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ON UPPER GAIN BOUND FOR TRADING STRATEGY BASED ON COINTEGRATION

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Abstract: A long-run trading strategy based on cointegration relationship between prices of two commodities is considered. A linear combination of the prices is assumed to be a stationary AR(1) process. In some range of parameters, AR(1) process is obtained by discrete sampling of Ornstein-Uhlenbeck process. This allows to calculate approximate number of transactions in long run trade horizon and obtain approximate upper bound for possible gain.

Key words and phrases: cointegration, AR(1) process, Ornstein-Uhlenbeck process, trading strategy.

INTRODUCTION

The Engle-Granger [Engle, Granger 1987] idea of cointegration deepened understanding of two central properties of many economic time series – nonstationarity and time-varying volatility. Two nonstationary series may be related so that the values of one of them can not go (after appropriate scaling) too far from the values of the second. This relationship may often be observed for prices of two commodities (e.g. crude oil and heating oil). When we consider series of differences between appropriately scaled prices of such commodities, we observe that it reverts to its mean. In this paper we investigate trading strategy based on this phenomenon. We assume AR(1) structure of the series of differences. However, it often appears that it is much easier to investigate properties of discrete time series through their continuous counterparts - continuous-time stochastic processes. There already exists large literature concerning continuous time ARMA and GARCH processes, also driven by Levy processes and fractionally integrated (see for example [Brockwell, Marquardt 2005]). We will use this approach and find continuous counterpart for AR(1) process - an Ornstein-Uhlenbeck process. Since statistical properties of Ornstein-Uhlenbeck process are subject of interest of many authors. we will be able to calculate all necessary quantities using their results.

We will use results of Thomas [Thomas 1975] and Ricciardi & Sato [Riccardi, Sato 1988] concerning first hitting time of Ornstein-Uhlenbeck process (for more recent survey see [Alili et al 2005]).

Author presumes that the used approach – investigation of properties of discrete time series through their continuous counterparts - may be very useful, since it is much easier to handle with continuous-time stochastic processes than with discrete-time ones. However, this approach shall be used with caution, since we always shall prove that the properties of continuous-time processes are good for approximation of the properties of discrete time series. As far as author knows, first step in this direction for the problem of approximating stopping times of crossing barriers by AR(1) discrete process with stopping times of hitting barriers by Ornstein-Uhlenbeck process was author's paper [Łochowski 2007].

INTRODUCTION OF APPROXIMATE GROSS GAIN

Let $(P_n, n \geq 1)$ and $(Q_n, n \geq 1)$ be two non-stationary time series representing evolution of the prices of futures contracts for two commodities P and Q. We will assume that $(P_n, n \geq 1)$ and $(Q_n, n \geq 1)$ are cointegrated i.e. for some positive α, β the process $R_n = \alpha P_n - \beta Q_n$ is stationary. Moreover, we will assume that it is mean zero AR(1) process, i. e.

$$R_{n+1} = \gamma R_n + Z_n, \tag{1}$$

where $(Z_n, n \geq 1)$ is i.i.d. sequence, independent from R_1, with $Z_1 \sim N(0, \sigma^2)$ ($N(\mu, \sigma^2)$ denotes here normal distribution with mean μ and variance σ^2). An equivalent form of (1) is

$$\Delta R_{n+1} = -(1-\gamma) R_n + \sigma \varepsilon_n, \tag{2}$$

where $\varepsilon_n = Z_n / \sigma, n \geq 1$.

It is easy to see that the stationarity of R_n holds iff $\gamma \in (-1;1)$. Stationarity implies that $R_n \sim N(0, \sigma^2/(1-\gamma^2))$ and $\mathrm{Cov}(R_n, R_{n+h}) = \gamma^h \sigma^2/(1-\gamma^2)$.

From stationarity of R_n one may derive long-run trading strategy based on selling α commodity P contracts and buying β commodity Q contracts when R_n exceeds certain threshold value a and doing opposite, when R_n goes below $-a$. If we enter the market with α contracts of commodity P and are interested in leaving it with the same volume of commodity P contracts after long time horizon

T then the gross gain obtained from the strategy equals $2a \cdot N(a)$, where $N(a)$ denotes the number of pairs of transactions:

- when $R_n = \alpha P_n - \beta Q_n \geq a$ sell α commodity P contracts and simultaneously buy β commodity Q contracts,
- when $R_n = \alpha P_n - \beta Q_n \leq -a$ buy α commodity P contracts and simultaneously sell β commodity Q contracts.

The problems which one faces deciding for the described strategy is the estimation of $N(a)$ and then the choice of an optimal threshold value a.

Firstly we will try to estimate $N(a)$ for large T and positive, fixed (but not too small) a. Let $\tilde{T}_1, \tilde{T}_2$ be the following stopping times

$$\tilde{T}_1 = \inf\{n \ : \ R_n \geq a\},\ \tilde{T}_2 = \inf\{n \geq \tilde{T}_1 \ : \ R_n \leq -a\}.$$

Let us assume that the process R_n has small jumps (much smaller then a) and it is obtained by discrete sampling of a continuous-time process with continuous trajectories, $(U_t, t \geq 0)$, i. e. $R_n = U_n$ for $n = 1, 2, ?$. Let T_1, T_2 be continuous counterparts to $\tilde{T}_1, \tilde{T}_2$, i. e.

$$T_1 = \inf\{t \ : \ U_t \geq a\},\ T_2 = \inf\{t \geq T_1 \ : \ U_t \leq -a\}.$$

In fact, from continuity of U_t we have

$$T_1 = \inf\{t \ : \ U_t = a\},\ T_2 = \inf\{t \geq T_1 \ : \ U_t = -a\}.$$

Let us denote $T(a) = E(T_2 - T_1)$. From the assumption about R_n we get $\tilde{T}_1 \approx T_1, \tilde{T}_2 \approx T_2$ (cf. [Łochowski 2007]). Now from theory of renewal processes (cf. [Rolski et al 1998]) and symmetry of the process R_n we may approximate number of transactions $N(a)$ in long time horizon by $T / \left(2T\left(a\right)\right)$. Thus the gross gain in long time horizon equals $2a \cdot N\left(a\right) \approx a \cdot T / T\left(a\right)$. Based on this reasoning let us define.

Definition. Approximate gross gain, *$AGG(a, T)$, for a threshold value $a > 0$ and time horizon $T > 0$ is defined by the formula*

$$AGG(a,T) = \frac{T \cdot a}{T(a)}.$$

ORNSTEIN-UHLENBECK PROCESS AS A CONTINUOUS VERSION OF AR(1) PROCESS

The natural candidate for process U_t is Ornstein-Uhlenbeck process being a solution of the following stochastic differential equation - the continuous counterpart of (2)

$$dU_t = -(1-\gamma)U_t dt + \sigma dW_t, \tag{3}$$

where W_t denotes a standard Wiener process.

Equation (3) has the following solution

$$U_t = e^{-(1-\gamma)t}U_0 + \sigma\int_0^t e^{-(1-\gamma)(t-s)}dW_s.$$

From bilinearity of covariance, independence of increments of Wiener process and then from isometry formula for stochastic integrals we have

$$Cov\left(\int_0^t e^{-(1-\gamma)(t-s)}dW_s, \int_0^u e^{-(1-\gamma)(u-s)}dW_s\right)$$
$$= e^{-(1-\gamma)(t+u)}Cov\left(\int_0^t e^{(1-\gamma)s}dW_s, \int_0^u e^{(1-\gamma)s}dW_s\right)$$
$$= e^{-(1-\gamma)(t+u)}\int_0^{\min(t,u)} e^{2(1-\gamma)s}ds = e^{-(1-\gamma)(t+u)}\frac{e^{2(1-\gamma)\min(t,u)}-1}{2(1-\gamma)}.$$

Assuming that U_0 is independent from $(W_t, t \geq 0)$, we get

$$\mathrm{Cov}(U_t, U_{t+h}) = e^{-(1-\gamma)(2t+h)}\mathrm{Var}(U_0) + \sigma^2 e^{-(1-\gamma)(2t+h)}\frac{e^{2(1-\gamma)t}-1}{2(1-\gamma)}.$$

If $U_0 \sim N(0, \sigma^2/(2(1-\gamma)))$ then $U_t \sim N(0, \sigma^2/(2(1-\gamma)))$ and $\mathrm{Cov}(U_t, U_{t+h}) = \sigma^2 e^{-(1-\gamma)h}/(2(1-\gamma))$.

Comparing the distribution of U_n with the distribution of R_n we see that they are different. But when $\gamma \in (0;1)$, taking process $(V_t, t \geq 0)$ defined by the equation

$$dV_t = -\ln(1/\gamma)V_t dt + \sigma\sqrt{\frac{2\ln(1/\gamma)}{1-\gamma^2}}dW_t$$

with $V_0 \sim N(0,\sigma^2/(1-\gamma^2))$ independent from $(W_t, t \geq 0)$, we obtain such a process that for positive integers $n_1, n_2, ?, n_k$ vector $(V_{n_1}, V_{n_2}, ?, V_{n_k})$ has the same distribution as vector $(R_{n_1}, R_{n_2}, ?, R_{n_k})$.

Remark. Ornstein-Uhlenbeck process may also be introduced as a time-space scaled Wiener process. Defining

$$\tilde{V}_t = \frac{\sigma}{\sqrt{1-\gamma^2}} e^{-\ln(1/\gamma)t} W(e^{2\ln(1/\gamma)t})$$

we get a process with the same finite distributions as the process $(V_t, t \geq 0)$.

UPPER BOUND FOR APPROXIMATE GROSS GAIN

Now we are ready to calculate $T(a)$. Let us denote

$$T_{b,c} := \inf\{t \geq 0 \,:\, V_t = c \,|\, V_0 = b\}.$$

From results of Thomas [Thomas 1975] and Ricciardi & Sato [Riccardi, Sato 1988] as well as from scaling properties of V_t we have that for $a > 0$

$$E(T_{a,0}) = \frac{\sqrt{\pi}}{2\ln(1/\gamma)} \int_0^{a\sqrt{1-\gamma^2}/\sqrt{2\sigma^2}} \left(1+\operatorname{erf}(t)\right) e^{t^2} dt$$

and

$$E(T_{0,a}) = \frac{\sqrt{\pi}}{2\ln(1/\gamma)} \int_0^{a\sqrt{1-\gamma^2}/\sqrt{2\sigma^2}} \left(1+\operatorname{erf}(-t)\right) e^{t^2} dt,$$

where $\operatorname{erf}(t)$ stands for error function defined as $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$. From strong Markov property of V_t and since $\operatorname{erf}(t) + \operatorname{erf}(-t) = 0$

we see that

$$T(a) = E(T_{a,0}) + E(T_{0,-a}) = E(T_{a,0}) + E(T_{0,a})$$
$$= \frac{\sqrt{\pi}}{\ln(1/\gamma)} \int_0^{a\sqrt{1-\gamma^2}/\sqrt{2\sigma^2}} e^{t^2}\, dt.$$

Since $\int_0^u e^{t^2} dt = \int_0^u \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} dt = \sum_{k=0}^{\infty} \frac{u^{2k+1}}{(2k+1)\cdot k!}$, we have

$$T(a) = \sum_{k=0}^{\infty} d_{2k+1} a^{2k+1},$$

where

$$d_{2k+1} = \frac{\sqrt{\pi}}{\ln(1/\gamma)} \frac{1}{(2k+1)\cdot k!} \left(\frac{1-\gamma^2}{2\sigma^2} \right)^{k+1/2} > 0, \text{ for } k = 0,1,?.$$

Now we have

$$\frac{a}{T(a)} = \frac{a}{d_1 a + d_3 a^3 + \cdots} = \frac{1}{d_1 + d_3 a^2 + \cdots}$$

and we see that approximate gross gain, $AGG(a,T) = \frac{T\cdot a}{T(a)}$, is a decreasing function of a. We also have

$$\sup_{a>0} \frac{a}{T(a)} = \lim_{a?0} \frac{a}{T(a)} = \frac{1}{d_1} = \sqrt{\frac{2}{\pi}} \frac{\ln(1/\gamma)\sigma}{\sqrt{1-\gamma^2}}.$$

The above calculations imply

Theorem. *If* $\gamma \in (0;1)$ *the approximate gross gain,* $AGG(a,T)$, *for any positive* a *and* T *is bounded from above by* T/d_1, *i. e.*

$$AGG(a,T) \le \sqrt{\frac{2}{\pi}} \frac{\ln(1/\gamma)\sigma}{\sqrt{1-\gamma^2}} \cdot T. \quad (4)$$

Moreover, there is no positive a *for which the above value is attained.*

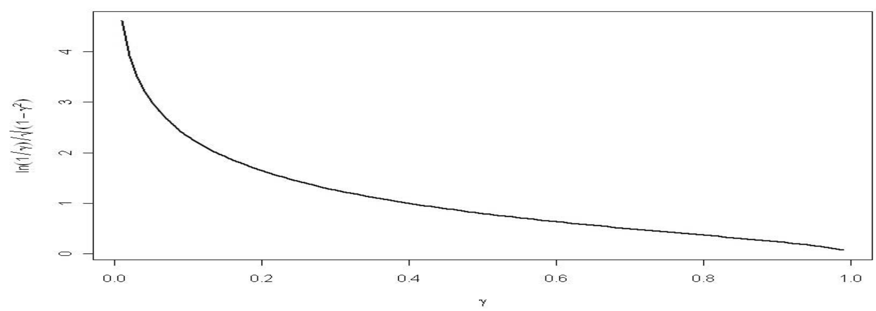
FINAL REMARKS

The results of the previous section are in some sense negative, since there is no optimal postive threshold value a maximizing the approximate gross gain. Moreover, the smaller is $a > 0$, the greater is $AGG(a,T)$. But for very small threshold values the relationships $\tilde{T}_1 \approx T_1, \tilde{T}_2 \approx T_2$ on which our reasoning was based, may fail. Moreover, when a is small we have to change our positions very often which may be not technically possible and if we take into account transaction costs then it would appear that for small a our trading startegy leads to loss.

However, it seems that the formula for upper gain bound in our simple model has natural interpretation in terms of quantities appearing on the right side of (4). The maximal posible gain is proportional to the duration T of the investment and to standard deviation σ of the random term Z_n. It is known phenomenon (used sometimes in so called volatility trading), that the bigger volatility, represented here by σ, the bigger profits in short term are possible.

The more sophisticated seems to be the dependence of $AGG(a,T)$ on parameter γ. This parameter determines the speed of reverting AR(1) process to its mean value. The bigger γ the longer time is needed for AR(1) process to revert to its mean. The dependence between γ and $AGG(a,T)$ is represented by the function $\gamma \mapsto \frac{\ln(1/\gamma)}{\sqrt{1-\gamma^2}}$. This is decreasing function on the interval $(0;1)$ and its graph is presented below.

Figure 1. Graph of the function $\gamma \mapsto \frac{\ln(1/\gamma)}{\sqrt{1-\gamma^2}}$.



REFERENCES

Alili, L., Patie, P. and Pedersen, J. L. (2005). Representation of the first hitting time density of an Ornstein-Uhlenbeck process. Stoch. Models 21 967--980.

Brockwell, P. J. and Marquardt, T. (2005). Lévy driven and fractionally integrated ARMA processes with continuous time parameter Statistica Sinica 15 474--494.

Engle, R. F. and Granger, C. W. J. (1987). Cointegration and Error-Correction: Representation, Estimation and Testing. Econometrica 55 251--276.

Łochowski, R. (2007) Stopping time for Ornstein-Uhlenbeck process vs. stopping time for discrete AR(1) process, reprint available from web page http://akson.sgh.waw.pl/~rlocho/stoppingtime2.pdf

Rolski T., Schmidli H., Schmidt V. and Teugels J. (1998) Stochastic Processes for Insurance and Finance. Wiley, Chichester New York Weinheim Brisbane Singapore Toronto.

Ricciardi, L. M. and Sato, S. (1988). First-passage time density and moments of the Ornstein-Uhlenbeck process. J. Appl. Probab. 25 43--57.

Thomas, M. U. (1975). Some mean first-passage time approximations for the Ornstein-Uhlenbeck process. J. Appl. Probab. 12 600--604.