated by an investment in the future of probability of p_2.

The results of the equation (5), (6) for the hypothetical cases of the investment are shown in Figures 2, 4, 6, but they need to accept respective assumptions for the remaining parameters. The basis for this research is the theoretical investment and its two different variants (Table 2). All simulation studies, which were used in the work, has been implemented in MATLAB programming environment.

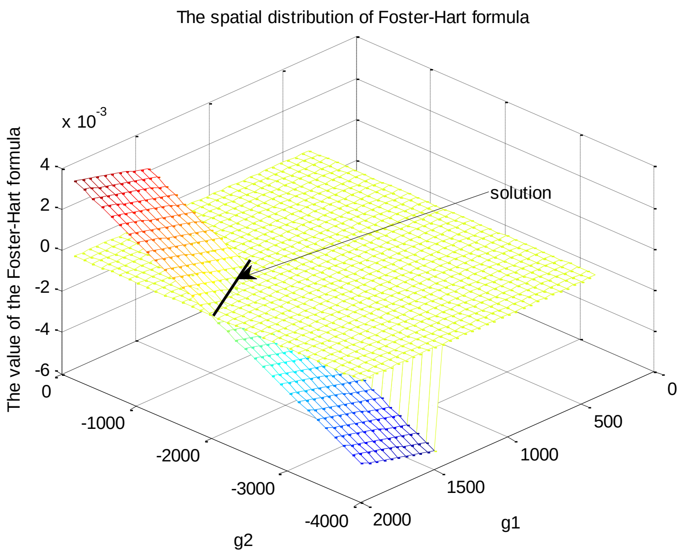
Table 2. Parameters of the sample investment and its analyzed variants

Parameters investments in the basic version	
Parameter	Value
R(g)	250,000
p_1 ($p_2 = 1 - p_1$)	0.5
Range of g_1	(1,500; 2,000)
Range of g_2	(-1,491; -1,984)
Variant number 1	
R(g)	250,000
p_1 ($p_2 = 1 - p_1$)	0.4; 0.5; 0.6; 0.7
Variant number 2	
R(g)	15,000; 20,000; 30,000; 90,000; 900,000
p_1 ($p_2 = 1 - p_1$)	0.5

Source: own calculations

Table 2 contains no g_1, g_2 values for variants 1 and 2, since the figures are based on other adopted parameters of the Foster-Hart formula. Key features differentiating presented cases of the investment is expressed by two-dimensional and three-dimensional charts. The first experiment, carried out for a number of investments in the basic version, the *R(g)*, and the probabilities are fixed, is shown in Figure 1.

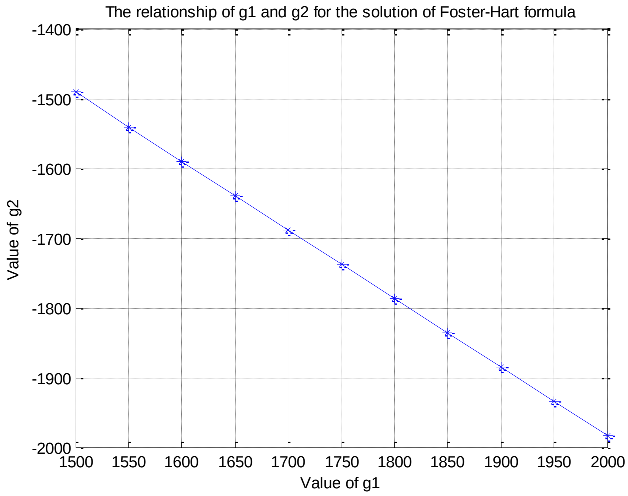
Figure 1. A three-dimensional graph of the relationship between the parameters of the analyzed investment



Source: own study based on the results of the MATLAB program

Figure 1 includes not only the graphical representation of the relationship values of the investment g_1, g_2 but also areas where no solution of the Foster-Hart formula is adopted for *R(g)*. To enhance the presentation, this solution has been selected not only on the presented three-dimensional graph, but also two-dimensional graph (Figure 2).

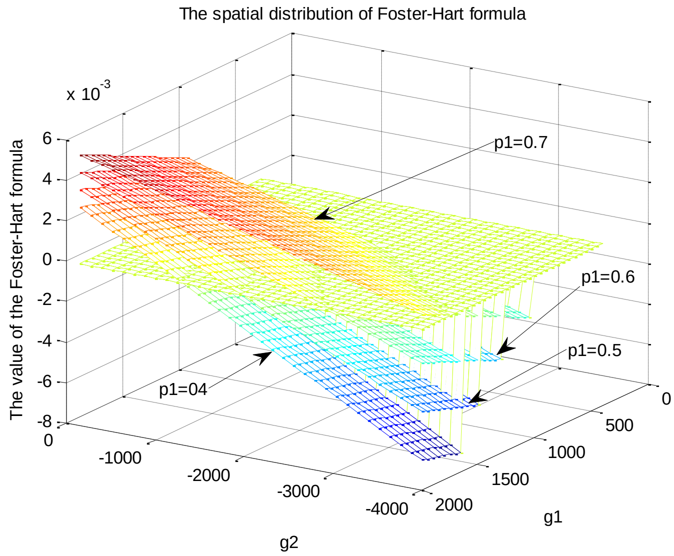
Figure 2. The diagram showing the two-dimensional solution of the Foster-Hart formula depending on the g_1 and g_2 size for the investment in the basic



Source: own study based on the results of the MATLAB program

Figure 2 shows the detailed embodiment of Figure 1 indicated as a line. If one assumes that the probability of positive and negative future revenue are the same, that is 50%, then the pair of these values, where the outcome of the characterized formula reaches a ”0” value , form a line that is depicted in Figure 2. The points not belonging to this line, are not the solution of the Foster-Hart formula, they reflect combinations of the investment, for which the value of this formula is different from zero. Analyzing the pattern (3), one should pay attention to the fact that the Foster-Hart formula is sensitive to the probability values, however the formula may get a solution only with certain pairs of values of future income of the investment. One does not need more to explain that the simulation charts containing the results of numerical experiments with changed probabilities of expected income, can be considered to be cognitively interesting (for the first variant), especially since the results of the Foster-Hart formula take a different graphic form (Figure 3 and Figure 4).

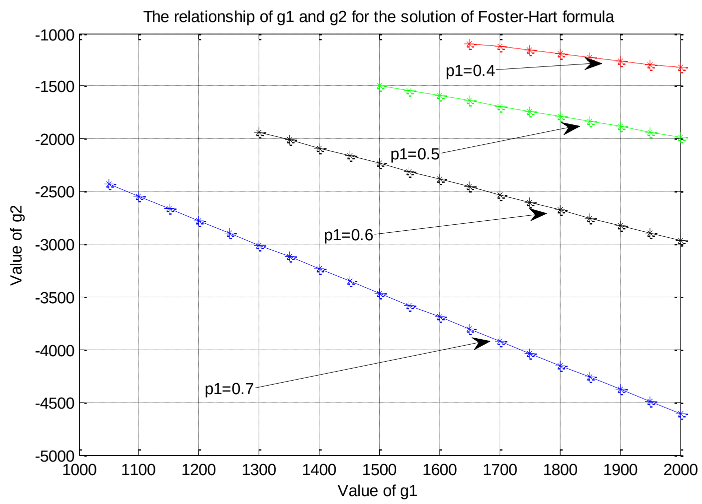
Figure 3. A three-dimensional graph of the relationship between the parameters of the first analyzed variant of the investment



Source: own study based on the results of the MATLAB program

For the accurate interpretation of the Foster-Hart formula in the analysis of the first variant, it is worth correlating the three-dimensional Figure 3 with the two-dimensional Figure 4, and in that drawing the most important conclusions.

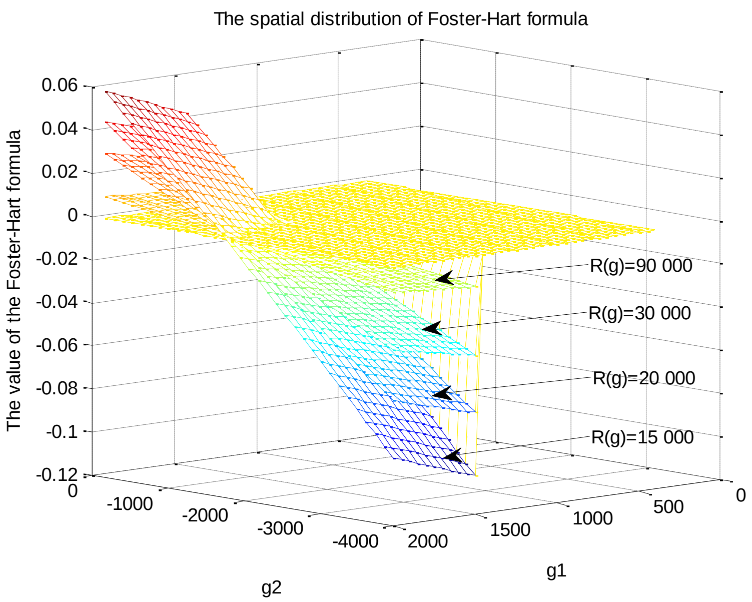
Figure 4. Diagram showing the two-dimensional solution of the Foster-Hart formula depending on the g_1 and g_2 size for the first variant



Source: own study based on the results of the MATLAB program

As it is mentioned, the graphs taking into account the change in probabilities represent a completely different investment situation, expressed even by four angled faces in Figure 3, which however in different places intersect with the horizontal plane, thereby reflecting different combinations of revenue (positive and negative) for providing a zero value of Foster – Hart formula. This application is shown in Figure 4, which summarizes the different solutions reflecting the relationships range of investment income from the specified values of probabilities (p_1 = *0.4, 0.5, 0.6, 0.7* and respectively p_2 = *0.6, 0.5, 0.4, 0.3*). The last investment variant (Table 2) take into account the impact of changes in the value of *R(g)* on the range of expected income (analyzed five possible values). This approach makes the solution for *R(g)* amounting to "900,000" on the three-dimensional graph is imperceptible at the scale adopted at the "*Z*" axis (Figure 5). This is due to the fact that the plane of the case investment is near the horizontal plane.

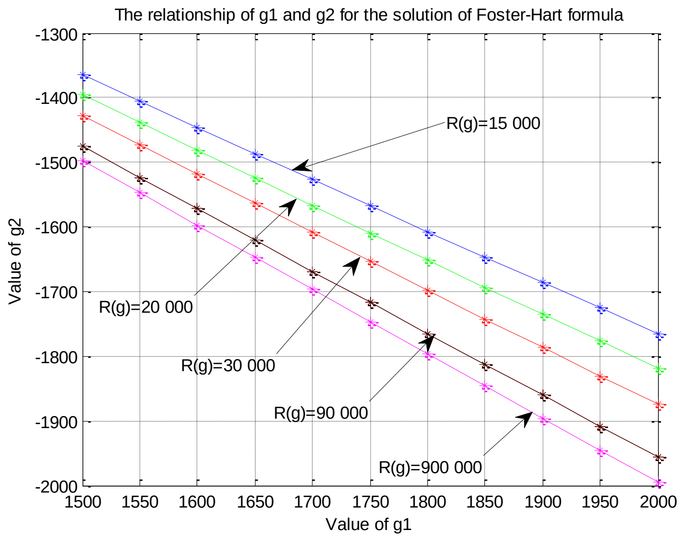
Figure 5. A three-dimensional graph of the relationship between the parameters of the second analyzed variant of the investment



Source: own study based on the results of the MATLAB program

The solution for this investment option is also presented in the form of two-dimensional graph (Figure 6), which already includes the case when *R(g)* is "900,000".

Figure 6. Graph showing the two-dimensional solution of the Foster-Hart formula depending on the value of g_1 and g_2 for the second variant



Source: own study based on the results of the MATLAB program

If the investment is characterized by certain probabilities, which are solid at varying values of *R(g)*, the oblique plane of Figure 5 in different places intersect the horizontal plane. As a result, different combinations of income (positive and negative) provide zero value of the Foster-Hart formula. This case is cognitively interesting for this reason that it allows for an analysis of the selected investments from the point of the investors' assets, whose size can vary considerably. Analyzing the last variant it is easy to see that with a relatively large values of *R(g)*, the ranges of positive and negative income are very little different. Quite different it is in the case of *R(g)* smaller by at least an order of magnitude as the differences of such ranges are already visible.

SUMMARY

The aim of this publication is the spatial graphical interpretation of the Foster-Hart formula. This subject was taken up because of the nature of this formula, which begins to arouse an interest among economists in the world, and which still devotes too little attention. Conducted considerations can be regarded as valuable mainly for this reason that they allow for the graphical assessment of individual investments or even entire portfolios that consist of financial instruments or other investment assets. Though hypothetical data were analyzed, still each investment can be written in a language in which it has been done in the publication.

In theory, there are infinitely many cases of investment, which can be analyzed graphically. Due to changes in the parameters of the Foster-Hart's equation, the choice is limited to three categories. Therefore, attention is paid to the theoretical project and its two variants. In summary, the analysis cites a number of facts indicating that the changes in investor assets are associated with significant changes in potential revenue, generated by the investment, which can be accepted by them without the fear of bankruptcy. Noteworthy is the fact that the probability of received income also affects the range of expected income, which can be accepted with certain assets of the investor. Summing up the above observations and taking into account the characteristics of the Foster-Hart measurement, it is clear that conducted considerations can be regarded as useful for the provision of the type of wealth management and in the preparation of investment products to investors. Moreover, the advantage of the graphical presentation is that one can obtain an immediate estimate of the investment case. This work can also become a base for separate studies related to the analysis of diversified portfolios in terms of Söhnholz, Rieken and Kaiser [Söhnholz & Rieken & Kaiser 2010].

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