THE PORTFOLIO OF FINANCIAL ASSETS OPTIMIZATION. DIFFERENT APPROACHES TO ASSESS RISK

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Abstract: Modern research has led to the rejection of the hypothesis of a normal distribution for financial asset returns. Under these conditions, the portfolio variance loses part of its informativity and can not serve as a good risk measure. The central aim of this work is the development and justification of a new technique of portfolio risk measure. We analyzed weekly stock returns of four largest German concerns: Deutsche Telekom, Siemens AG, Bayer AG and BMW. It is shown that asset returns are not normally distributed, but with good precision follow Laplace distribution (double exponential distribution). Using Laplace distribution function, we obtained the analytical expressions for VaR and CVaR risk measures and made calculations of risk measure using these approaches. Using modified Markowitz model the efficient frontiers of portfolios were constructed.

Keywords: portfolio of assets, expected return, risk measure, variance, Value-at-Risk, conditional Value-at-Risk

JEL classification: G11

INTRODUCTION

Due to its complexity, economic systems are constantly in a state of uncertainty. This uncertainty always gives rise to the risk [Vitlinskyy 1996]. This may be the risk of profit loss, risk of expenses, the risk of unused opportunities, etc. The causes of uncertainty and the resulting risk are accidental economic processes, inaccuracy, incompleteness and asymmetry of economic information.

One of the important tools for risk management is diversification [Sharpe et al. 1995; Bazylevych et al. 2011]. In practice, diversification is often realized by building a portfolio of financial assets. The portfolio theory originates from the

works of H. Markowitz [Markowitz 1952; Markowitz 1991]. The main characteristics of portfolio in this theory are mathematical expectation of return and variance (as a risk measure). This method, known as the classical theory of portfolio, relies on hypotheses about the normality of returns distribution for assets included in the portfolio, and their non-autocorrelation.

The hypotheses of classical portfolio theory are criticized in modern financial research. In works by R. Blattberg, T. Bollerslev, R. Engle the presence of "heavy tails" was discovered in the distribution of financial assets [Bollerslev 1990; Engle 1995]. Under these conditions, variance loses some part of its informativity. At present, VaR is considered a more reliable indicator of risk and its expansion to a coherent risk of CVaR and its modification [Hohlov 2012; Baumol 1963; Pflug 2000]. The question of the choice of a rational structure of the portfolio of financial assets on the basis of these measur'es is considered in the works by Alexander G. J., Baptista M. A. [Alexander, Baptista 2004] and Zabolotskyy T. [Zabolotskyy 2016a, 2016b]. Thus, the problem of optimizing financial assets portfolio remains relevant due to the openness of the question of an adequate method for measuring risk.

USED DATA

The main goal of this work is to perform a comparative analysis of financial portfolio optimization using different risk assessment techniques. We investigated the portfolio formed by stocks of four German companies: Deutsche Telekom, Siemens AG, Bayer AG, BMW. There are several approaches to determining the stock returns. We use the definition of stock returns as the ratio of the stock price by the end of the time interval to the stock price of its start. As the most adequate stock price, we considered the closing price (Adj Close). The series of stock returns will look like

$$r_t = \frac{y_{t+1}}{y_t}, \ \ t = 1,2,\ldots,T-1. \tag{1}$$

Siemens AG stock returns, calculated by the equation (1) and statistics (https://finance.yahoo.com/) are presented in Figure 1. Similarly, it is possible to characterize the stock returns of other three corporations included in the portfolio. Using the weekly price data from 01 January 2008 to 06 November 2017, we obtained statistical characteristics for stock returns of all four corporations (Table 1).

Table 1. Statistical characteristics of stock returns

	Bayer AG	BMW	Deutsche Telecom	Siemens AG
minimum	0.811	0.759	0.778	0.766
maximum	1.129	1.175	1.262	1.177
average	1.002	1.003	1.000	1.001
standard deviation	0.038	0.047	0.037	0.040

Source: https://finance.yahoo.com/

Figure 1. Stock returns of Siemens AG
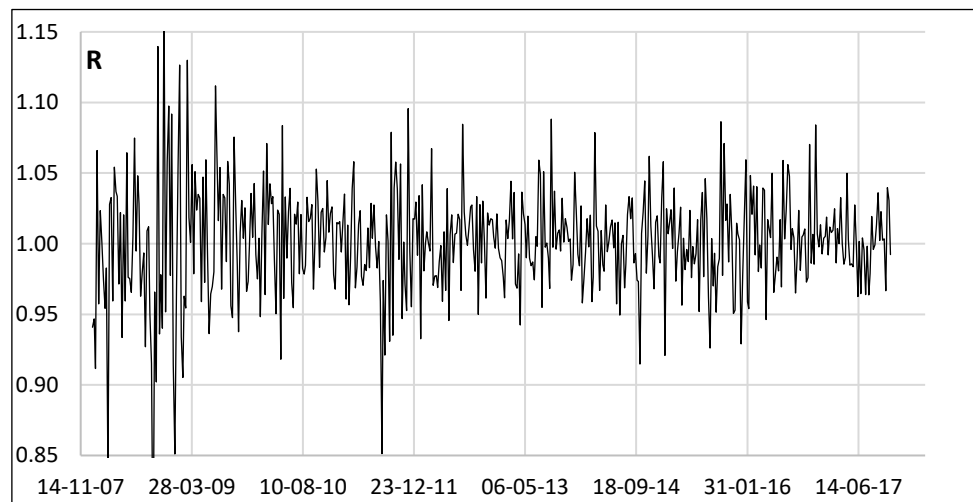
Source: own preparation

IDENTIFICATION OF THE STOCK RETURNS DISTRIBUTION

Markowitz model is based on the assumption of a normal distribution of financial assets. The probability density of the normal distribution is

$$f(r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-\mu)^2}{2\sigma^2}} , \qquad (2)$$

where μ is the mean or expectation of the stock returns distribution, σ is the standard deviation, and σ^2 is the variance. To test the hypothesis of the normal distribution of stock returns, we used Pearson's criterion of agreement and Kolmogorov-Smirnov criterion. For applying the first criterion, it is necessary to calculate Pearson statistics using the formula

$$Q^2 = \sum_{i=1}^{k} \frac{(n_i - m_i)^2}{m_i}, \tag{3}$$

and compare it with tabular values $\chi^2_{kp}(\alpha, k-3)$. Here k is the number of intervals, m_i - the theoretical number of the random variable values in the i-th interval, n_i - the actual number of the random variable values in the i-th interval, $\alpha = 0.05$ - the level of significance of the test. In our case $\chi^2_{kp}(0.05, 7-3) = 9.49$, $Q^2 = 119.12$. Since $Q^2 > \chi^2_{kp}$, the hypothesis of normal distribution is rejected. A similar conclusion was obtained using Kolmogorov-Smirnov criterion.

The investigation of the stock returns of three other corporations also led to the rejection of normal distribution hypothesis. The main reason for the deviation from the normal distribution is the presence of "heavy tails" in stock returns distribution. This means that the probability of occurrence of extreme (very large or very small) values of stock returns is much higher than assumed by the normal distribution. Consequently, we can not apply Markowitz model to optimize the stock portfolio.

To construct a new portfolio model, it is necessary to identify stock retutns distribution and choose an adequate risk measure. Computer experiments showed that the stock returns of all four corporations are described with good precision by Laplace distribution (double exponential distribution) [Lapach et al. 2002].

The random variable with Laplace distribution has a density:

$$f(r) = \frac{b}{2} \exp\left(-b|r - \mu|\right). \tag{4}$$

Here r - stock return, μ - the mathematical expectation of the stock return, b - the coefficient that determines the excess distribution. Laplace distribution density is similar to normal distribution, but Laplace distribution has thicker tails compared with normal distribution (Figure 2). The graph is based on calculations performed using statistical data (https://finance.yahoo.com/). The task of the distribution identification is reduced to the optimal choice of parameter b. The parameter b of Laplace distribution has been selected by minimizing Pearson statistics (3).

Figure 2. Identification of Siemens AG return distribution. Solid line - actual distribution, dashed line - Laplace distribution, dotted line - normal distribution
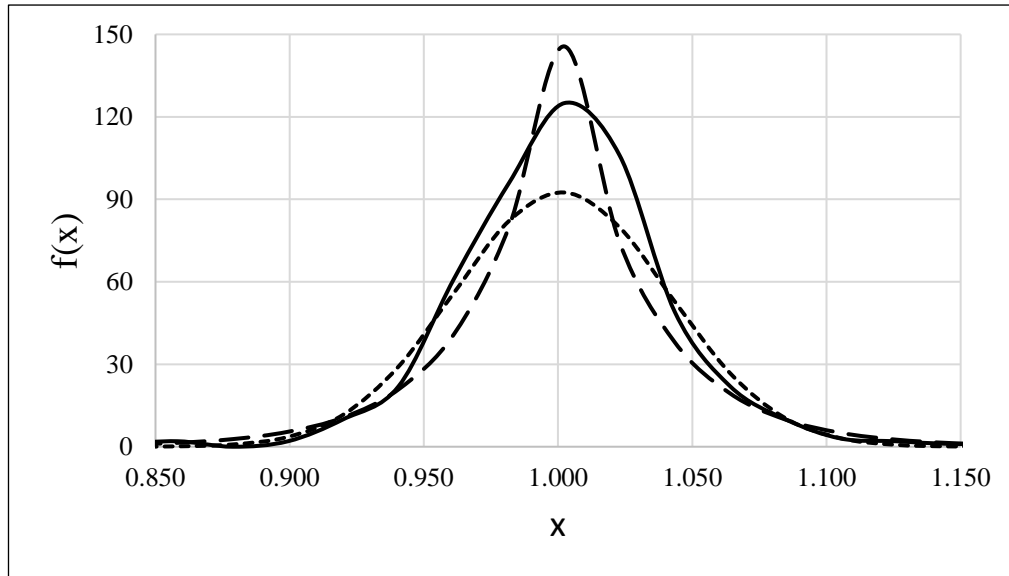
Source: own preparation

The integral Laplace distribution function has the following form

$$F(r) = \begin{cases} \dfrac{1}{2}e^{b(r-\mu)}, r \leq \mu \\ 1 - \dfrac{1}{2}e^{-b(r-\mu)}, r > \mu. \end{cases} \tag{5}$$

Checking the hypothesis about the Laplace distribution for stock returns according to Pearson criterion has confirmed the validity of the hypothesis (Table 2).

Table 2. Checking the hypothesis about Laplace distribution according to Pearson criterion

	Bayer	BMW	Deutsche Telecom	Siemens
b	32.14	27.06	33.97	32.15
μ	1.002	1.003	1.000	1.001
Q^2	22.66	21.91	26.50	25.12
χ_{kp}^2	27.59	26.30	27.59	27.59

Source: own study

RISK ASSESSMENT TECHNIQUE

Markowitz first pointed out that in constructing the portfolio of assets it is necessary to take into account not only the portfolio return but also the portfolio risk [Markowitz 1952]. In Markowitz model, the risk of i-th stock is considered as the mean-square deviation σ_i of return from its mathematical expectation. To assess the portfolio risk, it is necessary to evaluate the correlation between its components. Financial assets with high positive correlation increase the portfolio risk; financial assets, between which the correlation is weak or negative reduce the portfolio risk. The portfolio risk σ_p is determined by the function of mean-square deviation:

$$\sigma_p = \sqrt{\sum_{i=1}^{T}\sum_{j=1}^{T}\left(w_i \times \sigma_i \times w_j \times \sigma_j \times \rho_{ij}\right)} \qquad (6)$$

where: w_i, w_j - the percentage of assets in the portfolio; σ_i, σ_j - risk of assets (standard deviation of return); ρ_{ij} - Pearson correlation coefficient between the return of two assets.

In our research we follow Markowitz techniques. But the rejection of the normal distribution requires a different risk measure, that is different from the variance. In modern financial practice better risk measures are quantile-based measures. The most popular of them is the so-called Value-at-Risk (VaR) [Hohlov 2012]. VaR shows the maximal level of losses with the probability α. The parameter α is known as a confidence level. The values for α which are usually chosen are 0.9; 0.95; 0.99. To calculate the exact quantile value, it is necessary to know the distribution function of stock return F(x) (Figure 3).

At a certain confidence level of α for VaR, the risk of a financial asset with a return of X_t is [Zabolotskyy 2016a]

$$VaR_\alpha(X_t) = -\sup\{x \in \Re : F_x \leq 1-\alpha\}. \qquad (7)$$

Using the form of Laplace distribution function (5), we can find an analytic expression for risk degree at a given comfidence level α. From equality $e^{b(x-\mu)} = 2\alpha$ we define

$$VaR_\alpha = \mu + \ln 2\alpha / b. \qquad (8)$$

Figure 3. Determination of bound for risk zone VaR$_\alpha$ at level $\alpha = 5\%$ (Siemens AG)
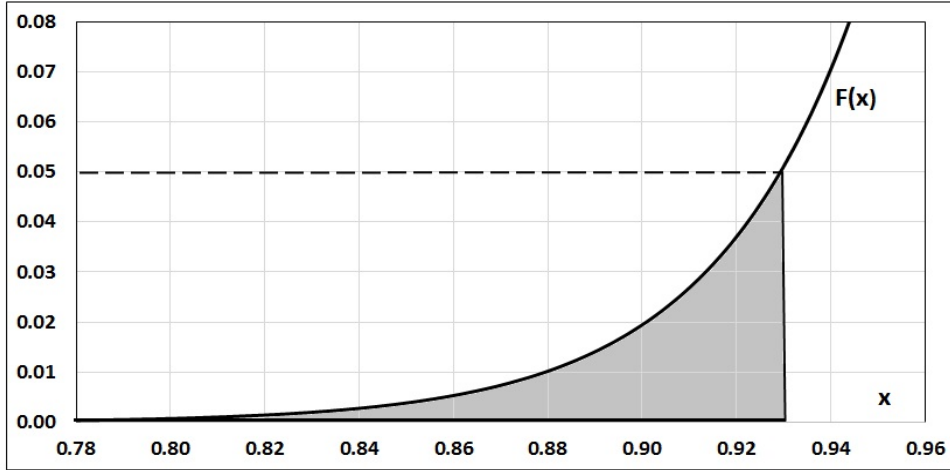
Source: own preparation

The value VaR_α specifies the limit value of the random variable x, below which the risk zone is located. To estimate the risk measure, we chose the distance from the mathematic expectation of return to the limit of the risk zone

$$V = -\frac{\ln 2\alpha}{b}. \tag{9}$$

The disadvantage of VaR is that it is not subadditive. Therefore, it is often used in its improved version - the so-called conditional VaR ($CVaR$). $CVaR_\alpha$ represents average losses with a probability of $1-\alpha$. If the function of the density of distribution $f(r)$ for the return of financial asset is known, $CVaR$ at the confidence level α can be calculated as follows:

$$CVaR_\alpha = \int_0^{VaR_\alpha} rf(r)dr \bigg/ \int_0^{VaR_\alpha} f(r)dr. \tag{10}$$

Using the expression for the function of the density of distribution (4), equality (8) and after integrating, we obtained the following expression for $CVaR_\alpha$

$$CVaR_\alpha = \left(\alpha\left(\mu + \frac{\ln 2\alpha}{b} - \frac{1}{b} \right) - \frac{1}{2b}e^{-b\mu} \right) \bigg/ \left(\alpha - \frac{1}{2}e^{-b\mu} \right). \tag{11}$$

By analogy with the previous case, to estimate the risk portfolio, we chose the distance from the mathematical expectation portfolio return to the bound of risk zone

$$V = \mu - CVaR_\alpha. \tag{12}$$

The values of the bound risk zone (Var, CVaR) and the risk measure V calculated by us are shown in Table 3.

Table 3. Estimation of stock risk by method of quantile zones ($\alpha = 0.95$)

	Bayer	BMW	Deutsche Telecom	Siemens
VaR	0.930	0.917	0.932	0.930
V var (%)	7.16	8.51	6.78	7.16
CVaR	0.900	0.881	0.903	0.899
V cvar (%)	10.15	12.16	9.69	10.18

Source: own study

PORTFOLIO OPTIMIZATION

Assuming that stock returns $r_i(t)$ are poorly stationary random processes, each of which is characterized by mathematical expectations μ_i and a degree of risk V_i, then for portfolio optimization, a modified Markovic model can be used. In this case, the mathematical description of the problem at the maximum portfolio return will have the form:

$$
\begin{cases}
R_p = w_i \times \mu_i \to \max; \\
V_p = \sqrt{\sum_{i=1}^{4}\sum_{j=1}^{4}\left(w_i \times V_i \times w_j \times V_j \times \rho_{ij}\right)} \le V_{req}; \\
w_i \ge 0; \sum w_i = 1.
\end{cases}
\tag{13}
$$

We used an approach similar to the Markovits approach to assess portfolio risk V_p, but instead of a standard deviation of stock return on the risk measure V_i we got. In contrast to the mean square deviation that describes the average deviation of stock return from its mathematical expectation, the risk measure V_i evaluates the deviation of VaR (CVaR) from the mathematical expectation of stock return. The correctness of such approach to optimizing the portfolio is analyzed in detail in the monograph of Zabolotskyy [Zabolotskyy 2016a]. The mathematical description of the problem for a minimum portfolio risk will have the form:

$$
\begin{cases}
V_p = \sqrt{\sum_{i=1}^{4}\sum_{j=1}^{4}\left(w_i \times V_i \times w_j \times V_j \times \rho_{ij}\right)} \to \min; \\
R_p = w_i \times \mu_i \ge R_{req}; \\
w_i \ge 0; \sum w_i = 1.
\end{cases}
\tag{14}
$$

Here w_i is the weight of the i-th financial asset in portfolio, V_p - general portfolio risk, V_{req} - recommended portfolio risk, R_p - overall portfolio return, R_{req} - recommended portfolio return. To optimize the portfolio, we will use the average stock return $\mu_1, \mu_2, \mu_3, \mu_4$, previously found risk estimates V_1, V_2, V_3, V_4, and a pseudo-covariance $cov(r_i, r_j) = p_{ij} \cdot V_i \cdot V_j$, where $\rho_{ij}, i = 1.4; j = 1.4$ is a Pearson correlation coefficient between the two time series of stock return.

Let's show the difference between a randomly formed portfolio and an optimal stock portfolio. Let us form portfolio within stocks of four companies (Deutsche Telekom, BMW, Bayer AG and Siemens AG), having the equal share of investment to them $w_1 = 0.25; w_2 = 0.25; w_3 = 0.25; w_4 = 0.25$. Such a portfolio (with the use of risk measure VaR) will have the following characteristics: $R_p = 1.0054; V_p = 5.97\%$ (the point on the graph – Figure 4). We will show that these characteristics will not be optimal. Indeed, using model (13) and recommended risk level $V_{req} = 5.97\%$, we get the maximum possible portfolio return $R_p = 1.0067$. If we use model (14) and recommended return level $R_p = 1.0054$, we obtain the minimum possible risk level $V_p = 5.82\%$.

Using the obtained above stock risk estimates (Table 3), we constructed the set of optimal portfolios (the efficient frontier). Each such portfolio gives maximum return at the established risk level. For the first time, the concept of optimal portfolios set was introduced by Markowitz [Markowitz 1952]. The technique for constructing the set of optimal portfolios was as follows. Initially, a portfolio structure with a minimum risk level and a minimum portfolio return was determined (model (14)). In the second step, we determined the portfolio structure with maximum portfolio return and maximum portfolio risk (model (13)). Then, by changing the risk value from the minimum to the maximum value in step 0.05 and using the model (13), we received the set of optimal portfolios. The graphic illustration of this set is shown in Figure 4.

Figure 4. The set of optimal portfolios (the risk measure Var)
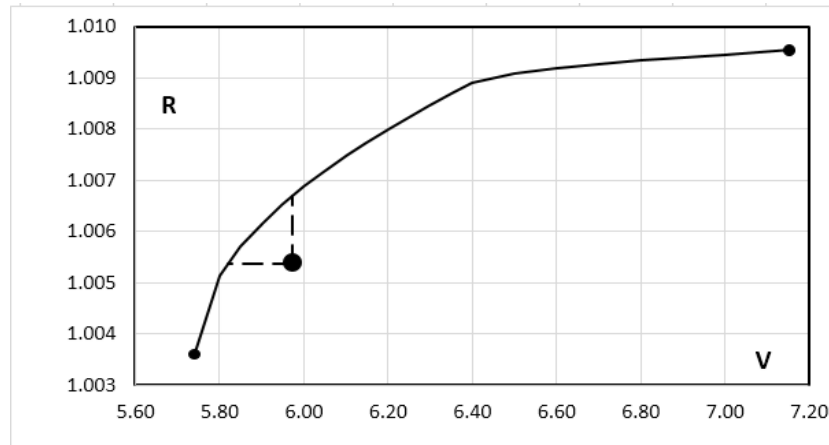
A round point represents a portfolio with equal shares of investment

Source: own preparation

     The Table 4 presents the portfolio structure for each of the optimal solutions obtained by us using stock weekly price data (https://finance.yahoo.com/). The graph and the table confirms the well-known statement that a higher return level always requires a higher risk degree.

Table 4. The set of optimal portfolios (the risk measure Var)

w_1	w_2	w_3	w_4	V_p	R_p
0.277	0.034	0.433	0.256	5.738	1.0035
0.326	0.035	0.301	0.337	5.800	1.0051
0.356	0.035	0.219	0.389	5.900	1.0061
0.378	0.035	0.160	0.427	6.000	1.0069
0.397	0.034	0.111	0.458	6.100	1.0075
0.413	0.034	0.068	0.485	6.200	1.0080
0.435	0.003	0.035	0.528	6.300	1.0085
0.435	0.003	0.000	0.563	6.400	1.0089
0.314	0.000	0.000	0.686	6.500	1.0091
0.244	0.000	0.000	0.756	6.600	1.0092
0.188	0.000	0.000	0.812	6.700	1.0093
0.140	0.000	0.000	0.860	6.800	1.0093
0.097	0.000	0.000	0.903	6.900	1.0094
0.000	0.000	0.000	1.000	7.162	1.0095

Source: own study

CONCLUSION

We have shown that the stock returns of Bayer AG, BMW, Deutsche Telekom, Siemens AG are not subject to normal distribution, but they can be described by Laplace distribution. Using the Laplace distribution function, we obtained the analytical expressions for VaR and CVaR risk measures and performed calculations of the risk assessment of considered stocks using two approaches: Var and CVaR. As a result of optimization, the set of optimal portfolios was constructed for both cases.

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