

## PRICING EUROPEAN OPTIONS IN SELECTED STOCHASTIC VOLATILITY MODELS

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**Abstract:** In this paper four methods of calculating characteristic functions and their application to selected stochastic volatility models are considered. The methods applied are based on the assumption that the prices of European calls are evaluated numerically by means of the Gauss-Kronrod quadrature. Such approach is used to investigate computational efficiency of pricing European calls. Particular attention in this matter is paid to the speed of generating theoretical prices of the analyzed contracts.

**Keywords:** option pricing, the Heston model, the Bates model, characteristic functions

**JEL classification:** C02, G13

### INTRODUCTION

The first model of pricing European options in continuous time was introduced by Black & Scholes [1973]. Although the model is still widely used by many practitioners its structure has long ceased to meet the requirements of modern financial markets. Such view should be considered correct for at least two reasons.

Firstly, there is enough evidence suggesting the existence of so-called stylized facts. According to Cont [2001] this term should be associated with seemingly random variations of asset prices that have non-trivial statistical properties. A different definition of stylized facts is proposed by Challet et al. [2001], who

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<sup>1</sup> The views expressed in the article by the author are the personal views of the author and do not express the official position of the institution in which he is employed.

identify the above term with empirical statistical regularities in financial data (which are not explained in terms of cause and effect - ed. by the author). Taylor [2005] suggests a similar approach stating that stylized facts are general properties expected to be present in any set of returns. Rogers & Zhang [2011] present a different opinion about it. According to them, stylized facts are a set of independently recognized characteristics relating to various instruments, markets and periods of time. According to R. Cont [2001], stylized facts are properties that are common across a wide range of instruments, markets and periods of time.

The following stylized facts are most often analyzed in the financial literature:

- absence of autocorrelation, except for very short time intervals where microstructural effects start to play a role,
- (conditional) heavy tails in the (conditional) distribution of returns,
- gain/loss asymmetry (prices of financial assets and stock index values are slower to go up and faster to go down; the opposite relationship is observed for exchange rates),
- aggregational gaussianity (as one increases the time interval over which returns are calculated, the distribution of returns converges to a normal one),
- intermittency of returns and volatility clustering (time series of returns are irregular and volatility shows tendency to cluster over time),
- slow decay of autocorrelation in absolute returns,
- leverage effect (there is a negative relationship between stock returns and both historical and implied volatility),
- volume/volatility correlation (trading volume and volatility are negatively correlated),
- asymmetry in time scales (coarse-grained measures of volatility predict a fine-scale volatility better than vice versa).

It is worth noting that stylized facts are not tantamount to market anomalies. Stylized facts refer to the immanent properties of the financial market forming the foundation for building some scientific theories. The market anomalies are regarded as inexplicable phenomena contradicting the existing concepts and views. Such statements seem to be in line with the opinion offered by T. Lux [2009]. It is important to notice that although microstructural effects, heavy tails in the distributions of returns, volatility clustering etc. are treated as immanent properties of the variables which do not fit within the restrictive framework of some existing models, e.g. the Black-Scholes model, they are sometimes captured by other models, e.g. stochastic volatility models.

Secondly, as a result of the constant technological progress, the computing power of computers is systematically increasing. It means that the range of computational techniques that can be used in practice for the valuation of capital assets or derivatives is expanding. The mathematical tools that have gained particular importance in this context are the characteristic functions. They can be applied to many option pricing models, including the stochastic volatility models.

The aim of the article is to show that the process of pricing European options in selected stochastic volatility models can be improved in terms of computational speed. The article is organized as follows. In the first section two stochastic volatility models of pricing European options are investigated, i.e. the Heston model [Heston 1991] and the Bates model [Bates 1996]. In the second section, some approaches to calculating characteristic functions are presented. In the third section, the computational speed of pricing European options is investigated. Finally, in the fourth section, this article has been summarized and some major conclusions have been drawn.

## SELECTED STOCHASTIC VOLATILITY MODELS

In this section, two stochastic volatility models of pricing European options are analyzed: the Heston model and the Bates model.

### The Heston model

In the Heston model, the dynamics of the underlying asset price  $S$  and the volatility process  $\sigma^2$  are governed by two stochastic differential equations:

$$dS_t = \mu S_t dt + \sqrt{\sigma_t^2} S_t dW_{1,t}. \quad (1)$$

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + v\sqrt{\sigma_t^2}dW_{2,t}. \quad (2)$$

where:  $S_t$  denotes the spot price of the underlying asset at time  $t$ ,  $\mu$  is the constant (drift),  $\sigma_t^2$  is the instantaneous variance,  $\theta$  is the long-term variance,  $\kappa$  is the mean-reversion rate, and  $v$  is the volatility of the variance process. It is worth noting that under Feller's condition, i.e.  $2\kappa\theta \geq v^2$ , the variance process remains always positive. The Brownian motions  $W_1$  and  $W_2$  are correlated with a constant  $\rho$ .

The formula for the price of a European call in the Heston model takes the following form:

$$C^H(s_t, \sigma_t^2, t) = S_t P_1^H(s_t, \sigma_t^2, \tau) - e^{-r\tau} K P_2^H(s_t, \sigma_t^2, \tau). \quad (3)$$

where:  $\tau = T - t$ ,  $r$  is the risk-free rate,  $K$  is the exercise price,  $P_1^H(s_t, \sigma_t^2, \tau)$  and  $P_2^H(s_t, \sigma_t^2, \tau)$  are the probabilities of expiring European call in-the-money.

Although  $P_1^H(s_t, \sigma_t^2, \tau)$  and  $P_2^H(s_t, \sigma_t^2, \tau)$  are not known they can be easily extracted from the characteristic functions, for  $j = 1, 2$ , i.e.:

$$P_j^H(s_t, \sigma_t^2, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-i\xi \ln K} \phi^{j,H}(\xi, s_t, \sigma_t^2)}{i\xi} \right) d\xi. \quad (4)$$

where:  $\Re(\cdot)$  is the real part of the subintegral function,  $i$  is the imaginary unit of the complex number,  $\phi^{j,H}(\xi, s_t, \sigma_t^2)$  is the characteristic function of  $s_t = \ln S_t$  (corresponding to  $P_j^H(s_t, \sigma_t^2, \tau)$ ). The remaining notation is the same as previously introduced.

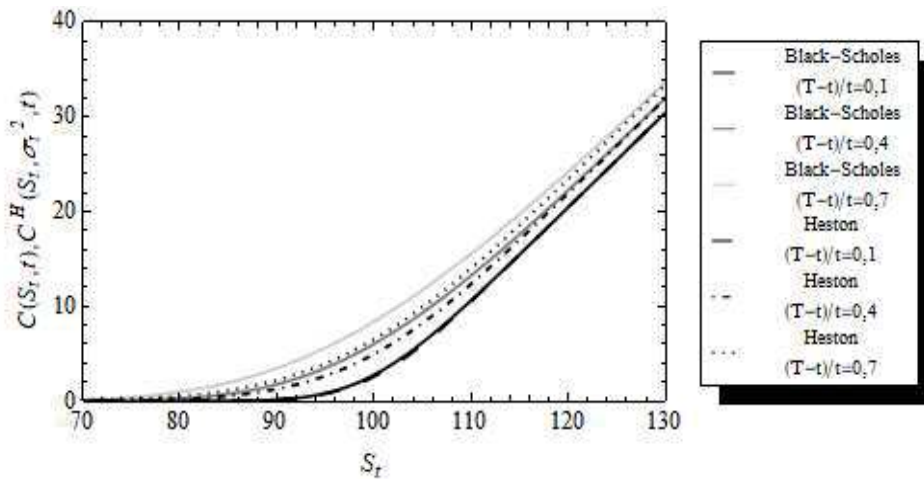
Theoretical price of a European call can be obtained under the assumption that the general form of the characteristic functions of  $s_t$  (corresponding to probabilities  $P_j^H$ , for  $j = 1, 2$ ), is the following:

$$\phi^{j,H}(\xi, s_t, \sigma_t^2) = e^{C_j(\xi, \tau) + D_j(\xi, \tau) \sigma_t^2 + \mathbb{I} \xi s_t}. \quad (5)$$

where:  $C_j(\xi, \tau) = r \mathbb{I} \xi \tau + \frac{a}{v^2} \left[ (b_j - v \rho \mathbb{I} \xi + d_j) \tau - 2 \ln \left( \frac{1 - g_j e^{d_j \tau}}{1 - g_j} \right) \right]$ ,  $D_j(\xi, \tau) = \frac{b_j - v \rho \mathbb{I} \xi + d_j}{v^2} \left( \frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}} \right)$ ,  $u_1 = \frac{1}{2}$ ,  $u_2 = -\frac{1}{2}$ ,  $a = \kappa \theta$ ,  $b_1 = \kappa + \lambda - v \rho$ ,  $b_2 = \kappa + \lambda$ ,  $g_j = \frac{b_j - v \rho \mathbb{I} \xi + d_j}{b_j - v \rho \mathbb{I} \xi - d_j}$ , zaś  $d_j = \sqrt{(v \rho \mathbb{I} \xi - b_j)^2 - v^2 (2u_j \mathbb{I} \xi - \xi^2)}$ .

The figure below presents the payoff functions of a European call in the Heston model ( $C^H(S_t, \sigma_t^2, t)$ ) compared to the payoff functions of a European call in the Black-Scholes model ( $C(S_t, t)$ ) assuming that:  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ ,  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$  for different periods remaining to expiration, i.e.  $\frac{T-t}{T} \in \{0.1; 0.4; 0.7\}$ .

Figure 1. Payoff functions of a European call in the Heston model and the Black-Scholes model assuming that:  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ ,  $\frac{T-t}{T} \in \{0.1; 0.4; 0.7\}$ ,  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$



Source: own preparation

It is worth noting that pricing of options using the method described above is inefficient. In eq. 3-4 there are two integrals and each of them has to be evaluated numerically. It means that the computational effort necessary for the valuation of options is huge. A straightforward solution to the problem lies in the application of

alternative methods in which only one characteristic function is used. This issue will be discussed in next section of this article.

### The Bates model

In the Bates model [Bates 1996], the underlying asset price process is determined by two equations, i.e.:

$$dS_t = (\mu - \lambda\mu_J)S_t dt + \sqrt{\sigma_t^2}S_t dW_{1,t} + JS_t dN_t. \quad (6)$$

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + v\sqrt{\sigma_t^2}dW_{2,t}. \quad (7)$$

where:  $S_t$  denotes the spot price of the underlying asset at time  $t$ ,  $\mu$  is the instantaneous expected rate of appreciation of the underlying asset,  $\lambda$  is the annual frequency of jumps,  $J$  is the random percentage jump conditional on a jump occurring,  $N_t$  is the Poisson counter with intensity  $\lambda$ ,  $\sigma_t^2$  is the instantaneous variance of the price process. It is worth noting that:  $1 + J \sim \mathcal{LN}(\mu_S, \sigma_S^2)$ . Moreover, the Brownian motions  $W_1$  and  $W_2$  are correlated with a constant  $\rho$  and the relationship between  $\mu_J$  and  $\mu_S$  is the following:  $\mu_J = e^{\left(\mu_S + \frac{\sigma_S^2}{2}\right)} - 1$ .

The price of a European call in the Bates model can be evaluated in the same manner as in the Heston model. Thus, the theoretical framework discussed herein is simple on the one hand, while, on the other hand, the drawbacks of the applied approach concerning computational inefficiency remain the same.

Similarly to the Heston model, the methodology of pricing options in the Bates model is based on characteristic functions. For the purpose of further analysis it is assumed that the characteristic function needed for the valuation of European options is as follows:

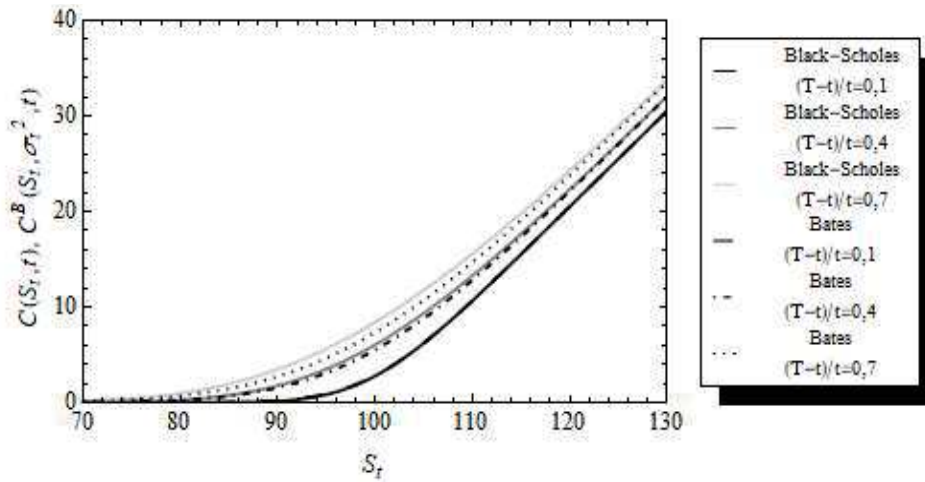
$$\phi^{2,B}(\xi, S_t, \sigma_t^2) = e^{C_2(\xi, \tau) + D_2(\xi, \tau)\sigma_t^2 + \mathcal{P}(\xi)\lambda\tau + \mathbb{I}\xi S_t} \quad (8)$$

where:  $\mathcal{P}(\xi) = -\mu_J \mathbb{I}\xi + \left( (1 + \mu_J)^{\mathbb{I}\xi} e^{\sigma_S^2 \left(\frac{\mathbb{I}\xi}{2}\right) (\mathbb{I}\xi - 1)} - 1 \right)$ , and both  $C_2(\xi, \tau)$  and  $D_2(\xi, \tau)$  can be concluded from the Heston model. The remaining notation is the same as in the previous case.

The formulas that can be used for pricing European options are presented in the next section of this article.

The figure below presents the payoff functions of a European call in the Bates model ( $C^B(S_t, \sigma_t^2, t)$ ) compared to the payoff functions of a European call in the Black-Scholes model ( $C(S_t, t)$ ) assuming that:  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ . Additionally, it is assumed that:  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$ ,  $\mu_S = -0.05$ , and  $\sigma_S = 0.00004$  for different periods remaining to expiration, i.e.  $\frac{T-t}{T} \in \{0.1; 0.4; 0.7\}$ .

Figure 2. Payoff functions of a European call in the Bates and the Black-Scholes model assuming that:  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ ,  $\frac{T-t}{T} \in \{0.1; 0.4; 0.7\}$ ,  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$ ,  $\mu_S = -0.05$  and  $\sigma_S = 0.00004$



Source: own preparation

Some alternative methods of pricing European options by characteristic functions are presented below.

## CHARACTERISTIC FUNCTIONS

There are numerous ways of deriving characteristic functions for the purpose of pricing European options. In this article particular attention is paid to the formulas developed by Carr & Madan [1999], Attari [2004], Bates [2006] and Orzechowski [2018], only. On the basis of the formulas, the theoretical values of European calls in the Heston model and the Bates model are determined. It should be noted that in all equations presented below it is assumed that  $t = 0$ . The remaining notation remains consistent with the one introduced previously.

### The Heston model

1. The Carr & Madan approach [Carr, Madan 1999] for  $\alpha = 1$ :

$$C^H(S_0, \sigma_0^2, 0) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \Re \left( e^{-\mathbb{I}\xi k} \frac{e^{-rT} \phi^{2,H}(\xi - (\alpha+1)\mathbb{I}, S_0, \sigma_0^2)}{\alpha^2 + \alpha - \xi^2 + \mathbb{I}(2\alpha+1)\xi} \right) d\xi. \quad (9)$$

2. The Attari approach [Attari 2004]:

$$C^H(S_0, \sigma_0^2, 0) = S_0 \left( 1 + \frac{e^l}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}(\xi + \mathbb{I})} \psi^{2,H}(\xi, s_0, \sigma_0^2) \right) d\xi \right) + e^{-rT} K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}\xi} \psi^{2,H}(\xi, s_0, \sigma_0^2) \right) d\xi \right). \quad (10)$$

where:  $l = \ln\left(\frac{K}{S_0 e^{rT}}\right)$ .

3. The Bates approach [Bates 2006]:

$$C^H(S_0, \sigma_0^2, 0) = S_0 - e^{-rT} K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi \ln\left(\frac{K}{S_0}\right)}}{\mathbb{I}\xi(1 - \mathbb{I}\xi)} \right) \varphi^{2,H}(\xi, s_0, \sigma_0^2) d\xi \right). \quad (11)$$

4. The Orzechowski approach [Orzechowski 2018]:

$$C^H(S_0, \sigma_0^2, 0) = \frac{1}{2} S_0 + e^{-rT} \frac{1}{\pi} \int_0^\infty \Re \left( e^{-\mathbb{I}\xi k} \frac{\phi^{2,H}(\xi - \mathbb{I}, s_0, \sigma_0^2)}{\mathbb{I}\xi(\mathbb{I}\xi + 1)} \right) d\xi. \quad (12)$$

It is worth noting that:  $\phi^{1,H}(\xi, s_0, \sigma_0^2)$ ,  $\phi^{2,H}(\xi, s_0, \sigma_0^2)$ ,  $\psi^{2,H}(\xi, s_0, \sigma_0^2)$  and  $\varphi^{2,H}(\xi, s_0, \sigma_0^2)$  are characteristic functions corresponding to  $P_1^H(s_t, \sigma_t^2, \tau)$  and  $P_2^H(s_t, \sigma_t^2, \tau)$ , respectively. The functions corresponding to  $P_2^H(s_t, \sigma_t^2, \tau)$  are determined by the following equations:

$$\phi^{2,H}(\xi, s_0, \sigma_0^2) = e^{C_2(\xi, T) + D_2(\xi, T)\sigma_0^2 + \mathbb{I}\xi s_0}. \quad (13)$$

$$\psi^{2,H}(\xi, s_0, \sigma_0^2) = \phi^{2,H}(\xi, s_0, \sigma_0^2) e^{-\mathbb{I}\xi s_0 - \mathbb{I}\xi rT}. \quad (14)$$

$$\varphi^{2,H}(\xi, s_0, \sigma_0^2) = \phi^{2,H}(\xi, s_0, \sigma_0^2) e^{-\mathbb{I}\xi s_0}. \quad (15)$$

### The Bates model

1. The Carr & Madan approach [Carr, Madan 1999] for  $\alpha = 1$ :

$$C^B(S_0, \sigma_0^2, 0) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \Re \left( e^{-\mathbb{I}\xi k} \frac{e^{-rT} \phi^{2,B}(\xi - (\alpha + 1)\mathbb{I}, s_0, \sigma_0^2)}{\alpha^2 + \alpha - \xi^2 + \mathbb{I}(2\alpha + 1)\xi} \right) d\xi. \quad (16)$$

2. The Attari approach [Attari 2004]:

$$C^B(S_0, \sigma_0^2, 0) = S_0 \left( 1 + \frac{e^l}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}(\xi + \mathbb{I})} \psi^{2,B}(\xi, s_0, \sigma_0^2) \right) d\xi \right) + e^{-rT} K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}\xi} \psi^{2,B}(\xi, s_0, \sigma_0^2) \right) d\xi \right). \quad (17)$$

where:  $l = \ln\left(\frac{K}{S_0 e^{rT}}\right)$ .

3. The Bates approach [Bates 2006]:

$$C^B(S_0, \sigma_0^2, 0) = S_0 - e^{-rT} K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-\mathbb{I}\xi \ln\left(\frac{K}{S_0}\right)}}{\mathbb{I}\xi(1-\mathbb{I}\xi)} \right) \varphi^{2,B}(\xi, s_0, \sigma_0^2) d\xi \right). \quad (18)$$

4. The Orzechowski approach [Orzechowski 2018]:

$$C^B(S_0, \sigma_0^2, 0) = \frac{1}{2} S_0 + e^{-rT} \frac{1}{\pi} \int_0^\infty \Re \left( e^{-\mathbb{I}\xi k} \frac{\phi^{2,B}(\xi - \mathbb{I}, s_0, \sigma_0^2)}{\mathbb{I}\xi(\mathbb{I}\xi + 1)} \right) d\xi. \quad (19)$$

It is worth noting that:  $\phi^{2,B}(\xi, s_0, \sigma_0^2)$ ,  $\psi^{2,B}(\xi, s_0, \sigma_0^2)$  as well as  $\varphi^{2,B}(\xi, s_0, \sigma_0^2)$  are characteristic functions determined by the following equations:

$$\phi^{2,B}(\xi, s_0, \sigma_0^2) = e^{C_2(\xi, T) + D_2(\xi, T)\sigma_0^2 + \mathcal{P}(\xi)\lambda T + \mathbb{I}\xi s_0} \quad (20)$$

$$\psi^{2,B}(\xi, s_0, \sigma_0^2) = \phi^{2,B}(\xi, s_0, \sigma_0^2) e^{-\mathbb{I}\xi s_0 - \mathbb{I}\xi rT}. \quad (21)$$

$$\varphi^{2,B}(\xi, s_0, \sigma_0^2) = \phi^{2,B}(\xi, s_0, \sigma_0^2) e^{-\mathbb{I}\xi s_0}. \quad (22)$$

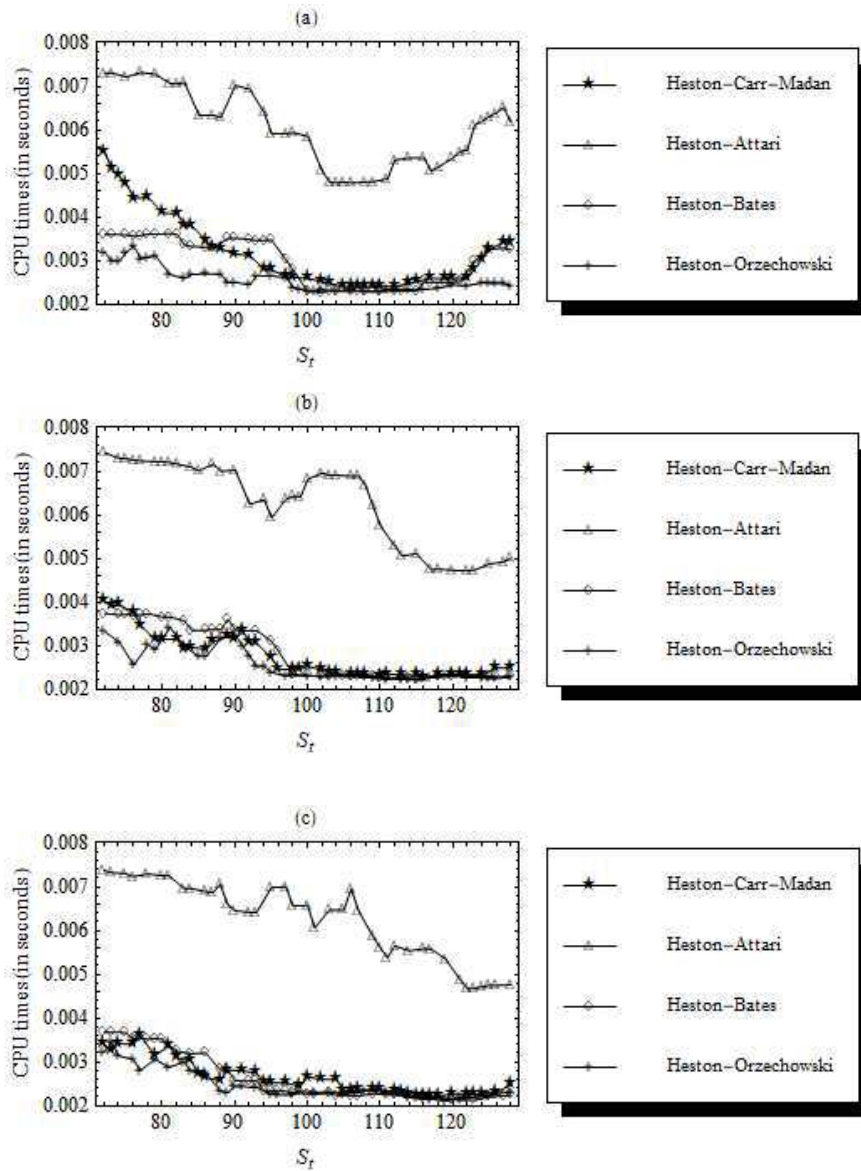
## RESULTS

In order to investigate which of the approaches proposed in the previous section is the most efficient one in terms of computational speed, appropriate codes are developed in the Mathematica 10.2. It is assumed that integrals in the formulas for the prices of European calls are evaluated numerically by means of the Gauss-Kronrod quadrature. Graphs are smoothed by averaging runs of five elements. The package is launched on a computer with Intel i7- 1070 CPU @ 2.90 GHz processor with RAM memory of 32 GB. Cache memory is cleared before starting codes allowing for the valuation of options.

The results of the research carried out are shown in the graphs below - see Figures 3 and 4.

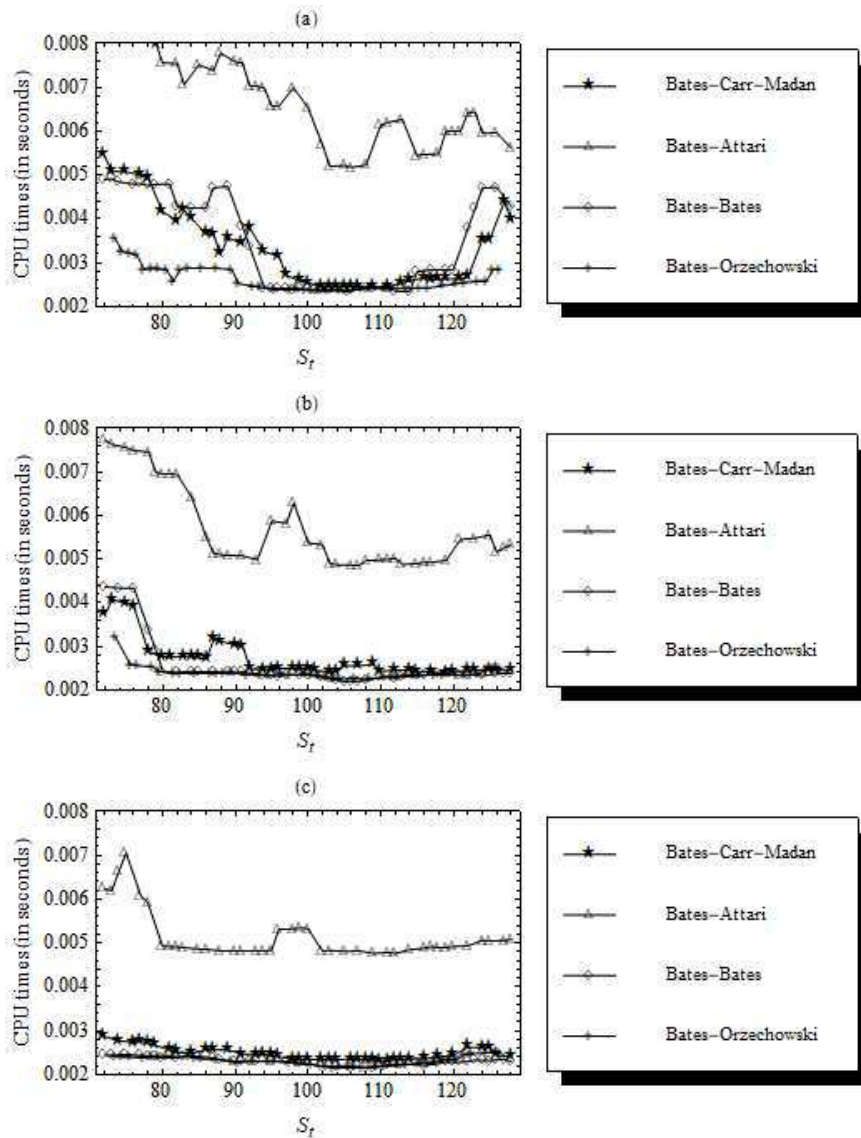


Figure 3. Computational speed in the Heston model assuming that  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ ,  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$  for (a)  $\frac{T-t}{T} = 0.1$ , (b)  $\frac{T-t}{T} = 0.4$  and (c)  $\frac{T-t}{T} = 0.7$



Source: own preparation

Figure 4. Computational speed in the Bates model assuming that  $S_t \in [70, 130]$ ,  $K = 100$ ,  $\sigma_t = 0.2$ ,  $r = 5\%$ ,  $v = 0.04$ ,  $\kappa = 1.5$ ,  $\lambda = 3$ ,  $\theta = 0.04$ ,  $\rho = 0.8$ ,  $\mu_S = -0.05$  and  $\sigma_S = 0.00004$  for (a)  $\frac{T-t}{T} = 0.1$ , (b)  $\frac{T-t}{T} = 0.4$  and (c)  $\frac{T-t}{T} = 0.7$



Source: own preparation

The results obtained show that the speed of pricing European options in the Heston model and the Bates model depends on the way the characteristic functions are calculated. The closer the time to expiration the better (in terms of computational speed) is the method developed by Orzechowski [Orzechowski 2018]. The closer the moment of writing European options, the more ambiguous become the differences between alternative methods of pricing derivatives. An exception to the rule applies to the Attari method. In both option pricing models this approach allows for the slowest pricing of the contracts being considered (regardless the time to expiration).

## SUMMARY

In this article some selected methods of determining characteristic functions were applied to the valuation of options. Based on the results obtained it can be concluded that the characteristic function developed by Orzechowski [Orzechowski 2018] allows for the fastest pricing of European options in the Heston model and the Bates model. This is true under the assumption that the prices of the contracts are evaluated numerically by means of the Gauss-Kronrod quadrature. It should be noted that although the results are vulnerable to the time to expiration, nevertheless abovementioned conclusion should be maintained in its current form.

Further research should focus on analyzing the speed of pricing European options in other stochastic volatility models. Particular attention in this matter should be paid to the selection of both numerical methods used to approximate theoretical values of the analyzed instruments and characteristic functions of  $s_t$ .

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