EXPECTED SHORTFALL AND HARELL-DAVIS ESTIMATORS OF VALUE-AT-RISK

Leszek Gadomski [1], Vasile Glavan [1, 2]
[1] Collegium Mazovia in Siedlce;
[2] University of Natural Sciences and Humanities in Siedlce,
Institute of Mathematics and Physics
e-mails: Leszek.Gadomski@mazovia.edu.pl, vglavan@uph.edu.pl

Abstract: The most widely used estimator for the *Value-at-Risk* is the corresponding order statistic. It relies on a single historic observation date, therefore it can exhibit high variability and provides little information about the distribution of losses around the tail. In this paper we purpose to replace this estimator of *VaR* by an appropriately chosen estimator of the Expected Shortfall. We also consider the Harrel-Davis estimator of *VaR* and give some comparative analysis among these estimators.

Key words: Risk management, tail loss, VaR, Expected Shortfall, Harrel-Davis estimator

INTRODUCTION

Risk measures appeared as a response to the necessity of quantifying the risk of potential losses on some asset, or a portfolio of assets. Among them Value-at-Risk (*VaR*) has become a standard risk measure for financial risk management due to its conceptual simplicity, easy of computation, and ready applications.

Most banks calculate daily 99% confidence interval *VaR* figures. To do this they look at a discrete distribution of simulated revenues. *VaR* at the 99% confidence level is estimated, for example, by the 14th worst loss across 1305 daily observations from 5yr historical data. It relies on a single historic observation date and therefore can exhibit high variability. This both reduces its efficiency and provides little information about the distribution of losses around the tail.

The process of risk management requires not only estimating the *VaR* but also examining the sensitivity of its positions comprising the portfolio. Taking a single order statistic such as the 14th worse loss may be inadequate for this

purpose. Computing a weighted average of the dates in the tail will produce more robust risk analysis. The use of quantile estimators ensures a more stable and accurate measure of tail losses and regulatory capital requirement.

This paper offers a coherent estimate for *VaR* and makes no distributional assumptions whatsoever in doing so. One of the major limitations of *VaR*, and it has been severely criticized through the crisis for not being additive. Using *HD* as an estimator for *Var* solves this problem, while still managing to keep *VaR* as a risk measure, which is a hard requirement by regulations and Basel rules for capital calculations. In our opinion, using the *HD* estimator solves the problem of *VaR* not being coherent, while at the same time adheres to Basel rules by keeping *VaR* for capital calculation.

VaR AS RISK MEASURE

Current regulations from finance (Basle II) or insurance (Solvency II) business formulates risk and capital requirements in terms of quantile based measures (see, e.g., Dowd and Blake 2006). The upper quantile of the loss distribution is called Value-at-Risk (*VaR*). In other words, *VaR* is defined as the maximum potential loss in value of a portfolio over a given holding period within a fixed confidence level: the riskier the portfolio, the larger are the minimal losses during the holding period and for a certain probability level.

More formally, given a random variable Y and a probability level $\alpha \in (0,1)$ denote by $Q(Y,\alpha)$ the α-quantile , i.e.,

$Q(Y,\alpha) = \inf\{y \in R, P(Y \leq y) > \alpha\} = F_Y^{-1}(\alpha)$, where "minus one" denotes the right-continuous generalized inverse of the cumulative distribution function F.
Recall that in the Gaussian model $Y \approx N(\mu, \sigma^2), Q(Y,\alpha) = \mu + u_\alpha \sigma$.

For a confidence level $\alpha \in (0,1)$ the Value-at-Risk at level α for log-returns X is defined as

$VaR(X,\alpha) = Q(-X,\alpha)$.

There are some conceptual problems with *VaR* , an important one is that the *Value-at-Risk* disregards any loss beyond the *VaR* level, also called the problem "of the tail risk". As mentioned in [Artzner et al. 1999], *VaR* has major drawback by not being coherent. By a coherent risk measure [Artzner et al. 1999] mean any real-valued function ρ of real-valued random variables X, which models the losses, and with the following characteristics:

- $X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$ a.s. (*Monotonicity*)
- $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (*Subadditivity*)
- $\rho(\lambda X) = \lambda \rho(X)$ (*Positive homogeneity*)
- $\rho(X + \lambda) = \rho(X) - \lambda$ (*Translation equivariance*).

The *Value-at-Risk* does not meet subadditivity in some cases. For a counter-example see, e.g., (Dowd and Blake 2006).

In response to a coherent equivalent to *VaR*, a series of *VaR*-related risk measures were proposed. Among them the Expected Shortfall as an alternative to *VaR* is mentioned [Rockafeller and Uryasev, 2000].

The Expected Shortfall [Acherbi and Tasche, 2002] at level α is defined as

$$ES(X,\alpha) = E(-X|-X > VaR(X,\alpha)).$$

It equals the conditional expected loss given that it exceeds $VaR(X,\alpha)$, and is also called *Tail Value-at-Risk* by [Artzner et al.1999], *Conditional Tail Expectation* [Wirch and Hardy, 1999], or C*onditional Value-at-Risk* [Rockafeller and Uryasev, 2000]. An alternative definition of *ES* is the mean of the tail distribution of the *VaR* losses.

EMPIRICAL ESTIMATORS FOR *VaR* AND *ES*

Let $X_{(n)}, X_{(n-1)}, \ldots, X_{([n\alpha+1])}, X_{([n\alpha])}, \ldots X_{(2)}, X_{(1)}$ denote the log-returns of a portfolio in the sample period arranged in increasing order. Then, for a sample large enough, the estimator of $VaR(X,\alpha)$, at a given level of confidence will be the statistics $X_{([n\alpha+1])}$. Similarly,

$$EstimateES(X,\alpha) = \frac{X_{(1)} + X_{(2)} + \ldots + X_{[n\alpha+1]}}{[n\alpha+1]}.$$

The estimator of $VaR(X,\alpha)$, as the corresponding sample statistic, has the advantage of simplicity and no specific distributional assumption. It is an unbiased estimator, but neither efficient, nor consistent. The above mentioned estimator for the Expected Shortfall, is an unbiased, efficient and consistent estimator.

ESTIMATING *VAR* USING *ES*

A different way to estimate the 99% percentile of the distribution is to use an Expected Shortfall approach. As we are not looking at 99% *ES*, but estimating the *VaR* percentile using *ES*, we need to determine the confidence level for which *ES* is equivalent to a 99% *VaR*. This approach has no closed solution and the equivalent confidence interval is dependent on the distribution assumptions of the underlying losses.

Assuming the losses are Normally Distributed

Let: $X \sim N(0,1)$ and recall the corresponding cumulative distribution function, and density, respectively:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad VaR_p(X) = -F^{-1}(p)$$

In this case the Expected Shortfall has the following form:

$$ES(X, p) = E\big[X \big| X \leq VaR_p\big] = E\big[X \leq VaR_p\big] \cdot \frac{1}{P\big(X \leq VaR_p\big)} =$$

$$\int_{-\infty}^{VaR_p} y \cdot f(y) dy \cdot \frac{1}{F\big(VaR_p\big)} = f\big(VaR_p\big) \cdot \frac{1}{(1-p)}$$

The problem is to find a probability p for which $ES_p = VaR_{99}$. Since this problem has no closed solution, we have to find a numerical one. The numerical solution p for a $N(0,1)$ distribution is then applied to our discrete distribution, in order to find the number of the worst observations we need to use for the ES_p calculation. The numerical solution is $p = 94.72\%$. [13]

If the empirical distribution of our losses has fatter tails than that of the normal distribution, then the actual confidence interval for using *ES* as an estimate for *VaR* will be lower than the one used for the normal distribution. Hence the *ES* using the 'actual' confidence interval is lower than that calculated by using the Gaussian distribution, and hence empirical *VaR*(99%) is lower than *ES*(94.72%). Most studies show that financial time series exhibit fat tail, hence using the normal distribution confidence interval is conservative whenever we have a fat tail.

Examples of other distributions: t-student distribution

Under the assumption that the losses follow a t-student distribution, we have that the equivalent confidence interval *x*, (*ES*(*x*%) = *VaR*(99%)) is lower than 94.72%, the confidence interval for the normal distribution. The confidence level *x* converges towards 94.72% as the degrees of freedom increase, as expected, because the *t*-student distribution converges in distribution to the Gaussian one as the degrees of freedom approach infinity.

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[13] Notation: Let N be the total number of observations in our historical window.
Number of worst historical observations to be used for *ES* calculation equals $\mathrm{int}(N(1-p))+1$.

Even though, the family of *t*-student distributions has fatter tails than normal distribution does, it would be unrealistic to assume that the loss distribution follows a t-student distribution. It is widely accepted that empirical loss distribution varies significantly depending on the positions in the portfolio and hence one cannot make reasonable assumptions that it follows a certain t-student distribution with a specific degree of freedom.

Table 1. Confidence intervals vs. degrees of freedom for t-student distributions

Degrees of freedom	Confidence interval X such that $ES(X\%) = VaR(99\%)$
1.1	86.1%
1.5	94.2%
2	96.0%
3	96.7%
4	96.9%
5	97.0%
8	97.2%
15	97.3%
50	97.4%
200	97.4%
1000	97.4%

Source: own calculations

ESTIMATING VAR USING THE HARELL-DAVIS ESTIMATOR

The *Harrel-Davis* quantile estimator was proposed by [Harrell and Davis, 1982]. It makes no assumptions about the underlying loss distribution (just that the observations are i.i.d). It is in general close to an *ES* measure, just that the weights are not a step function, but given by a beta function. The *Harrel-Davis* estimator is in essence the bootstrap estimator of the expected value of the *(n+1)p-th* order statistic, with *p* - the quantile and *n* the sample size. It is based on the fact that as the sample size increases, the expected value of the $(n+1)p$-th order statistic converges to the *p* quantile. Another advantage of using the *HD* estimator is that it gives confidence intervals regarding how good the *VaR* estimator is. The *Harrel-Davis* estimators are defined as follows:

$$HD_y = \sum_{k=1}^{n} w_k x_k \quad \text{with}$$

$$w_k = \frac{1}{\beta\{(n+1)p,(n+1)(1-p)\}} \int_{(k-1)/n}^{k/n} y^{(n+1)p-1}(1-y)^{(n+1)(1-p)-1} dy =$$

$$= I_{k/n}\{(n+1)p,(n+1)(1-p)\} - I_{(k-1)/n}\{(n+1)p,(n+1)(1-p)\}$$

where $I_{k/n}\{(n+1)p,(n+1)(1-p)\}$ is the incomplete beta function.

Figure 1 plots the *HD* weights for estimating the 0.99 quantile from a sample of 1305 observations. Note that unlike the 14th worse loss estimator, which places the total weight on the 1292th order statistic in this case, the *HD* estimator distributes the weights among a range of order statistics. It is worth noting that the weights depend only on the sample size and on the quantile.

Figure1.The weights of the *HD* estimator for p=0.99 and N=1305
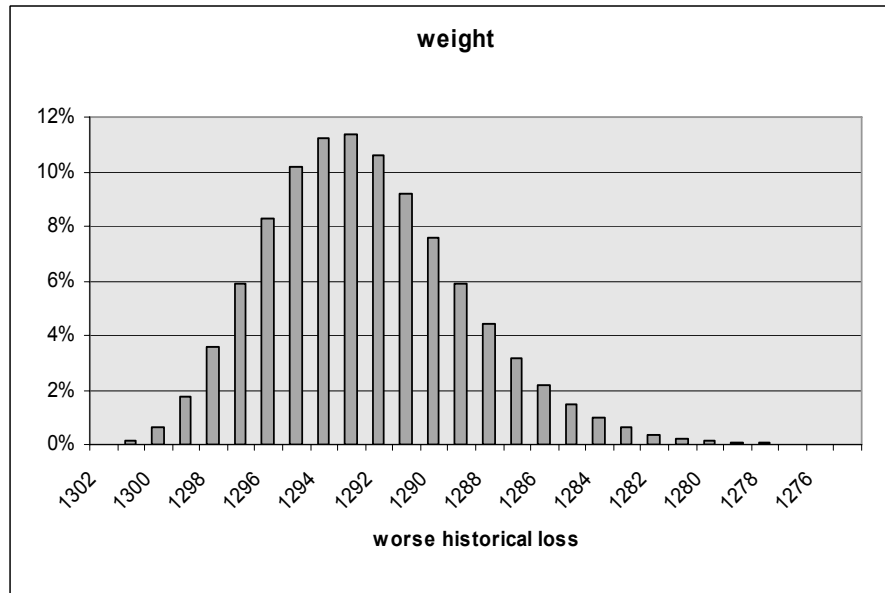
Source: own calculations

In what follows we calculate the *VaR*(99%) estimates for some major stock indices, using 1305 daily observations from 5yr historical data taken from Bloomberg data services. The estimates in Table 2 represent *VaR* numbers as a percentage loss of the entire portfolio if we are to hold it entirely in the respective stock / index. The estimates *ES*(94.7%) are the numbers using the *ES* as an estimate, the 14th worse loss represents the actual *VaR* number when we have 5 years history of observations which is 1305 data points as relative returns, and in this case *VaR*(99%) is the 14th worse loss of the empirical distribution, while *HD* is the *VaR* estimate using the Harell-Davis estimator.

Table 2. *VaR*(99%) estimates by ES(94.7%), 14th worse loss and *HD*

|	AEX	ATG	ATX	BFX	BGLI	BHSE	BMV	BSI	BUX	BVSP	CAC	CCSI	CFG25
ES (94.7%)	-5,17%	-5,03%	-5,91%	-4,42%	-5,47%	-2,50%	-4,57%	-4,36%	-5,34%	-5,93%	-4,75%	-5,00%	-3,20%
14th worse loss	-5,12%	-5,37%	-5,35%	-4,31%	-5,19%	-2,36%	-4,72%	-3,73%	-4,89%	-5,61%	-4,58%	-4,75%	-3,32%
HD Estimate	-5,14%	-5,34%	-5,75%	-4,45%	-5,39%	-2,43%	-4,74%	-3,96%	-5,16%	-6,06%	-4,76%	-4,93%	-3,35%
Stock Indices	AEX	ATG	ATX	BFX	BGLI	BHSE	BMV	BSI	BUX	BVSP	CAC	CCSI	CFG25

Source: own calculations

The stock indices in Table 2 have the following description:

AEX Amsterdam Exchanges Index
ATG Athens General Composite
ATX Austrian Traded Index
BFX BEL 20 Index
BGLI Bulgarian Index (WDR Sofia 30)
BHSE Bahrain All Share Index
BMV IPC General Index
BSI Beirut Stock Index
BUX Budapest S.E. Index
BVSP Bovespa Index
CAC CAC 40 Index
CCSI Egyptian Stock Index
CFG25 Casablanca 25

The following figure represents graphically table 2. Here *ETL* stands for Expected Tail Loss, another term for *ES*. From this diagram one can observe that, as was mentioned above, the 14th worse loss underestimates the *Value-at-Risk* , while *ES*(94.7%) and *HD* agree quite well as estimators, mainly due to sample size of observed data and to the asymptotic normality of the underlying statistics for calculating Expected Shortfall.

Figure 2. Graphical representation of Table 2

Estimates for VaR 99% (quantile)
for 10day combined SimPnL
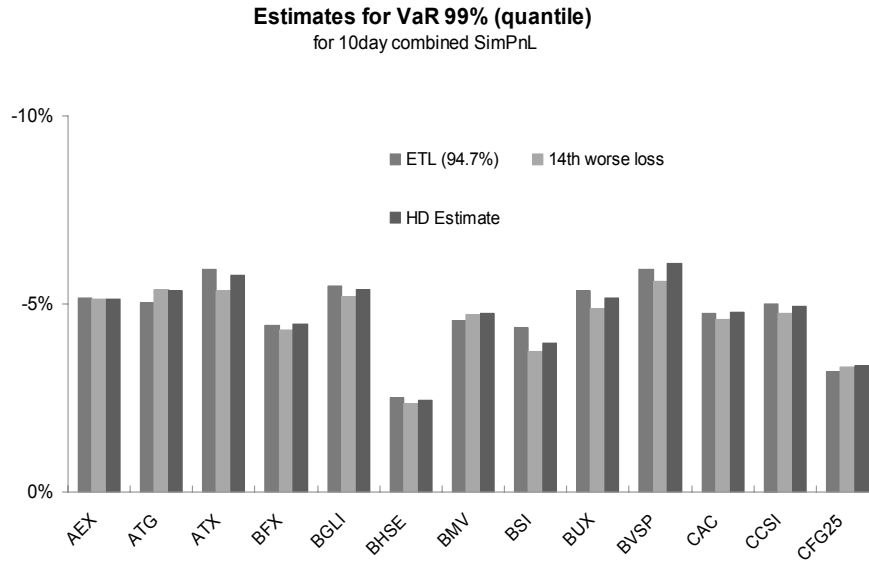
Source: own calculations

CONCLUSIONS

VaR as risk measure together with the corresponding sample statistic as empirical quantile estimator are widely used in the financial risk management, due to their conceptual simplicity, easy of computation and using no specific distributional assumption. At the same time, *VaR* suffers major drawback not taking into account the losses beyond the *VaR* level, on one hand, and not being coherent, on another hand. Also, taking a single order statistic as estimator, it can exhibit high variability. Expected Shortfall avoids these shortcomings and the (uniform) average of data in the tail gives a more stable and accurate estimate of the tail losses. In this paper, for given α, we solve the equation $ES_p = VaR_\alpha$ to find the confidence level *p* with the aim to replace the *[nα+1]* sample statistic by the uniformly averaged data from the tail as estimator for *VaR* at the α confidence level. We also discuss the *Harell-Davis* estimators as *beta*-averaged data from the tail and give some comparative analysis of these three estimators. Using *HD* as an estimator for *Var* solves the problem of *VaR* not being coherent, while at the same time adheres to Basel rules by keeping *VaR* for capital calculation.

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