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SYMMETRY PROPERTIES OF MODIFIED BLACK-SCHOLES EQUATION

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Abstract: This paper concerns the classical and conditional symmetries of the Black-Scholes equation. Modifications of the Black-Scholes equation have also been considered and their maximal algebras of invariance have been found. Examples of creation operators for the Black-Scholes eigenvalue problem have been provided.

Keywords: Black-Scholes equation, Lie symmetries, creation operator

JEL classification: C69, G150, G170

INTRODUCTION

The remarkable Black-Scholes equation [Black & Scholes 1973, Merton 1971, Merton 1973, Black 1976] that describes the time evolution of the option prices of a European call under the Black-Scholes model belongs to a group of successful mathematical models related to the derivatives market. As such, it has been vigorously investigated from many different points of view. In particular, a transformation to the heat (diffusion) equations has been found that resulted in a convenient formula for a solution in terms of the standard normal cumulative distribution functions [Black 1976].

One of the most successful methods or perhaps *the* most successful method of analysis of partial differential equations (linear or nonlinear) is based on finding the *symmetries* of a given equation. In particular, thanks to the works by Lie and also Ovsyannikov, semi-algorithmic, effective methods of finding symmetry transformations have been found that are based on solving an overdetermined system of linear partial differential equations for the coefficients of the so-called generator

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of symmetry [Ovsyannikov 1962, Olver 1986, Bluman & Kumei 1989, Stephani 1990]. The generators of symmetry allow for finding special solutions to a given equation by lowering its dimensionality and obtaining new solutions from already known ones. The symmetry generators obtained in the classical Lie sense form a Lie algebra. It provides insight into the deep structure of the equation. If the Lie algebras for two different equations (or systems of equations) are isomorphic, they provide a means to find invertible transformations between these two equations [Bluman & Kumei 1989].

There are good reasons to believe that the symmetry analysis has become more or less *de rigueur* in the modern theory (and practice) of partial differential equations. The original Black-Scholes equation has been investigated from the point of view of its symmetries in [Gazizov & Ibragimov 1998, Bordag 2015] among others. As for the BS equation with variable coefficients, only partial results exist (to our best knowledge), to which we add here our contribution.

One of the limitations imposed on applications of the Black-Scholes formula under realistic circumstances is the assumption that its two important parameters, risk-free rate, and volatility, are constant. A version of the Black-Scholes equation with time-varying coefficients has been derived by Merton already in 1973 [Merton 1973]. Interesting mathematical insights associated with non-constant parameters have been obtained, in particular, in [Harper 1994, Willmot et al. 1999, Rodrigo & Mamon 2006, Naz & Naeem 2020, Lyu & Wang 2017].

Thus, our objective has been to analyze the variable-coefficient (i.e. modified) Black-Scholes equations from the point of view of its symmetries. We allow the coefficients to depend on both independent variables. In particular, we identify the dependence of the coefficients that allow for maintaining the maximal symmetry of the Lie sense.

The main body of this work is organized as follows. In Section 2 we derive the classical and conditional symmetries of the standard Black-Scholes equation using commutators instead of prolongations used in [Gazizov & Ibragimov 1998]. The creation operator for the corresponding eigenvalue problem with varying risk-free interest rates has been obtained in Section 3. In Section 4 we find symmetries of the logarithmic dependence of the risk-free interest rate on the value of a security. Section 5 contains some concluding remarks.

STANDARD BLACK-SCHOLES EQUATION AND ITS SYMMETRIES

By *standard* Black-Scholes equation we will understand the following partial differential equation:

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 v}{\partial s^2}+rs\frac{\partial v}{\partial s}-rv=0.$$

Classical symmetries

The method for calculating symmetries of the Black-Scholes equation as employed in GI has been based on the so-called second prolongation of the symmetry generator. However, for the case of *linear* equations, a simpler method exists. It takes into account more directly the fact that a vector field is a symmetry if it transforms a hypersurface of solutions of a partial differential equation into itself. By "hypersurface of solutions" we mean the hypersurface in the space spanned by the independent variables, dependent variables, and the derivatives of the latter as specified by the constraint given by the differential equation.

Let $\mathcal{F}$ be a set of all triple-differentiable functions of two variables (t, s). Let $Q = Q(t, s, \partial_t, \partial_s)$ be a second-order linear partial differential operator with the domain $\mathcal{F}$. By a *(first-order) symmetry (operator)* of the partial differential equation $Qf = 0$, $f \in \mathcal{F}$, we will understand a first-order partial differential operator $L = L(t, s, \partial_t, \partial_s)$ such that [Miller 1984, Fushchych & Nikitin 1987]:

$$[Q, L]f = k(t, s)Qf, \tag{1}$$

for any f, where $[Q, L] = QL - LQ$ is the commutator of the operators Q and L, while $k(t, x)$ is a function of the arguments (t, s). Thus, the operator L transforms the set of solutions of the equation $Qf = 0$ into itself.

The definition contained in Eq. (1) provides effective means to determine the symmetries. Indeed, let Q be the operator $\partial_t - B$, and let us represent L as:

$$L = \tau(t, x)\,\partial_t + \xi(t, x)\,\partial_s + \phi(t, x), \tag{2}$$

where the functions τ, ξ, and ϕ are at least twice differentiable. Substituting $\partial_t - B$ for Q and the representation (2) into Eq. (1) and equating to zero the coefficients standing at the consecutive derivatives of the function f we obtain the following set of *determining equations* for the functions τ, ξ and ϕ:

$$s^2\sigma^2\frac{\partial\tau}{\partial s} = 0, \tag{3}$$

$$s^2\sigma^2\frac{\partial\xi}{\partial s} - s\sigma^2\xi - \frac{1}{2}s^2\sigma^2 k = 0, \tag{4}$$

$$\frac{\partial\tau}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\tau}{\partial s^2} + rs\frac{\partial\tau}{\partial s} - k = 0, \tag{5}$$

$$\sigma^2 s^2\frac{\partial\phi}{\partial s} + \frac{\partial\xi}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\xi}{\partial s^2} + rs\frac{\partial\xi}{\partial s} - r\xi - rsk = 0, \tag{6}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\phi}{\partial s^2} + rs\frac{\partial\phi}{\partial s} + rk = 0. \tag{7}$$

Due to the favorable structure of the above determining equations and their overdetermined character, it is possible to obtain their general solutions without difficulties. Indeed, we immediately realize that both τ and k must be independent of s with $k = k(t) = \partial_t \tau$. Integration of (4) leads to:

$$\xi(t,s) = s\left(\overleftarrow{\xi}(t) + \frac{1}{2}k(t)\log(s)\right) \tag{8}$$

while the integration of (6) yields:

$$\phi(t,s) = \overleftarrow{\phi}(t) - \frac{\dot{\overleftarrow{\xi}}(t)\log(s)}{\sigma^2} - \frac{\dot{k}(t)\log^2(s)}{4\sigma^2} + \frac{1}{2}\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)\log(s)k(t), \tag{9}$$

where the “dot" over a symbol indicates the derivative over time t.
Finally, with the substitution of the above partial solutions to Eq. (7) and equating to zero the coefficients standing at $\log(s)^2$, $\log(s)$, and the s-independent term, we obtain:

$$k(t) = k_0 + k_1 t,$$

$$\overleftarrow{\xi}(t) = \xi_0 + \xi_1(t), \tag{10}$$

$$\overleftarrow{\phi}(t) = \phi_0 - -\frac{\left(\sigma^2+2r\right)^2}{16\ \ ^2}k_1 t^2 - \frac{\left(\sigma^2+3r\right)^2)}{8\sigma^2}k_0 t + -\frac{\sigma^2-2r}{2\sigma^2}\xi_1 t + \frac{1}{4}k_1 t. \tag{11}$$

The expression for τ is obtained by simple integration over t:

$$\tau = \tau_0 + k_0 t + \frac{1}{2}k_1 t^2. \tag{12}$$

Thus, by the above straightforward calculations we have proved the following proposition:
Theorem 1. *The most general first-order symmetry operator of the Black-Scholes equation is given by*

$$L = \tau\,\partial_t + \xi\,\partial_s + \phi,$$

where the functions τ, ξ, and ϕ are given by (12, 8, 10, 9, 11).

We conclude that the algebra of invariance operators is generated by the following operators:

$$L_0 = \partial_t,$$

$$L_1 = t\,\partial_t + \frac{1}{2}s\log(s)\,\partial_s - \frac{\left(\sigma^2+2r\right)^2}{8\sigma^2}t + (\frac{r}{2\sigma^2} - \frac{1}{4}\log(s),$$

$$L_2 = \frac{1}{2}t^2\,\partial_t + \frac{1}{2}st\log(s)\,\partial_s - \frac{\left(\sigma^2+2r\right)^2}{16\ \ ^2}t^2 + \frac{t}{4} - \frac{\log^2(s)}{4\sigma^2},$$

$$L_3 = s\frac{\partial}{\partial s},$$

$$L_4 = st\frac{\partial}{\partial s} + \left(\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)t - \frac{\log(s)}{\sigma^2}\right),$$

$$L_5 = I.$$

Although the shape of the above operators may seem a bit complicated, they form a Lie algebra with a fairly simple commutation table:

$$\begin{aligned}
[L_0, L_1] &= L_0 - \frac{(\sigma^2 + 2r)^2)}{8\sigma^2}L_5, \\
[L_0, L_2] &= L_1 + \frac{1}{4}L_5, \\
[L_0, L_3] &= 0, \\
[L_0, L_4] &= L_3 + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)L_5, \\
[L_0, L_5] &= 0, \\
[L_1, L_2] &= L_2, \\
[L_1, L_3] &= -\frac{1}{2}L_3 - \left(\frac{r}{2\sigma^2} - \frac{1}{4}\right)L_5, \\
[L_1, L_4] &= 0, \\
[L_1, L_5] &= 0, \\
[L_2, L_3] &= -\frac{1}{2}L_4, \\
[L_2, L_4] &= 0, \\
[L_2, L_5] &= 0, \\
[L_3, L_4] &= -\frac{1}{\sigma^2}L_5, \\
[L_3, L_5] &= 0, \\
[L_4, L_5] &= 0.
\end{aligned}$$

The Lie algebra rederived above is isomorphic to the so-called extended Schrödinger algebra of invariance of both the one-dimensional Schrödinger (with at most quadratic potential) and one-dimensional heat (diffusion) equation. It is precisely this isomorphism that allows for the well-known transformation from the Black-Scholes to the heat equation.

Conditional symmetries

Let Q be a second-order partial differential operator with the domain as above. By a *conditional symmetry* of the partial differential equation $Qf = 0$ we will understand a first-order partial differential operator L such that

$$[Q, L] = k(t,s)Q + m(t,s)L, \tag{13}$$

where $m(t, s)$ is another first-order partial differential operator. The intuition behind the above definition is as follows. The number of symmetries can be enlarged if we add a condition to the solutions. Namely, even though the hypersurface $Qf = 0$ is not invariant under transformations generated by L, a subset U of that hypersurface can be invariant if it satisfies certain additional conditions ($Lf = 0$ in this case). That subset U is, therefore, defined by the two conditions: $Qf = 0$, $Lf = 0$.
The general form of the operator $m(t, s)$ is:

$$m(t,s) = \mu(t,s)\frac{\partial}{\partial t} + \nu(t,s)\frac{\partial}{\partial s} + \rho(t,s).$$

However, this form is, in fact, *too* general - on employing it we obtain a family of uncountably many symmetry operators parametrized by the arbitrary functions μ, ν without any effective means to calculate them explicitly. Thus, we will restrict our discussion to the case of m reduced to the function ρ multiplied with the identity operator. What is more, we will assume that ρ is constant.
From (13) we obtain the following set of determining equations. The first two of them coincide with (3), (4) while the remaining three are just a little bit more complicated:

$$\frac{\partial \tau}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \tau}{\partial s^2} + rs\frac{\partial \tau}{\partial s} - \rho\tau - k = 0,$$

$$\sigma^2 s^2 \frac{\partial \phi}{\partial s} + \frac{\partial \xi}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \xi}{\partial s^2} + rs\frac{\partial \xi}{\partial s} - \xi\rho - r\xi - rsk = 0,$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \phi}{\partial s^2} + rs\frac{\partial \phi}{\partial s} - \rho\phi + rk = 0.$$

The results appear to be somewhat trivial at the first sight. The generators obtained from the above equations are those of the previous subsection but multiplied by the factor $\exp(\rho t)$. This seemingly innocent factor changes quite radically the structure of the set of generators: they no longer form a Lie algebra because the commutators contain the factor $\exp(2\rho t)$. However, it is to be noted that the commutators of the symmetry operators found are still symmetries.

CREATION OPERATOR FOR EIGENVALUE PROBLEM

The problem considered above sends us to another consideration of non-vanishing interest. Let us consider the following eigenvalue problem for the Black-Scholes operator:

$$B(v) = -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} - rs\frac{\partial v}{\partial s} + rv = \lambda v,$$

where λ is an eigenvalue (complex, in principle). And let us look for a first-order ordinary differential operator $a^{\dagger}$ such that:

$$[B, a^{\dagger}] = ka^{\dagger}. \tag{14}$$

Both the notation of $a^{\dagger}$ and the name by which we call it here - the creation operator - are borrowed, of course, from quantum mechanics. The merit of the creation operator, if it exists, is that it transforms one solution of the eigenvalue problem into another belonging to, in general, a different eigenvalue.

Let us represent $a^{\dagger}$ in the same way as the symmetry operator L before:

$$a^{\dagger} = \xi(s)\frac{d}{ds} + \phi(s).$$

Upon substitution of the above expression into (14) we obtain the immediately obtain the following solutions for ξ and ϕ:

$$\xi(s) = \xi_0 s,$$

$$\phi(s) = \phi_0 - \frac{k\xi_0 \log(s)}{\sigma^2}.$$

Unfortunately, there are further constraints for the solution ϕ that results in:

$$\xi_0\left(\frac{k^2\log(s)}{\sigma^2} + \frac{kr}{\sigma^2} - \frac{k}{2}\right) - \frac{k\xi_0}{2} = 0.$$

from which it follows that either $k = 0$ or both ξ_0 and ϕ_0 vanish. We must, therefore, conclude that the standard Black-Scholes operator has no non-trivial creation operators. Now, the following question naturally arises: can we improve the Black-Scholes operator in such a way that a creation operator is admitted.

Let us make a somewhat exotic assumption that the risk-free interest rate r depends on the value of s, $r = r(s)$. Under this assumption, the expressions for ξ and ϕ are only slightly changed:

$$\xi = \xi_0 s,$$

$$\phi = \phi_0 + \xi_0\frac{r(s) - k\log(s)}{\sigma^2}.$$

The constraint that must be fulfilled by ϕ leads to the following non-linear second-order differential equation for $r(s)$:

$$-\frac{1}{2}\xi_0 s^2 r''(s) - \xi_0 s\left(\frac{r(s)}{\sigma^2} + 1\right)r'(s) + \xi_0\frac{k^2\log(s)}{\sigma^2} - k\phi_0 - \frac{k\xi_0}{2} = 0.$$

It is not difficult to find a special solution to the above equations that, by the way, simplifies ϕ, namely:

$$r(s) = a\log(s) + b.$$

We immediately find:

$$k = a,$$

$$\phi_0 = -\xi_0\left(\frac{b}{\sigma^2} + 1\right).$$

Thus, we have found a modified Black-Scholes equation with a non-trivial creation operator. The modification is very minor here.

SYMMETRIES IN THE CASE OF NON-CONSTANT RISK-FREE INTEREST RATE

In this section, we will find symmetries of a modified Black-Scholes equation in which the risk-free interest rate r depends either on s or on time.

Motivated by the results for the creation operator, we pay particular attention to the case:

$$r(s) = a\log(s) + b.$$

The general form of the symmetry generator is as before. The determining equations take the following form:

$$\sigma^2 s\frac{\partial\tau}{\partial s} = 0,$$

$$\sigma^2 s^2\frac{\partial\xi}{\partial s} - \sigma^2 s\xi - \frac{1}{2}\sigma^2 s^2 k = 0,$$

$$\frac{\partial\tau}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\tau}{\partial s^2} + sr\frac{\partial\tau}{\partial s} - k = 0,$$

$$\sigma^2 s^2\frac{\partial\phi}{\partial s} + \frac{\partial\xi}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\xi}{\partial s^2} + sr\frac{\partial\xi}{\partial s} - s\frac{\partial r}{\partial s}\xi - r\xi - srk = 0,$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\phi}{\partial s^2} + sr\frac{\partial\phi}{\partial s} + \frac{\partial r}{\partial s}\xi + rk = 0.$$

They differ very little from the determining equations for the standard Black-Scholes equation, but the dependence of r of s spoils the symmetry quite radically. But with the logarithmic dependence of r on s we obtain the following solution:

$$k(s,t) = k(t) = c_1 e^{2at} + c_2 e^{-2at},$$

$$\tau(s,t) = \tau(t) = c_0 + \frac{1}{2a}(c_1 e^{2at} - c_2 e^{-2a}\ ),$$

$$\xi(s,t) = s\left(\xi_0(t) + \frac{1}{2}k(t)\log(s)\right),$$

$$\phi(s,t) = \phi_0(t) + \frac{b\xi_0}{\sigma^2} + \frac{1}{4\sigma^2}\log(s)^2\left(2bk - \frac{\partial k}{\partial t}\right) + \frac{1}{4\sigma^2}\left(4b\xi_0 + 2ak - 4\frac{\partial \xi_0}{\partial t} - k\right).$$

The dependence of xi_0 on t is given by:

$$\xi_0(t) = \frac{1}{4a}(\sigma^2 + 2b)(c_1e^{2at} + c_2e^{-2at}) + c_3e^{at} + c_4e^{-at},$$

while $\phi(t)$ is obtained by the integration of the formula:

$$\frac{d\phi_0}{dt} + \frac{1}{2}\frac{d\xi_0}{dt} - \frac{1}{4}\frac{dk}{dt} + \frac{1}{2\sigma^2}(2b + \sigma^2)a\xi_0(t) + \frac{1}{8}\left(8\sigma^2 + \frac{4b^2}{\sigma^2} + 4a + 4b\right)k(t) = 0.$$

The above symmetry generator allows, of course, for an attempt to perform similarity reduction and find non-trivial solutions to the modified Black-Scholes equation. This, however, would be only meaningful if our logarithmic dependence of the risk-free interest rate on s can be in any sense justified.

We have also studied the dependence on the risk-free interest rate on time, $r = r(t)$. We have found, however, that even the simplest, linear time dependence largely reduces the symmetry algebra: the function $tau(t)$ depends in this case on one (instead of three) arbitrary parameters. Let us notice that Naz and Naeem [Naz, Naeem 2020] provided an interesting treatment of the case time dependence of *both* r and σ^2 with $r(t) = (1/2)\sigma^2(t)$ using the so-called potential symmetries.

We plan to provide a more comprehensive treatment of similarity reductions in another paper where multidimensional versions of the Black-Scholes equation with time-dependent risk-free interest rates will also be studied.

CONCLUDING REMARKS

In this work, we have considered the standard as well as modified Black-Scholes equations from the point of view of their symmetries. Known results about it have been rederived using a different method. An eigenvalue problem for the Black-Scholes operator has also been considered for a non-trivial, logarithmic dependence of the risk-free interest rate. For the same dependence of that quantity, the symmetry of the modified Black-Scholes equations has been found. Work is in progress to provide a more comprehensive exposition of the symmetries of variable-coefficient versions of that equation and to extend the symmetry approach to the parameter-dependent multidimensional Black-Scholes equation.

As is usually the case with variable-coefficient partial differential equations, only for very peculiar dependence of the coefficients on independent variables can we obtain non-trivial symmetries? This can be in some sense remedied if a small parameter is present in the equation; the resulting approximate symmetries can be richer than the set of "exact" symmetries. This case will also be addressed in our future work.

REFERENCES

Black F. S. (1976) The Pricing of Commodity Contracts. Journal of Financial Economics, 3(1-2), 167-179.

Black F.S., Scholes M. (1973) The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637-654.

Bluman G. W., Kumei S. (1989) Symmetries and Differential Equations. Springer, New York.

Bordag L. A. (2015) Geometrical Properties of Differential Equations: Applications of the Lie Group Analysis in Financial Mathematics. World Scientific, Singapore.

Fushchych W. I., Nikitin A. G. (1987) Symmetries of Maxwell's Equations. Springer, Berlin.

Gazizov R. K., Ibragimov N. H. (1998) Lie Symmetry Analysis of Differential Equations in Finance. Nonlinear Dynamics, 17, 387-407, https://doi.org/10.1023/A:1008304132308.

Harper J. (1994) Reducing Parabolic Partial Differential Equations to Canonical Form. European Journal of Applied Mathematics, 5, 159-164.

Lyu W., Wang Y. (2017) Black-Scholes Equation with the Variable Risk-free Interest Rate. 2nd International Conference on Advances in Management Engineering and Information Technology (AMEIT 2017).

Merton R. C. (1971) Optimum Consumption and Portfolio Rules in a Continuous Time Model. Journal of Economic Theory, 3(4), 373-413.

Merton R. C. (1973) Theory of Rational Option Pricing. Bell Journal of Economics, 4(1), 141-183.

Miller W. Jr (1984) Symmetry and Separation of Variables. Cambridge University Press.

Naz R., Naeem I. (2020) Exact Solutions of a Black-Scholes Model with Time-Dependent Parameters by Utilizing Potential Symmetries. Discrete and Continuous Dynamical Systems, Series S, 13(10), 2841-2851.

Olver P. J. (1986) Applications of Lie Groups to Differential Equations. Springer, New York.

Ovsyannikov L. V. (1962) Group Properties of Differential Equations. USSR Academy of Sciences, Siberian Branch, Novosibirsk [in Russian].

Rodrigo M.R., Mamon R.S. (2006) An alternative approach to solving the Black–Scholes equation with time-varying parameters. Applied Mathematics Letters 19(4), 398-402.

Stephani H. (1990) Differential Equations: Their Solution Using Symmetries. Cambridge University Press.

Wilmott P, Howison S., Dewynne J. (1999) The Mathematics of Financial Derivatives. Cambridge University Press.