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ON STOCK TRADING WITH STOCK PRICE DRIFT AND MARKET IMPACT

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Abstract: The drift in the stock price and the occurring of the transaction costs in the stock market can significantly affect the profitability of the investment in the stock. In the article the model of the market is described with the stock price drift and two sources of the transaction costs: bid-ask spread and market impact. In the considered model, the trading strategy which maximizes the expected amount of money received from selling the shares of the stock of the market participant subject to the constraint of the constant trading velocity is explicitly determined. The numerical example is also included.

Keywords: transaction cost, market impact, stock price drift, trading speed

JEL classification: C6, G11

INTRODUCTION

Liquidity risk is one of the categories of the financial risk. The degree of liquidity is an important parameter of a stock traded on the stock exchange. It refers to the ease with which the transactions on the market can be executed. A good measure of the market liquidity is the level of transaction costs (trading costs, costs of trading) which occur on the market. The major sources of transaction costs usually taken into consideration in financial investment management are commissions (and similar payments) , bid-ask spreads and market impact [Elton et al. 1999]. An important source of transaction cost is a bid-ask spread which can be defined as the difference between the highest bid price of the stock and the stock lowest ask price. The bid-ask spread is used as proxy for the stock liquidity [Barucci 2003]. Another important source of transaction costs is

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a market impact (price impact). In case of stock trading, it can be defined as a change in a price of the stock, induced the transaction execution. This change is unfavourable to the initiator of the trade. If the initiator of the trade is a buyer of the stock, then the trade execution can cause the increase in the stock price. If the trade initiator is a stock seller, than the effect of the transaction can decrease the price of the stock. The empirical study of the market impact is shown, for example, in [Zarinelli et al. 2014]. In theoretical finance market impact modelling can be used to build realistic financial market models and explain the empirical phenomena which seem to contradict the efficiency of the financial market [Czekaj et al. 2001].

Transaction cost is an important factor determining the financial profit of the investment in the stock market. Another important factor determining the financial effect of the investment in the stock can be the stock price drift.

In the article, the trading strategy which maximizes the expected amount of money obtained from selling the shares of the stock of the market participant with the constraint of the constant trading velocity is explicitly determined. The theoretical formulas are implemented in numerical computations.

THE MARKET PRICE OF THE STOCK

Denote by $S(t)$ the market price of the stock at time t. Consider the market participant (the stock seller) who expects that there will be drift μ in the market price of the stock and this expectation is expressed as follows:

$$E(S(t)) = S_0(1+\mu t)\,\mathrm{for}\, t \in [0,T] \tag{1}$$

where $E(S(t))$ denotes the expected value of $S(t)$ and T is a parameter such that $T > 0$. The cause of the drift μ can be the effect of the reaction of the stock market investors on the information regarding the financial condition of the company that issued the stock shares, or the forecast of its future profits. For example, the positive drift may be generated by the information that the financial perspectives of the considered company are better than it was expected by the market investors. The negative drift may be, for example, generated by the announcement that the stock dividend will be lower than it was predicted by the stock investors. The market price of the stock can be calculated as the following average:

$$S(t) = \frac{S^{bid}(t) + S^{ask}(t)}{2} \tag{2}$$

where $S^{bid}(t)$ denote the highest bid price of the stock at time t by and $S^{ask}(t)$ symbolizes the lowest bid price of the stock at time t. Assume that the bid-ask spread $S^{bid}(t) - S^{ask}(t)$ is given as follows:

$$S^{bid}(t) - S^{ask}(t) = 2\lambda S(0) \tag{3}$$

where λ is the non-negative coefficient.
The discrete time model of the market impact and the drift in the stock market can be found in [Almgren, Chriss 2000].

THE MODEL OF THE STOCK SELLING

Assume that the market participant has Y shares of the stock. The parameter Y can be expressed as the fraction of the stock average traded volume in time T. The stock seller wants to maximize the amount of money obtained from selling the stock between the moments 0 and T. The selling strategy of the market participant which is executed in the time interval (t_1, t_2) is characterized by the non-random and non-decreasing function of time x such that $x(t_1) = 0$ and $x(t_2) \le Y$. The value $x(t)$ denotes the number of the shares of the stock sold up to time t. Assume that x is differentiable and denote the derivative of x at time t by $x^{'}(t)$. Denote by $S^{trade}(t)$ the trade price at time t when the strategy x is executed. If the selling strategy x is implemented, then $S^{trade}(t)$ satisfies the formula:

$$S^{trade}(t) = S(t) - \lambda S(0) - S(0)cx^{'}(t) \tag{4}$$

where c is the non-negative proportionality ratio which can be estimated, provided the suitable data sets of transactions and the stock prices are available .

The formula (4) means that the stock seller realizing the trading strategy $x(t)$, pays for selling one share of the stock the cost which is the sum of the half of the spread $\lambda S(0)$ and the market impact cost given by $S(0)cx^{'}(t)$.

The described here influence of the trade speed on the trading price of the stock seems to be similar to the dependence between the trade price and the trade velocity from [Almgren et al. 2005].

Assume that the strategy x satisfies the following constraint:

$$1 - \lambda + \mu t - cx^{'}(t) > 0 \text{ for each } t \in (t_1, t_2). \tag{5}$$

The constraint (5) is imposed in order the value of the expected value of $S^{trade}(t)$ in the equality (4) to be positive. Moreover, assume that the value of $S^{trade}(t)$ defined by (4) is positive with probability 1. When executing x at an infinitesimal time interval dt from the interval (t_1, t_2) starting from the moment $t \in (t_1, t_2)$, the trade price of one share of the stock is $S^{trade}(t)$ and the number of the stock shares sold equals to $x^{'}(t)dt$. In consequence, by (1) and (4) the expected amount of money $a(x)$ received by executing the selling strategy x is given as follows:

$$a(x)=S_0(1-\lambda)X+S_0\mu\int_{t_1}^{t_2} tx^{'}(t)dt-S_0c\int_{t_1}^{t_2} x^{'}(t)x^{'}(t)dt. \tag{6}$$

Let $X=x(t_2)$.

Consider the problem of finding the selling strategy x such that that the value of $a(x)$ is maximized, with the constraint:

$$x^{'}(t)=\frac{X}{t_2-t_1}\text{ for each } t\in(t_1,t_2). \tag{7}$$

The constraint (7) simplifies the problem of maximizing $a(x)$. Moreover, on the market where the realized trading speed can be different from the trade velocity intended by the stock seller, the negative influence of the constraint (7) on the maximization of empirical value of $a(x)$ can be moderate in practice.

By (6) and (7) it follows that

$$a(x)=S_0(1-\lambda)X+S_0\mu X\frac{t_2+t_1}{2}-S_0c\frac{X^2}{t_2-t_1}. \tag{8}$$

Define the function $f(t_1,t_2,X)$ as follows:

$$f(t_1,t_2,X)=S_0(1-\lambda)X+S_0\mu X\frac{t_2+t_1}{2}-S_0c\frac{X^2}{t_2-t_1}. \tag{9}$$

Denote by $\tilde{x}$ the selling strategy $\tilde{x}$ maximizing $a(x)$ subject to the constraint (7). Moreover, let $\tilde{t}_1$ symbolize the start of the execution of the selling strategy $\tilde{x}$ and $\tilde{t}_2$ denote the end of the execution of the trading strategy $\tilde{x}$.

By (5), (7) and (8), $\tilde{t}_1$, $\tilde{t}_2$ and the number of the stock shares $\tilde{x}(\tilde{t}_2)$ sold by the market participant executing the selling strategy $\tilde{x}$ are the values of t_1, t_2 and X, respectively which maximize $f(t_1,t_2,X)$ with the constraints:

$$0\le t_1<t_2\le T, \tag{10}$$

$$0\le X\le Y, \tag{11}$$

$$1-\lambda+\mu t-c\frac{X}{t_2-t_1}>0\text{ for each } t\in(t_1,t_2). \tag{12}$$

The inequality $X \leq Y$ follows from the assumption that the number of shares of the stock sold is not bigger than Y which symbolizes the number of the stock possessed by the market participant at the start of the selling strategy. If the constraint (12) is not satisfied then, there exists $t^* \in (t_1, t_2)$ such that $1 - \lambda + \mu t - c\frac{X}{t_2 - t_1} < 0$ for $t \in (t_1, t^*)$ and because $a(x) = S_0 \int_{t_1}^{t_2} (1 - \lambda + \mu t - c\frac{X}{t_2 - t_1})\frac{X}{t_2 - t_1} dt$, the strategy determined by of t_1, t_2 and X is not optimal. Consequently, if the values of t_1, t_2 and X maximize $f(t_1, t_2, X)$ with the constraints (10) and (11) then the constraint (12) is also satisfied. Thus, the values of t_1, t_2 and X maximizing $f(t_1, t_2, X)$ under the constraints (10) and (11) characterize the selling strategy maximizing $a(x)$ with the constraint (7).

The strategy $\tilde{x}$ can be determined by using some functions derivatives.

THE CASE $\mu > 0$

If the drift μ is positive then the selling strategy maximizing $a(x)$ with the constraint (7) is such that

$$t_2 = T \;. \tag{13}$$

Thus, $\tilde{x}$ is obtained by finding the values of t_1 and X such that the value of $f(t_1, T, X)$ is maximized with the constraints:

$$0 \leq t_1 < T \tag{14}$$

and (11).

Let $g(t_1, X) = f(t_1, T, X)$. By (9) and (13) it follows that

$$g(t_1, X) = S_0(1 - \lambda)X + S_0\mu X\frac{T + t_1}{2} - S_0 c\frac{X^2}{T - t_1} \;. \tag{15}$$

Denote by $g_{t_1}(t_1, X)$ the derivative of $g(t_1, X)$ with respect to t_1.

Denote by $t_1^g(X)$ the value of t_1 maximizing (15) with the constraint (14) for the fixed value of X. It is easily seen that $t_1^g(X)$ is given as follows:

$$t_1^g(X) = \left(T - \sqrt{\frac{2cX}{\mu}}\right)^+ . \tag{16}$$

Let $h(X) = g\left(t_1^g(X), X\right)$. The value of $\tilde{x}(T)$ equals to the value of X such that the value of $h(X)$ is maximized with the constraint (11).

By (15) and (16) the function $h(X)$ is given as follows:

$$h(X) = \begin{cases} S_0(1\text{-}\lambda)X + \mu T S_0 X - \sqrt{2\mu c} S_0 X^{\frac{3}{2}} \text{ for } X \le \dfrac{\mu T^2}{2c} \\ S_0(1-\lambda)X + S_0 X \dfrac{\mu T}{2} - S_0 c \dfrac{X^2}{T} \text{ for } X > \dfrac{\mu T^2}{2c} \end{cases} . \tag{17}$$

Denote by X^h the value of X maximizing $h(X)$ in the interval $[0,\infty)$. It is not difficult to calculate that:

$$X^h = \begin{cases} X^h = \dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right) \text{for } \mu T \le 2(1-\lambda) \\ X^h = \dfrac{2}{\mu c}\left(\dfrac{1-\lambda+\mu T}{3}\right)^2 \text{ for } X > \dfrac{\mu T^2}{2c} \end{cases} . \tag{18}$$

Consequently, $\tilde{x}(\tilde{t}_2)$ is characterized as follows:

$$\tilde{x}(\tilde{t}_2) = \begin{cases} Y \text{ for } Y \le \dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right) \text{ and } \mu T \le 2(1-\lambda) \\ \dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right) \text{for } Y > \dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right) \text{and } \mu T \le 2(1-\lambda) \\ Y \text{ for } Y \le \dfrac{2}{\mu c}\left(\dfrac{1-\lambda+\mu T}{3}\right)^2 \text{ and } \mu T > 2(1-\lambda) \\ \dfrac{2}{\mu c}\left(\dfrac{1-\lambda+\mu T}{3}\right)^2 \text{for } Y > \dfrac{2}{\mu c}\left(\dfrac{1-\lambda+\mu T}{3}\right)^2 \text{ and } \mu T > 2(1-\lambda) \end{cases} . \tag{19}$$

Moreover, by (16):

$$\tilde{t}_1 = \left(T - \sqrt{\frac{2c\tilde{x}(\tilde{t}_2)}{\mu}}\right)^+ . \tag{20}$$

THE CASE $\mu = 0$

It is easily seen that in case $\mu = 0$:

$$\tilde{t}_1 = 0, \tag{21}$$

$$\tilde{t}_2 = T. \tag{22}$$

and by (9), (21) and (22) , the number of shares $\tilde{x}(T)$ is equal to X which maximizes $f(0,T,X)$ with the constraint (11).

Denote by X^0 the value of X maximizing $f(0,T,X)$ in the interval $[0,\infty)$.

The value of X^0 is given by the following formula:

$$X^0 = \frac{(1-\lambda)T + \frac{\mu T^2}{2}}{2c} \tag{23}$$

Consequently, the number of shares $\tilde{x}(\tilde{t}_2)$ is given as follows:

$$\tilde{x}(\tilde{t}_2) = \begin{cases} Y \text{ for } Y \le \dfrac{(1-\lambda)T + \frac{\mu T^2}{2}}{2c} \\ \dfrac{(1-\lambda)T + \frac{\mu T^2}{2}}{2c} \text{ for } Y > \dfrac{(1-\lambda)T + \frac{\mu T^2}{2}}{2c} \end{cases}. \tag{24}$$

In the considered model the number of stocks can be non-integers.

THE CASE $\mu < 0$

If the drift μ is negative then the selling strategy maximizing $a(x)$ subject to the constraint () is such that

$$\tilde{t}_1 = 0. \tag{25}$$

Thus, the strategy x such that $a(x)$ is maximized with the constraint (7) is obtained by finding the values of t_2 and X such that the value of $f(0,t_2,X)$ is maximized subject to the constraints:

$$0 < t_2 \le T \tag{26}$$

and (11).

Let $k(t_1, X) = f(0,t_2,X)$. By (9) and (25) it follows that

$$k(t_1, X) = S_0(1-\lambda)X + S_0\mu X\frac{t_2}{2} - S_0 c\frac{X^2}{t_2}. \tag{27}$$

Denote by $t_2^k(X)$ the value of t_2 maximizing (27) subject to the constraint (26) for the fixed value of X. It is easily seen that $t_2^k(X)$ is given as follows:

$$t_2^k(X)=\begin{cases}\sqrt{-\dfrac{2cX}{\mu}} \text{ for } X\le-\dfrac{\mu T^2}{2c}\\ T \text{ for } X>-\dfrac{\mu T^2}{2c}\end{cases}. \tag{28}$$

Let $l(X)=k\left(t_2^k(X),X\right)$. The value of $\tilde{x}(T)$ can equals to the value of X such that the value of $l(X)$ is maximized with the constraint (11).

By (27) and (28) the function $l(X)$ is given as follows:

$$l(X)=\begin{cases}S_0(1\text{-}\lambda)X-\sqrt{-2\mu c}S_0X^{\frac{3}{2}} \text{ for } X\le-\dfrac{\mu T^2}{2c}\\ S_0(1-\lambda)X+S_0X\dfrac{\mu T}{2}-S_0c\dfrac{X^2}{T} \text{ for } X>-\dfrac{\mu T^2}{2c}\end{cases}. \tag{29}$$

Denote by X^l the value of X maximizing $l(X)$ in the interval $[0,\infty)$. It is not difficult to obtain that:

$$X^l=\begin{cases}\dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right)\text{for}-\mu T\le\dfrac{2}{3}(1-\lambda)\\ X^l=-\dfrac{2}{\mu c}\left(\dfrac{1-\lambda}{3}\right)^2 \text{for}-\mu T>\dfrac{2}{3}(1-\lambda)\end{cases}. \tag{30}$$

Consequently, $\tilde{x}(\tilde{t}_2)$ is characterized as follows:

$$\tilde{x}(\tilde{t}_2)=\begin{cases}Y \text{ for } Y\le\dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right)\text{ and } -\mu T\le\dfrac{2}{3}(1-\lambda)\\ \dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right)\text{for } Y>\dfrac{T}{2c}\left(1-\lambda+\dfrac{\mu T}{2}\right)\text{ and } -\mu T\le\dfrac{2}{3}(1-\lambda)\\ Y \text{ for } Y\le-\dfrac{2}{\mu c}\left(\dfrac{1-\lambda}{3}\right)^2\text{ and } -\mu T>\dfrac{2}{3}(1-\lambda)\\ -\dfrac{2}{\mu c}\left(\dfrac{1-\lambda}{3}\right)^2\text{for } Y>-\dfrac{2}{\mu c}\left(\dfrac{1-\lambda}{3}\right)^2\text{ and } -\mu T>\dfrac{2}{3}(1-\lambda)\end{cases}. \tag{31}$$

Moreover,

$$\tilde{t}_2 = \begin{cases} \sqrt{-\frac{2c\tilde{x}\left(\tilde{t}_2\right)}{\mu}} \text{ for } \tilde{x}(T) \le -\frac{\mu T^2}{2c} \\ T \text{ for } \tilde{x}(T) > -\frac{\mu T^2}{2c} \end{cases}. \tag{32}$$

NUMERICAL EXAMPLE

In the Table 1 there are the results of computing $a(\tilde{x})$ for 300 pairs of (μ, Y) for the following exemplary values of the parameters T, S_0, λ and c: $T = 1$, $\lambda = 0.01$, $S_0 = 1$ and $c = 1$. The parameter Y^* is expressed as the fraction of the stock average traded volume in time T ($Y^* = \frac{Y}{V}$ where V is the average traded volume of the stock).

Table 1. The values of $a(\tilde{x})$ as the function of Y^* and μ

		μ										
		-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5
Y^*	0.1	0.089	0.090	0.091	0.093	0.095	0.098	0.105	0.113	0.121	0.130	0.139
	0.2	0.170	0.173	0.176	0.180	0.185	0.194	0.205	0.220	0.236	0.253	0.270
	0.3	0.245	0.251	0.257	0.264	0.274	0.288	0.304	0.324	0.347	0.371	0.395
	0.4	0.316	0.324	0.334	0.345	0.360	0.380	0.400	0.425	0.454	0.484	0.516
	0.5	0.383	0.395	0.408	0.424	0.445	0.470	0.495	0.524	0.558	0.595	0.633
	0.6	0.447	0.463	0.480	0.501	0.528	0.558	0.588	0.621	0.660	0.703	0.747
	0.7	0.508	0.527	0.550	0.576	0.609	0.644	0.679	0.716	0.760	0.807	0.858
	0.8	0.566	0.590	0.617	0.649	0.688	0.728	0.768	0.809	0.857	0.910	0.966
	0.9	0.621	0.650	0.682	0.720	0.765	0.810	0.855	0.900	0.952	1.010	1.071
	1	0.674	0.707	0.745	0.790	0.840	0.890	0.940	0.990	1.045	1.107	1.174
	1.1	0.724	0.763	0.806	0.858	0.913	0.968	1.023	1.078	1.136	1.203	1.274
	1.2	0.772	0.816	0.866	0.924	0.984	1.044	1.104	1.164	1.226	1.296	1.372
	1.3	0.818	0.868	0.924	0.988	1.053	1.118	1.183	1.248	1.314	1.388	1.468
	1.4	0.862	0.917	0.980	1.050	1.120	1.190	1.260	1.330	1.400	1.477	1.562
	1.5	0.904	0.965	1.035	1.110	1.185	1.260	1.335	1.410	1.485	1.565	1.654

Source: own computation

The arithmetic average of bid-ask spreads at the Warsaw Stock Exchange in years 2011-2013 was not very far from 2% [Kociński 2014]. Thus, in the considered example, the half of the bid-ask spread λ (expressed at the fraction of the stock price) is 0.01. The price impact cost seems to be more difficult to quantify than the bid-ask spread and the value of c equal to 0.1 is one of the reasonable choices to the exemplary calculations.

SUMMARY

In the article the model of the market, with the drift in the stock price, bid-ask spread and the market impact induced by trading, is described. In a framework of this model the problem of determining the trading strategy which maximizes the expected amount of money received from selling the shares of the stock of the market participant under the constraint of the constant trading speed is explicitly solved. From the numerical example included in the article it follows that the drift in the stock price may have a significant impact on the financial profitability of investing in the stock market. It is not difficult to see that in case $\mu > 0$ it may be advantageous for the market participant to buy the stock shares when maximizing $a(x)$. The trading strategy with purchasing and selling the shares of the stock may be more profitable for the market participant than the strategy such that x is a non-decreasing function of time. Moreover, the trading strategies which are random processes with the possibility of varying trading speed can be more profitable than the strategies where x is a non-random function of time.

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