QUANTITATIVE METHODS IN ECONOMICS
Volume XVIII, No.1, 2017, pp. 68 – 77

Received: 25.01.2017
Accepted: 05.04.2017

COMPARING PROPORTIONS OF SENSITIVE ITEMS IN TWO POPULATIONS WHEN USING POISSON AND NEGATIVE BINOMIAL ITEM COUNT TECHNIQUES

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Abstract: Sensitive attributes are extremely difficult to be measured directly. Recently new indirect methods of questioning, called Poisson and negative binomial item count techniques, have been proposed by [Tian et al. 2014]. This paper focuses on important problem of comparing proportions of sensitive items in two populations when using new indirect method. Proper statistical theory is introduced, including tests for equality of two sensitive proportions followed by derivation of their asymptotic power functions. Simulation studies are conducted to illustrate the problem.

Keywords: sensitive questions, Poisson and negative binomial item count techniques, latent data, two populations, asymptotic test

INTRODUCTION

Illegal and socially stigmatized behaviors, like tax evasion, addictions, sexual risk activities etc. are usually impossible to be measured via direct questioning. Therefore social science and mathematical statistics have developed some indirect methods of questioning, i.e. methods in which the sensitive question is not asked directly and therefore individual answer to this particular question cannot be recognized. Such a procedure guarantees privacy and allows truthful answers. One of the most popular methods of dealing with sensitive questions nowadays are various item count techniques [e.g. Gonzales-Ocantos et. al 2012, Kuha and Jackson 2013, Wolter and Laier 2014], which started with [Miller 1984] and are is still being

DOI: 10.22630/MIBE.2017.18.1.07 JEL: C15, C63

extensively developed [Imai 2011, Hussain 2012, Glynn 2013, Kuha and Jackson 2013, Tian et al. 2014].

Recent proposition by [Tian et al. 2014] assumes a new approach to item count procedure and introduces Poisson and negative binomial item count techniques. Authors propose to randomly assign respondents into control and treatment groups. In a control group respondents are asked one neutral question independent of the sensitive one with possible outcomes 0,1,2,… , which can be modeled by a counting variable *X*, e.g. "How many times have you been to the cinema last month?" In a treatment group respondents are presented with two questions: one neutral and exactly the same as in control group, *X*, and the other one sensitive, Z, with possible outcomes 0 or 1. e.g. "1. How many times have you been to the cinema last month?, 2. Have you bribed a police officer during last year? Assign 1 if yes and 0 if not. Please report only the sum of your answers." Thus respondents in a control group report *X*, and respondents in a treatment group report *Y*=*X*+*Z*. Sensitive variable Z is a latent one and is not directly observable. Because by definition *X* is a counting variable, two basic models are considered by [Tian et al. 2014] for *X*, Poisson and negative binomial. Selection of the proper model is possible after the survey is done on the basis of a control group by standard methods. In a control group of n_1 elements we observe vector $\left(X_{1,},\dots,X_{n_1}\right)$, and in a treatment group of n_2 elements we observe $\left(Y_{1,},\dots,Y_{n_2}\right)$, where $Y_j = X_{n_1+j} + Z_j$.

In the present paper we extend [Tian et al. 2014] method into two populations. The problem is crucial in statistical practice and to our best knowledge has not been developed yet. We introduce statistical tests based on asymptotic normality of unbiased estimators suitable for testing equality of sensitive proportions in two populations when using Poisson and negative binomial ICTs. We also provide asymptotic power of the proposed tests together with numerical illustration. Next to classic approach we also consider an alternative one. Finally we conduct a Monte Carlo simulation study to illustrate tests and EM algorithm performances.

CLASSIC POISSON MODEL

Preliminaries

We presume that two independent surveys are conducted in two populations with different control questions. We assume that in both cases control variable follows Poisson distribution. Therefore in population I we have a control variable $X_1 \sim Poisson(\lambda_1)$ independent of the sensitive variable $Z_1 \sim Bernoulli(\pi_1)$. Two independent random samples are observable: $\left(X_{11,},\dots,X_{1n_{11}}\right)$, $\left(Y_{11},\dots,Y_{1n_{12}}\right)$, where $Y_{1j} = X_{1n_{11}+j} + Z_{1j}$ and $n_1 = n_{11} + n_{12}$ is a total sample size. Analogously, in population II we have a control variable $X_2 \sim Poisson(\lambda_2)$ independent of the sensitive variable $Z_2 \sim Bernoulli(\pi_2)$, and two independent random samples of observable variables: $\left(X_{21,},\dots,X_{2n_{21}}\right)$ and $\left(Y_{21},\dots,Y_{2n_{22}}\right)$, $Y_{2j} = X_{2n_{22}+j} + Z_{2j}$,

$n_2 = n_{21} + n_{22}$. Unknown parameters λ_1, λ_2 can be assessed by $\bar{X}_1$ and $\bar{X}_2$ respectively. Unknown sensitive proportions π_1 and π_2 can be assessed by unbiased method of moments MM estimators $\hat{\pi}_1 = (\bar{Y}_1 - \bar{X}_1)$ and $\hat{\pi}_2 = (\bar{Y}_2 - \bar{X}_2)$ respectively. Therefore natural unbiased estimator of $\pi_2 - \pi_1$ is $\hat{\pi}_2 - \hat{\pi}_1 = (\bar{Y}_2 - \bar{X}_2) - (\bar{Y}_1 - \bar{X}_1)$ with variance:

$$D^2(\hat{\pi}_2 - \hat{\pi}_1) = \left(\frac{\lambda_1}{n_{11}} + \frac{\lambda_1 + \pi_1(1-\pi_1)}{n_{12}}\right) + \left(\frac{\lambda_2}{n_{21}} + \frac{\lambda_2 + \pi_2(1-\pi_2)}{n_{22}}\right). \tag{1}$$

Hypothesis testing

For the sake of definiteness let us focus on two sided test. Hypothesis testing problem of interest is H_0: $\pi_1 = \pi_2$ versus H_1: $\pi_1 \neq \pi_2$. Introduced test is based on asymptotic normality of unbiased estimator $\hat{\pi}_2 - \hat{\pi}_1$. Two-sided (restricted) test of size α is to reject H_0 if:

$$\frac{|\hat{\pi}_2 - \hat{\pi}_1|}{\sqrt{\left(\frac{\hat{\lambda}_1}{n_{11}} + \frac{\hat{\lambda}_1}{n_{12}}\right) + \left(\frac{\hat{\lambda}_2}{n_{21}} + \frac{\hat{\lambda}_2}{n_{22}}\right) + \hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_{12}} + \frac{1}{n_{22}}\right)}} > z_{1-\frac{\alpha}{2}}, \tag{2}$$

where $z_{1-\frac{\alpha}{2}}$ is the $\left(1 - \frac{\alpha}{2}\right)$th quantile of standard normal distribution, $\hat{\lambda}_1, \hat{\lambda}_2$ are control group sample means, and $\hat{\pi} = w_1\hat{\pi}_1 + (1 - w_1)\hat{\pi}_2$ is a restricted estimator of the joint sensitive proportion with

$$w_1 = \left(\frac{\hat{\lambda}_2}{n_{21}} + \frac{\hat{\lambda}_2}{n_{22}} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_{22}}\right) \Big/ \left(\frac{\hat{\lambda}_1}{n_{11}} + \frac{\hat{\lambda}_1}{n_{12}} + \frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_{12}} + \frac{\hat{\lambda}_2}{n_{21}} + \frac{\hat{\lambda}_2}{n_{22}} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_{22}}\right) \tag{3}$$

If alternative $\hat{\pi}_2 \neq \hat{\pi}_1$ is true, asymptotic power of the proposed test is:

$$1 - \Phi\left(z_{1-\frac{\alpha}{2}} \frac{\sqrt{\delta\lambda_1(1+k_1) + \lambda_2(1+k_2) + (\delta+1)(v_1\pi_1 + v_2\pi_2)(1 - v_1\pi_1 - v_2\pi_2)}}{\sqrt{\delta\lambda_1(1+k_1) + \lambda_2(1+k_2) + \delta\pi_1(1-\pi_1) + \pi_2(1-\pi_2)}} - \right.$$
$$\left. - \frac{|\pi_2 - \pi_1|\sqrt{n_{22}}}{\sqrt{\delta\lambda_1(1+k_1) + \lambda_2(1+k_2) + \delta\pi_1(1-\pi_1) + \pi_2(1-\pi_2)}}\right) \tag{4}$$

where $\delta = n_{22}/n_{12}$, $k_1 = n_{12}/n_{11}$, $k_2 = n_{22}/n_{21}$, $v_2 = 1 - v_1$ and

$$v_1 = \frac{\lambda_2(1+k_2) + \pi_2(1-\pi_2)}{\delta[\lambda_1(1+k_1) + \pi_1(1-\pi_1)] + \lambda_2(1+k_2) + \pi_2(1-\pi_2)} \tag{5}$$

General remarks

Theory presented in this section gives some important directions for practitioners. Although it is clear that indirect questioning demands much larger sample sizes, only exact mathematical formulas allow for definite analysis of the problem and proper survey design. Therefore below we present several numerical examples concerning power of test (2) calculated on the basis of formula (4). All examples are obtained for balanced samples, i.e. for $n_{11} = n_{12} = n_{21} = n_{22}$.

Table 1. Asymptotic power of test (2) for $\alpha = 0.05$ and different model parameters

	$\pi_1 = 0.10$ $\pi_2 = 0.20$			$\pi_1 = 0.10$ $\pi_2 = 0.25$		
sample size n_1=n_2	$\lambda_1 = 0.9$ $\lambda_2 = 1.1$	$\lambda_1 = 1.9$ $\lambda_2 = 2.1$	$\lambda_1 = 2.9$ $\lambda_2 = 3.1$	$\lambda_1 = 0.9$ $\lambda_2 = 1.1$	$\lambda_1 = 1.9$ $\lambda_2 = 2.1$	$\lambda_1 = 2.9$ $\lambda_2 = 3.1$
500	0.117	0.079	0.066	0.208	0.128	0.100
1000	0.191	0.119	0.093	0.368	0.213	0.158
2000	0.335	0.195	0.145	0.631	0.378	0.272

Source: own calculations

ALTERNATIVE POISSON MODEL

Preliminaries

Here we consider a situation when comparing two sensitive proportions is the main goal of the survey. Therefore the same control question is asked in two populations, to which the answer is independent of the population and can be modeled by a Poisson distribution (with the same parameter in two populations). In this case it is reasonable to resign from control samples to increase precision of difference between sensitive proportions estimation. Thus whole samples of n_1 and n_2 elements from population I and II respectively are allocated to treatment groups, where $Y_1 = X + Z_1$ and $Y_2 = X + Z_2$ are observable. Mathematical model is the following: $X \sim Poisson(\lambda)$ is independent of $Z_1 \sim Bernoulli\,(\pi_1)$ and $Z_2 \sim Bernoulli\,(\pi_2)$. Two independent samples of observable variables from two populations are available: $(Y_{11}, \dots, Y_{1n_1})$ and $(Y_{21}, \dots, Y_{2n_2})$. Unbiased MM estimator of $\pi_2 - \pi_1$ is $\hat{d} = \bar{Y}_2 - \bar{Y}_1$ with variance:

$$D^2(\hat{d}) = \frac{\lambda+\pi_1(1-\pi_1)}{n_1} + \frac{\lambda+\pi_2(1-\pi_2)}{n_2} \tag{6}$$

Hypothesis testing

Hypothesis testing problem of interest is H_0: $\pi_1 = \pi_2$ versus H_1: $\pi_1 \neq \pi_2$. Introduced test is based on asymptotic normality of unbiased estimator $\hat{d}$. Null hypothesis in this model implies equality of variances $D^2 Y_1 = D^2 Y_2$. Two sided test (restricted) of size α is to reject H_0 if:

$$\frac{|\bar{y}_2 - \bar{y}_1|}{\sqrt{S^2 \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} > z_{1-\frac{\alpha}{2}}, \tag{7}$$

where $S^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1+n_2}$ is a pooled sample variance and $S_i^2 = \frac{1}{n_i}\sum_{k=1}^{n_i}(Y_{ik} - \bar{Y}_i)^2$, $i = 1{,}2$. Let us further denote $n_2 = t n_1$. If alternative $\hat{\pi}_2 \neq \hat{\pi}_1$ is true, asymptotic power of the proposed test is:

$$1-\Phi\left(z_{1-\frac{\alpha}{2}}\frac{\sqrt{\lambda(t+1)+(\pi_1+\pi_2\mathrm{t})\left(1-\frac{\pi_1+\pi_2\mathrm{t}}{\mathrm{t}+1}\right)}}{\sqrt{[\lambda+\pi_1(1-\pi_1)]t+\lambda+\pi_2(1-\pi_2)}}-\frac{|\pi_2-\pi_1|\sqrt{\mathrm{n}_2}}{\sqrt{[\lambda+\pi_1(1-\pi_1)]t+\lambda+\pi_2(1-\pi_2)}}\right) \quad (8)$$

Although formula (7) takes familiar form, due to existence of control variable, implications for practitioners are not straightforward. In table 2 we present asymptotic power of test (7) obtained for selected model parameters and the same sample sizes $n_1 = n_2$.

Table 2. Asymptotic power of test (7) for $\alpha = 0.05$ and different model parameters

	$\pi_1 = 0.10$ $\pi_2 = 0.20$			$\pi_1 = 0.10$ $\pi_1 = 0.25$		
sample size	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
500	0.319	0.190	0.143	0.602	0.367	0.267
1000	0.558	0.335	0.243	0.881	0.630	0.473
2000	0.846	0.582	0.432	0.993	0.900	0.763

Source: own calculations

It is clear that asymptotic power of test (7) in alternative model is substantially larger, under similar parameters, than the one for classic model. But in alternative approach comparing proportions of sensitive items constitutes the main aim of the survey and no information about sensitive proportions in each population separately is available through MM estimation. To address this issue below we analyze ML estimation via EM algorithm, that allows for estimation all model parameters, including sensitive proportions in each population separately. The working version of ML estimation via EM algorithm for this particular model is discussed later in a simulation study.

ML estimation via EM algorithm

Likelihood function based on complete data in the analyzed model is:

$$L(\pi_1,\pi_2,\lambda;\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{z}_1,\boldsymbol{z}_2) =$$
$$\prod_{i=1}^{n_1}\frac{e^{-\lambda}\lambda^{y_{1i}-z_{1i}}}{(y_{1i}-z_{1i})!}\pi_1^{z_{1i}}(1-\pi_1)^{1-z_{1i}}\prod_{j=1}^{n_2}\frac{e^{-\lambda}\lambda^{y_{2j}-z_{2j}}}{(y_{2j}-z_{2j})!}\pi_2^{z_{2j}}(1-\pi_2)^{1-z_{2j}} \quad (9)$$

M step of EM algorithm results in:

$$\widehat{\lambda}_{ML} = \frac{1}{n_1+n_2}\left(\sum_{i=1}^{n_1}(y_{1i}-z_{1i})+\sum_{j=1}^{n_2}\left(y_{2j}-z_{2j}\right)\right), \quad (10)$$

$$\hat{\pi}_{1ML} = \frac{1}{n_1}\sum_{i=1}^{n_1}z_{1i}\,,\ \hat{\pi}_{2ML} = \frac{1}{n_2}\sum_{j=1}^{n_2}z_{2j}. \quad (11)$$

In E step values $\{z_{1i}\}_{i=1}^{n_1}$ and $\left\{z_{2j}\right\}_{j=1}^{n_2}$ are replaced by conditional expectations:

$$E(Z_1|Y_1;\pi_1,\pi_2,\lambda) = \frac{\pi_1 y_{1i}}{\pi_1 y_{1i}+\lambda(1-\pi_1)}\ , i = 1,\dots,n_1 \quad (12)$$

$$E(Z_2|Y_2;\pi_1,\pi_2,\lambda) = \frac{\pi_2 y_{2j}}{\pi_2 y_{2j}+\lambda(1-\pi_2)}, j = 1,\dots,n_2 \quad (13)$$

NEGATIVE BINOMIAL MODEL

Having presented foundation for Poisson model, obtaining analogous theory for negative binomial distribution is quite straightforward. Therefore we give here only selected closing formulas. Notation is exactly the same as in previous sections, the only difference is that here in classic negative binomial model we have $X_1 \sim NB(r_1, p_1)$, $X_2 \sim NB(r_2, p_2)$ and in alternative negative binomial model with the same control question in two populations $X \sim NB(r, p)$. In classic approach unbiased MM estimator of $\pi_2 - \pi_1$ is also $\hat{\pi}_2 - \hat{\pi}_1$, but now its variance is:

$$\frac{r_1 p_1}{(1-p_1)^2}\left(\frac{1}{n_{11}} + \frac{1}{n_{12}}\right) + \frac{\pi_1(1-\pi_1)}{n_{12}} + \frac{r_2 p_2}{(1-p_2)^2}\left(\frac{1}{n_{21}} + \frac{1}{n_{22}}\right) + \frac{\pi_2(1-\pi_2)}{n_{22}}. \tag{14}$$

Hypothesis testing problem of interest is H_0: $\pi_1 = \pi_2$ versus H_1: $\pi_1 \neq \pi_2$. Two-sided test (restricted) of size α based on asymptotic normality of unbiased estimator $\hat{\pi}_2 - \hat{\pi}_1$ is to reject H_0 if:

$$\frac{|\hat{\pi}_2 - \hat{\pi}_1|}{\sqrt{\frac{\hat{r}_1 \hat{p}_1}{(1-\hat{p}_1)^2}\left(\frac{1}{n_{11}} + \frac{1}{n_{12}}\right) + \frac{\hat{r}_2 \hat{p}_2}{(1-\hat{p}_2)^2}\left(\frac{1}{n_{21}} + \frac{1}{n_{22}}\right) + \hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_{12}} + \frac{1}{n_{22}}\right)}} > z_{1-\frac{\alpha}{2}} \tag{15}$$

where $\hat{r}_1, \hat{p}_1$ and $\hat{r}_2, \hat{p}_2$ are either MM or ML estimators of r_1, p_1 and r_2, p_2 based on control groups from two populations, $\hat{\pi} = w_1 \hat{\pi}_1 + (1 - w_1)\hat{\pi}_2$ is a restricted estimator of the joint sensitive proportion with:

$$w_1 = \left(\frac{\hat{r}_2 \hat{p}_2}{(1-\hat{p}_2)^2}\left(\frac{1}{n_{21}} + \frac{1}{n_{22}}\right) + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_{22}}\right) /$$
$$/ \left(\frac{\hat{r}_1 \hat{p}_1}{(1-\hat{p}_1)^2}\left(\frac{1}{n_{11}} + \frac{1}{n_{12}}\right) + \frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_{12}} + \frac{\hat{r}_2 \hat{p}_2}{(1-\hat{p}_2)^2}\left(\frac{1}{n_{21}} + \frac{1}{n_{22}}\right) + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_{22}}\right) \tag{16}$$

If alternative $\hat{\pi}_2 \neq \hat{\pi}_1$ is true, asymptotic power of the proposed test is:

$$1 - \Phi\left(z_{1-\frac{\alpha}{2}} \frac{\sqrt{A}}{\sqrt{B}} - \frac{|\pi_2 - \pi_1|\sqrt{\mathrm{n}_{22}}}{\sqrt{B}}\right), \tag{17}$$

where

$$A = \delta \frac{r_1 p_1}{(1-p_1)^2}(1 + k_1) + \frac{r_2 p_2}{(1-p_2)^2}(1 + k_2) +$$
$$+(\delta + 1)(a_1 \pi_1 + a_2 \pi_2)(1 - a_1 \pi_1 - a_2 \pi_2), \tag{18}$$

$$a_1 = \frac{\frac{r_2 p_2}{(1-p_2)^2}(1+k_2) + \pi_2(1-\pi_2)}{\delta\left[\frac{r_1 p_1}{(1-p_1)^2}(1+k_1) + \pi_1(1-\pi_1)\right] + \frac{r_2 p_2}{(1-p_2)^2}(1+k_2) + \pi_2(1-\pi_2)}, \tag{19}$$

$$B = \delta \frac{r_1 p_1}{(1-p_1)^2}(1 + k_1) + \frac{r_2 p_2}{(1-p_2)^2}(1 + k_2) + \delta \pi_1(1 - \pi_1) + \pi_2(1 - \pi_2) \tag{20}$$

and $a_2 = 1 - a_1$. In alternative approach, assuming that the same control variable X follows $NB(r, p)$, two sided test (restricted) of size α is exactly the same as the one defined in formula (7). If alternative $\hat{\pi}_2 \neq \hat{\pi}_1$ is true, asymptotic power of the proposed test is:

$$1-\Phi\left(z_{1-\frac{\alpha}{2}}\frac{\sqrt{\frac{rp}{(1-p)^2}(1+\mathrm{t})+(\pi_1+\pi_2\mathrm{t})\left(1-\frac{\pi_1+\pi_2\mathrm{t}}{\mathrm{t}+1}\right)}}{\sqrt{\left[\frac{rp}{(1-p)^2}+\pi_1(1-\pi_1)\right]t+\frac{rp}{(1-p)^2}+\pi_2(1-\pi_2)}}-\right.$$
$$\left.-\frac{|\pi_2-\pi_1|\sqrt{\mathrm{n}_2}}{\sqrt{\left[\frac{rp}{(1-p)^2}+\pi_1(1-\pi_1)\right]t+\frac{rp}{(1-p)^2}+\pi_2(1-\pi_2)}}\right) \quad (21)$$

Additionally, in classic approach also mixed model is possible, where $X_1 \sim Poisson(\lambda)$ and $X_2 \sim NB(r,p)$. Asymptotic test for equality of two sensitive proportions can be constructed analogously to the ones presented above.

SIMULATION STUDIES

First, a series of Monte Carlo simulation studies is conducted to assess asymptotic tests performances for Poisson classic and alternative models. 50 000 replications are used for every single set of model parameters. Summarized simulation results for type I error rates are given in Table 3. Both tests control Type I error satisfactory good for relatively small sample sizes with a tendency to minutely exceed the nominal $\alpha = 0.05$.

Table 3.Type I error rate for different Poisson model parameters, balanced designs and nominal $\alpha = 0.05$

sample size	parameters of control variables	$\pi_1 = \pi_2$			
		0.05	0.10	0.20	0.30
Classic model – Test (2)					
200	$\lambda_1 = 1, \lambda_2 = 2$	0.051	0.050	0.050	0.050
200	$\lambda_1 = 1, \lambda_2 = 3$	0.051	0.051	0.049	0.053
200	$\lambda_1 = 2, \lambda_2 = 3$	0.051	0.050	0.051	0.052
Alternative model – Test (7)					
200	λ =1	0.052	0.052	0.051	0.051
200	λ =2	0.051	0.051	0.051	0.051
200	λ =3	0.052	0.053	0.051	0.051

Source: own calculations

Next, empirical powers of the considered tests are obtained, i.e. the proportion of cases out of 50 000 where the null hypothesis is correctly rejected. In Table 4 empirical (E) and asymptotical theoretical (T) powers are juxtaposed together for selected model parameters and $\pi_1 = 0.10$, $\pi_2 = 0.20$. For each set of model parameters, absolute difference between achieved empirical power and asymptotical theoretical one is a decreasing function of a sample size, with some minor exception for each test, which is typical for simulations. For large sample size empirical power is very close to the asymptotical theoretical one.

Table 4. Empirical (E) and asymptotical theoretical (T) powers of tests (2) and (7) for $\alpha = 0.05$, $\pi_1 = 0.10$, $\pi_2 = 0.20$ and different Poisson model parameters

sample size	E	T	E	T	E	T
Classic model – Test (2)						
	$\lambda_1 = 1, \lambda_2 = 2$		$\lambda_1 = 1, \lambda_2 = 3$		$\lambda_1 = 1, \lambda_2 = 4$	
200	0.073	0.060	0.067	0.054	0.064	0.050
500	0.099	0.093	0.088	0.080	0.082	0.072
1000	0.145	0.144	0.123	0.119	0.112	0.104
2000	0.246	0.244	0.198	0.196	0.169	0.166
Alternative model –Test (6)						
	λ =1		λ =2		λ =3	
200	0.159	0.154	0.107	0.101	0.089	0.081
500	0.321	0.319	0.191	0.190	0.149	0.143
1000	0.556	0.558	0.332	0.335	0.243	0.243
2000	0.847	0.846	0.584	0.582	0.433	0.432

Source: own calculations

In the case of classic negative binomial model, rate of convergence to the limit distribution will depend on applied estimators of *r* and *p*. For estimating procedures see [e.g. Lloyd-Smith 2007]. As the problem goes beyond the purpose of this paper, here we give only exemplary simulation results for alternative negative binomial model. In Table 5 empirical power is juxtaposed with asymptotical theoretical one given in (21). For each set of model parameters 50000 replications are used.

Table 5. Empirical (E) and asymptotical theoretical (T) powers of test (7) for $\alpha = 0.05$, $\pi_1 = 0.10$, $\pi_2 = 0.20$ and different negative binomial model parameters

sample size	E	T	E	T	E	T
Alternative model – Test (6)						
	$r = 2, p = 0.2$		$r = 2, p = 0.3$		$r = 2, p = 0.4$	
200	0.216	0.209	0.140	0.135	0.103	0.095
500	0.447	0.445	0.278	0.274	0.177	0.176
1000	0.736	0.732	0.485	0.485	0.312	0.308
2000	0.955	0.954	0.779	0.776	0.545	0.541

Source: own calculations

Subsequently, a series of Monte Carlo simulation studies is conducted for Poisson alternative model to illustrate ML estimation via EM algorithm described in formulas (9)-(13). Simulation results are of particular importance in this case because no theoretical formulas for variances of estimators are available for EM algorithm. Moreover, in alternative model estimating each sensitive proportion separately is not even possible through MM estimation, thus no reference point is available. For every set of model parameters 10 000 replications are used. Summarized results for estimating difference $\pi_2 - \pi_1$ in alternative model are given in Table 6. Results indicate that the smaller sample size is (and the higher λ) the larger gain in efficiency is achieved from using ML estimation via EM algorithm as compared to simple MM estimation.

Table 6. Simulation MSE of MM and ML (via EM algorithm) estimators of difference $\pi_2 - \pi_1$ between two sensitive proportions for different Poisson model parameters

	$\pi_1 = 0.05$ $\pi_2 = 0.05$		$\pi_1 = 0.05$ $\pi_2 = 0.15$		$\pi_1 = 0.05$ $\pi_2 = 0.30$		$\pi_1 = 0{,}30$ $\pi_2 = 0{,}30$	
	MM	ML	MM	ML	MM	ML	MM	ML
λ	*n*=200							
$\lambda = 3$	0.0303	0.0272	0.0310	0.0273	0.0319	0.0276	0.0319	0.0294
$\lambda = 2$	0.0203	0.0183	0.0203	0.0181	0.0214	0.0182	0.0223	0.0204
$\lambda = 1$	0.0104	0.0093	0.0108	0.0096	0.0113	0.0096	0.0121	0.0111
n	$\lambda = 2$							
200	0.0203	0.0183	0.0203	0.0181	0.0214	0.0182	0.0223	0.0204
500	0.0081	0.0076	0.0084	0.0077	0.0084	0.0075	0.0087	0.0082
1000	0.0040	0.0039	0.0042	0.0039	0.0042	0.0038	0.0044	0.0042
2000	0.0021	0.0020	0.0020	0.0019	0.0021	0.0019	0.0022	0.0021

Source: own calculations

Table 7. Simulation MSE of ML (via EM algorithm) estimator of π_1 for different model parameters

π_1 π_2	0.05 0.05	0.15 0.15	0.30 0.30	0.05 0.15	0.15 0.05	0.05 0.30	0.30 0.05
λ	*n*=200						
$\lambda = 3$	0.0787	0.0575	0.0528	0.0632	0.0642	0.0505	0.0497
$\lambda = 2$	0.0508	0.0369	0.0373	0.0406	0.0408	0.0291	0.0303
$\lambda = 1$	0.0233	0.0182	0.0211	0.0176	0.0183	0.0131	0.0134
n	$\lambda = 2$						
200	0.0508	0.0369	0.0373	0.0406	0.0408	0.0291	0.0303
500	0.0288	0.0209	0.0237	0.0217	0.0225	0.0150	0.0152
1000	0.0193	0.0140	0.0151	0.0137	0.0141	0.0088	0.0089
2000	0.0124	0.0093	0.0082	0.0086	0.0086	0.0054	0.0052

Source: own calculations

In Table 7 results of simulation studies are presented for estimating π_1 via EM algorithm. MSE of a single proportion estimator is much higher as compared to MSE of a difference between two proportions estimator. Situation here is thus reversed to the classic one. Although alternative model is in favor when estimating difference between two sensitive proportions, the same does not apply for estimating individual sensitive proportions. MSE of ML (via EM algorithm) estimator of π_1 is very high.

SUMMARY

In the paper we have provided an extensive theory for testing equality of two sensitive proportions for two populations when using Poisson and negative binomial item count techniques, introduced earlier for a single population case in a seminal paper by [Tian et al. 2014]. To give practitioners some directions we have illustrated theoretical results by numerical calculations and simulation studies. All simulation results are consistent with theoretical models. Considering not only classic approach for two population problem, but also the alternative one, brought about some interesting results. In our opinion a compromised approach should be closely explored in a future research on multipurpose surveys concerning sensitive items.

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