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ASYMPTOTIC NASH EQUILIBRIA IN DISCOUNTED STOCHASTIC GAMES OF RESOURCE EXTRACTION

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Abstract: A class of two person nonzero-sum nonsymmetric stochastic games of capital accumulation/resource extraction is considered. It is shown that the Nash equilibrium in the discounted games has a limit when the discount factor tends to 1. Moreover, this limit is an epsilon-equilibrium in the discounted game with sufficiently large discount factor.

Key words: Nonzero-sum stochastic games. capital accumulation problems. resource extraction games. Nash equilibrium.

Introduction

Nonzero-sum dynamic games of capital accumulation (or resource extraction) studied in this paper belong to a class of stochastic games with uncountable state space. Unlike in the games with a countable state space (see [9]), the question of the existence of stationary Nash equilibria in such games has a positive answer only in some special cases of interest. For a survey of the existing literature on this topic the reader is referred to [1, 2, 7, 12, 13, 14] and [16]. The special case of these games is a class of concave games of resource extraction or capital accumulation. The pioneering work on this field is [10]. Similar class is studied in [1, 2, 3, 4, 12, 14] and [16]. Our main assumption on the transition probability function in the game says that it is a combination of finitely many probability measures on the state space with the coefficients depending on the investment. Similar form of it we can find in [2] and in [16].

In our model, we restrict our assumptions to two person game. Asymptotic properties of Nash equilibria in the discounted stochastic game, with respect to discount factor tending to 1, are the main results of this paper. We accept the assumptions, from the model considering in [14]. There is proven existence of Nash equilibria in finite and infinite horizon game. Moreover, there is also proven the uniform convergence of equilibria payoffs with respect to horizon of the game. The limit is an equilibrium payoff in infinite horizon game. The convergence of Nash equilibria is proven as well, but there is only pointwise convergence. In this paper we can complete those results and prove uniform convergence. This also leads us to uniform convergence of Nash equilibria and adequate equilibria payoffs with respect the discount factor tending to 1 in finite and in infinite horizon game. These limits it is shown to be Nash equilibrium and adequate equilibrium payoff in the undiscounted model. Similar problems are often described in the literature. However, unlike in [8, 11] and [17] we also obtain further results. Uniform convergence yields us more properties. Taking an arbitary small ε, the Nash equilibrium in the undiscounted model is ε - equilibrium in the discounted games with sufficiently large discount factor. The results obtained in this paper would be useful in the discounted model, if we do not know exact value of the discount factor. Then we could approximate Nash equilibria in discounted models by the Nash equilibrium in the undiscounted model.

This paper is organized in a following way: in the next section is presented a model with main assumptions. In the third section the model of auxillary one shot game and some properties of it are described. This model is a special case of that from Section 3 in [3]. The main results are discreibed in the fourth and fifth section. Fourth section is about finite horizon game and fifth about infinite horizon game. At the end of the last section there is an example in which the results obtained in this paper would be useful.

The model and basic assumptions

Consider an two-person nonzero-sum nonsymmetric stochastic game in which:

(i) $S \subset R_+ := [0, \infty)$ is a compact interval containing zero called the *state space* or the set of all possible stocks of a resource. Denote $S := [0, \bar{s}]$, for some strictly positive constant $\bar{s}$.

(ii) $A_i(s) \subset S$, $A_i(s) := [0, a_i(s)]$ is the *space of actions* available to player i in the state $s \in S$, where $a_i(\cdot)$ is called *capacity function.* Assume that $a_i(\cdot)$ is nonnegative, nondecreasing and continous function such that $a_1(s) + a_2(s) \leq s$. For $\bar{x} := (x_1, x_2)$ let

$$X(s) := A_1(s) \times A_2(s) \quad \text{and} \quad D = \{(s, \bar{x}) : s \in S, \bar{x} \in X(s)\}.$$

(iii) $u_i : S \to R_+$ is a bounded *instantaneous utility function* for player i.

(iv) q is a Borel measurable transition probability from D to S, called the *law of motion* among states. If s is a state (resource stock) at some stage of the game and the players select an $\bar{x} = (x_1, x_2) \in X(s)$, then $q(\cdot|s, x_1, x_2)$ is the probability distribution of the next stock.

Our further assumptions are:

C1: The utility function for player i $u_i : S \to R_+$ is a strictly concave twice continuously differentiable and increasing function such that $u_i(0) = 0$.

C2: The transition probability is of the form

$$q(\cdot|s, \bar{x}) = \sum_{l=1}^{L} g_l\,(s - x_1 - x_2)\,\lambda_l(\cdot|s) + g_0(s - x_1 - x_2)\delta_0(\cdot),$$

where

(a) δ_0 is the Dirac measure concentrated at 0,

(b) λ_l is a Borel measurable transition probability from S to S. Moreover,

- assume additionally that there exists a probability measure μ such that, μ stochasticaly dominates $\lambda_l(\cdot|s)$ for all $s \in S$, $l = 1, ..., L$, and holds

$$\int_S g_0(s)\mu(ds) > 0,$$

  or equivalentely

$$\int_S \sum_{l=1}^{L} g_l(s)\mu(ds) < 1.$$

- for each $l = 1, ..., L$ and for each Borel measurable and bounded function $v(\cdot)$ the function $s \to \int_S v(s')\lambda_l(ds'|s)$ is continous. Clearly λ_l is a stochasticaly continous measure.

(c) $g_0(0) = 1$ and for $l = 1 \ldots, L$ $g_l : S \to [0, 1]$ is strictly concave, increasing and twice continuously differentiable. Obviously $\sum_{l=0}^{L} g_l \equiv 1$.

Remark. By assumption **C2** it follows that 0 is an absorbing state since

$$q(\{0\}|0, 0, 0) = g_0(0)\delta_0(\{0\}) + \sum_{l=1}^{L} g_l(0)\lambda_l(\{0\}|0) = \delta_0(\{0\}) = 1.$$

Remark. Note that assumption **C2** it satisfied when

$$\sum_{l=1}^{L} g_l(\bar{s}) < 1.$$

Then as μ we can take a Dirac measure concentrated at $\bar{s}$.

Remark. Typical examples of the functions g_l are:

$$g_l(y) := \alpha(1 - e^{-y}) \quad \text{or} \quad g_l(y) := \alpha\sqrt{y}, \quad y = s - x_1 - x_2,$$

where $\alpha > 0$ are small enough, S is bounded from above.

Remark. Typical example of the collection of measures λ_l are

$$\lambda_l(A|s) := \int_A \rho_l(s, s')\nu_l(ds'),$$

where $\nu_l(\cdot)$ is some probability measure on S and $\rho_l(s, \cdot)$ is a density for each $s \in S$. Moreover, for each $s' \in S$ $\rho_l(\cdot, s')$ is a continous function.

The game is played in discrete time with past history as common knowledge for all the players. A *strategy* for a player is a Borel mapping which associates with each given history an action available to him. For $i = 1, 2$ let F_i be set of Borel measurable functions $f_i : S \to R_+$ such that $f_i(s) \in A_i(s)$ for each $s \in S$. A *Markov strategy* for player i is a sequence $\pi_i = (f_{i,1}, f_{i,2}, \ldots,)$ where each $f_{i,n}$ belongs to F_i. A *stationary strategy* for player i is a constant sequence π_i where each $f_{i,n} = f_i$ for some $f_i \in F_i$. Let $F := F_1 \times F_2$ be a set of all *profiles*. In the sequence, a stationary strategy $(f_i, f_i, ...)$ of player i will be identified with f_i.

Let $H^\infty := D \times D \times \ldots$ by the set of all possibles histories of the game endowed with the product σ-field. For any profile of strategies $\pi = (\pi_1, \pi_2)$, and every initial state $s \in S$, a probability measure P_s^π and a stochastic proces $\{s_n, \bar{x}_n\}$ are defined on H^∞ in the canonical way, where the random variables s_n and $\bar{x}_n = (x_{n1}, x_{n2}) \in X(s_n)$ describe the state and the actions chosen by the players, respectively, on the n-th stage of the game (see Chapter 7 in Bertsekas and Shreve (1978)). Thus, for each profile π of strategies and any initial state s, one can define the operator of the expected value E_s^π with respect to the probability measure P_s^π. In the n-stage β - *discounted* model the total expected utility for player i is

$$\gamma_{i,n}^\beta(\pi)(s) = E_s^\pi\left(\sum_{k=1}^n \beta^{k-1}u_i(x_{ki})\right).$$

The value $\beta \in (0, 1]$ is said to be a *discount factor*. If $\beta = 1$ then we have *undiscounted* model. Let us denote $\gamma_{i,n}(\pi)(s) := \gamma_{i,n}^1(\pi)(s)$. The total expected utility for player i in the infinite horizon game is

$$\gamma_i^\beta(\pi)(s) = E_s^\pi\left(\sum_{k=1}^\infty \beta^{k-1}u_i(x_{ki})\right) = \lim_{n\to\infty} \gamma_{i,n}^\beta(\pi)(s).$$

In the undiscounted model, the total expected utility for player i is

$$\gamma_i(\pi)(s) = E_s^\pi\left(\sum_{k=1}^\infty u_i(x_{ki})\right).$$

Remark. This criterion above makes a sens, because as shows Corollary 12, the series under the expected value are convergents are limited by a common value.

Let $\pi = (\pi_1, \pi_2)$ and σ_i be a strategy for player i. Then, as usual, (π_{-i}, σ_i) is the strategy profile π with π_i replaced by σ_i.

Definition 1. A strategy profile $\pi^\star = (\pi_1^\star, \pi_2^\star)$ is called a *Nash equilibrium* in the β - discounted stochastic game if and only if no unilateral deviations from it are profitable, that is, for every player i, his strategy σ_i and $s \in S$,

$$\gamma_i^\beta(\pi^\star)(s) \geq \gamma_i^\beta(\pi^\star_{-i}, \sigma_i)(s).$$

Similarly, Nash equilibria are defined in the finite horizon case and in the undiscounted case. Obviously, a strategy for player i in a n-step game consists of n components only.

Definition 2. Fix $\varepsilon > 0$. A strategy profile $\pi^\varepsilon = (\pi_1^\varepsilon, \pi_2^\varepsilon)$ is called an ε - *equilibrium* in the β - discounted stochastic game if and only if for every player i, his strategy σ_i and $s \in S$,

$$\varepsilon + \gamma_i^\beta(\pi^\varepsilon)(s) \geq \gamma_i^\beta(\pi^\varepsilon_{-i}, \sigma_i)(s).$$

Similarly we define ε - equilibrium in the finite horizon game. It is easy to see that unilateral deviation from ε - equilibrium can take the profit but no greater then ε.
Notation: If ρ is arbitrary game, then by $NE\,\rho$ we denote the set of Nash equilibria in this game.

Auxilliary one shot game

In this section we introduce an auxilliary one shot model $G(s, \eta)$ of two person game. The payoff function for player i $(i = 1, 2)$ is

$$w_i(\eta_i, s, x_1, x_2) = u_i(x_i) + \sum_{l=1}^{L} \eta_{i,l}(s) g_l\left(s - x_1 - x_2\right), \tag{1}$$

for $s \in S$, $x_i \in A_i(s)$, where each $\eta_{i,l} : S \to R_+$ is a continous function, $\eta_i := (\eta_{i,1}, ..., \eta_{i,L})$ and η is a matrix with the rows η_1 and η_2. Let $x^\eta(s) := (x_1^\eta(s), x_2^\eta(s))$ be a Nash equilibrium in $G(s, \eta)$. By Proposition 1 in [3] follows that this definition is well.

Lemma 3. For each continous η, the function $x_i^\eta(\cdot)$ is continous.

Proof. Let $s_n \to s_0$. Then by definition of x^η we have

$$\begin{aligned} w_1(\eta_1, s_n, x_1^\eta(s_n), x_2^\eta(s_n)) \quad &\geq \quad w_1(\eta_1, s_n, x_1, x_2^\eta(s_n)), \\ w_2(\eta_2, s_n, x_1^\eta(s_n), x_2^\eta(s_n)) \quad &\geq \quad w_2(\eta_2, s_n, x_1^\eta(s_n), x_2). \end{aligned} \tag{2}$$

for arbitrary $(x_1, x_2) \in X(s)$. Let $x^* := (x_1^*, x_2^*)$ be an arbitrary cumulation point of $(x^\eta(s_n))$. Consider subsequence of s_n which leads $(x^\eta(s_n))$ to x^*. By Assumption **C1** and **C2** and definition of w_i, taking a limit in (2) we obtain

$$\begin{aligned} w_1(\eta_1, s_0, x_1^*, x_2^*) &\geq w_1(\eta_1, s_0, x_1, x_2^*), \\ w_2(\eta_2, s_0, x_1^*, x_2^*) &\geq w_2(\eta_2, s_0, x_1^*, x_2). \end{aligned}$$

This implies that x^* is a Nash equilibrium in $G(s_0, \eta)$. As we have mentioned before this game has unique Nash equilibrium $x^\eta(s_0)$, hence $x^* = x^\eta(s_0)$. ■

Lemma 4. For all $i = 1, 2$ and $l = 1, ..., L$, let $\eta_{i,l}^n(\cdot)$ be a sequence of continous function on S and

$$\lim_{n\to\infty} \eta_{i,l}^n(\cdot) = \eta_{i,l}(\cdot)$$

in sup - norm on S. Then

$$\lim_{n\to\infty} x_i^{\eta^n}(s) = x_i^\eta(s), \tag{3}$$

and

$$\lim_{n\to\infty} w_i\left(\eta_i^n, s, x_1^{\eta^n}(s), x_2^{\eta^n}(s)\right) = w_i\left(\eta_i, s, x_1^\eta(s), x_2^\eta(s)\right), \tag{4}$$

in sup - norm on S.

Proof. *Step 1* First we prove uniformly convergence in (3). We show that there is exactly one cumulation point of the set of couples $K := \left\{(s_n, x^{\eta^n}(s_n)) : n \in N\right\}$, where $(s_1, s_2, ...)$ is a sequence such that

$$\sup_{s\in S} \left|x^{\eta^n}(s) - x^\eta(s)\right| = \left|x^{\eta^n}(s_n) - x^\eta(s_n)\right|.$$

This sequence exists by Lemma 3. Let (s^*, x^*) be an arbitrary cumulation point of the set K. Then there exisists a sequence containing in K such that

$$x^* := \lim_{k\to\infty} x^{\eta^k}(s_k) \quad \text{and} \quad s^* = \lim_{k\to\infty} s_k.$$

Note that in this equation above, we denote x^{η^k} instead $x^{\eta^{n_k}}$ and s_k instead s_{n_k} for simplify the notation. Note that

$$\begin{aligned} &w_1\left(\eta_1^k, s_k, x_1^{\eta^k}(s_k), x_2^{\eta^k}(s_k)\right) \geq w_1\left(\eta_1^k, s_k, x_1, x_2^{\eta^k}(s_k)\right), \\ &w_2\left(\eta_2^k, s_k, x_1^{\eta^k}(s_k), x_2^{\eta^k}(s_k)\right) \geq w_2\left(\eta_2^k, s_k, x_1^{\eta^k}(s_k), x_2\right), \end{aligned} \tag{5}$$

when $x_i \in A_i(s_k)$ is arbitrary. Since the sequence η^k uniformely converges to η, and η is continous, hence $\eta^k(s_k) \to \eta(s^*)$. By continouity of w_i and by (5) we obtain

$$\begin{aligned} &w_1\left(\eta_1, s^*, x_1^*, x_2^*\right) \geq w_1\left(\eta_1, s^*, x_1, x_2^*\right), \\ &w_2\left(\eta_2, s^*, x_1^*, x_2^*\right) \geq w_2\left(\eta_2, s^*, x_1^*, x_2\right), \end{aligned}$$

which means that x^* is Nash equilibrium in the game $G(s_0, \eta)$. By Proposition 1 in [3] there is exactly one Nash equilibrium in this game. Hence each cumulation point of the set K is in a form $(s^*, x^\eta(s^*))$ i.e. is on the graph of $x^\eta(\cdot)$. Hence and by Lemma 3, we immediately obtain that the unique cumulation point of the sequence

$$\left| x^{\eta^n}(s_n) - x^\eta(s_n) \right|$$

is 0. This implies that $x^{\eta_n}(\cdot)$ is uniformely convergent to $x^\eta(\cdot)$.

Step 2. The uniform convergence in (4) follows directly from uniform convergence of the sequences $x^{\eta^n}(\cdot)$ and $\eta^n_{i,l}(\cdot)$, and uniform continouity of the functions u_i and g_l. ■

In [14] it is proven following lemma.

Lemma 5. For $i = 1, 2$ $X_i = [0, d_i]$ be an action spaces. Let

- u_i satysfy condition **C1**,
- the functions $\xi_i : [0, d_1 + d_2] \to R_+$ and $\zeta_i : [0, d_1 + d_2] \to R_+$ are twice continously differentables, strictly concave and decreasing,
- for $t \in [0, d_1 + d_2]$ we have $\xi_i(t) \le \zeta_i(t)$ and $\xi_i'(t) \ge \zeta_i'(t)$.

Consider two games: ρ_1 in which the payoff function for player i is on the form

$$W_i^1(x_1, x_2) = u_i(x_i) + \xi_i(x_1 + x_2),$$

and ρ_2 with the payoff function

$$W_i^2(x_1, x_2) = u_i(x_i) + \zeta_i(x_1 + x_2)$$

for $(x_1, x_2) \in X_1 \times X_2$. Then there exists Nash equilibrium in the game ρ_1 say $x^* := (x_1^*, x_2^*)$ and Nash equilibrium in the game ρ_2 say $y^* := (y_1^*, y_2^*)$ such that

$$W_i^1(x_1^*, x_2^*) \le W_i^2(y_1^*, y_2^*).$$

By Lemma 5 we immediately obtain following lemma:

Lemma 6. If for each $l = 1, ..., L$, and $s \in S$ holds $\eta^1_{i,l}(s) \le \eta^2_{i,l}(s)$, then

$$w_i\left(\eta_i^1, s, x_1^{\eta^1}(s), x_2^{\eta^1}(s)\right) \le w_i\left(\eta_i^2, s, x_1^{\eta^2}(s), x_2^{\eta^2}(s)\right).$$

Proof. Fix $s \in S$. Note that the games $G(s, \eta^j)$ $j = 1, 2$ can be described as ρ_i where

$$W_i^1(x_1, x_2) := w_i(\eta_i^1, s, x_1, x_2) = u_i(x_i) + \xi_i(x_1 + x_2)$$

with

$$\xi_i(t) := \sum_{l=1}^{L} \eta^1_{i,l}(s) g_l\left(s - t\right),$$

and
$$W_i^2(x_1,x_2) := w_i(\eta_i^2, s, x_1, x_2) = u_i(x_i) + \zeta_i(x_1+x_2)$$
with
$$\zeta_i(t) := \sum_{l=1}^{L} \eta_{i,l}^2(s) g_l\,(s-t)\,, t := x_1 + x_2.$$
Clearly u_i, ξ_i, ζ_i are strictly concave, twice continously differentable and strictly monotone (u_i is increasing ξ_i and ζ_i are decreasing). Since $\eta_{i,l}^1(s) \leq \eta_{i,l}^2(s)$ for all $l = 1, ..., L$, hence $\xi(t) \leq \zeta(t)$ for all $t \in [0, d_1 + d_2]$. Note that
$$\xi_i'(t) = -\sum_{l=1}^{L} \eta_{i,l}^1(s) g_l'\,(s-t) \geq -\sum_{l=1}^{L} \eta_{i,l}^2(s) g_l\,(s-t) = \zeta_i'(t).$$
Therefore conditions of Lemma 5 are satisfied. By Proposition 3 in [3] Nash equilibria in both ρ_i are unique, hence the proof is complete. ■

Asymptotic Nash equilibria in the n - step model

In this section we consider finite horizon game. Define $B_0(S) := \{v : S \to R_+ : v(0) = 0\}$. For every $(v_1, v_2) \in B_0(S) \times B_0(S)$, $s \in S$, $\beta \in (0,1]$, we define auxiliary two person one shot game $\Gamma(\beta, v_1, v_2, s)$, in which the payoff function for each player i is
$$k_i(\beta, v_i, s, x) = u_i(x_i) + \beta \int_{S_+} v_i(s') q(ds'|s,x),$$
where $x = (x_1, x_2) \in X(s)$. Since $v_i \geq 0$, by the assumptions **C1** and **C2** the payoff function this game has the same form as in section 3 in [3]. Hence by Proposition 1 in [3] we can conclude that for every $s \in S$, this game have an unique proper Nash equilibrium $NE\Gamma(\beta, v_1, v_2, s)$.
Obviously $NE\Gamma(\beta, v_1, v_2, s) = (0,0)$ for $s = 0$.

Let $\beta \in (0,1]$ be a discount factor, and n be a horizon of the finite step game. For $i = 1, 2$ i $s \in S$ let $f_{i,1}^{\beta}(s) := a_i(s)$, and
$$v_{i,1}^{\beta}(s) := \max_{a_i \in A_i(s)} u_i(a_i) = u_i(f_{i,1}^{\beta}(s)).$$
Clearly $v_{i,1}^{\beta} \in B_0(S)$. If $v_{i,0}^{\beta}(s) := 0$ for arbitrary $s \in S$, then
$$\bar{f}_1^{\beta} := (f_{1,1}^{\beta}(s), f_{2,1}^{\beta}(s)) = NE\Gamma(\beta, v_{1,0}^{\beta}, v_{2,0}^{\beta}, s).$$
Therefore $\bar{f}_1^{\beta}$ is a Nash equilibrium in the one-step game, $v_{i,1}^{\beta}$ is an equilibrium function for the player i and $v_{i,1}^{\beta} = k_i(\beta, v_{i,0}, s, (a_1(s), a_2(s)))$. Analogously as in [14] and in section 4 in [2], we can define $f_{i,2}^{\beta}, ..., f_{i,n}^{\beta} \in F_i$ and $v_{i,2}^{\beta}, ..., v_{i,n}^{\beta} \in B_0(S)$ in the following way
$$\begin{aligned} \bar{f}_k^{\beta} := (f_{1,k}^{\beta}, f_{2,k}^{\beta}) &:= NE\Gamma(\beta, v_{1,k-1}^{\beta}, v_{2,k-1}^{\beta}, s) \quad \text{and} \\ v_{i,k}^{\beta}(s) &:= k_i(\beta, v_{i,k-1}^{\beta}(s), s, \bar{f}_k^{\beta}(s)), \end{aligned}$$

where $s \in S$ and $k = 2, ..., n$. By Proposition 1 in [3], these definitions above are well. Let $\pi_i^{(n),\beta}$ be a n-step strategy for the player i which is defined as

$$\pi_i^{(n),\beta} = (f_{i,1}^{\beta *}, f_{i,2}^{\beta *}, ..., f_{i,n}^{\beta *}) := (f_{i,n}^{\beta}, f_{i,n-1}^{\beta}, ..., f_{i,1}^{\beta}).$$

(Clearly, $f_{i,k}^{\beta} = f_{i,n-k+1}^{\beta *}$.) Let $\pi^{(n),\beta} := \left(\pi_1^{(n),\beta}, \pi_2^{(n),\beta}\right)$. We denote $f_{i,n}^{\star} := f_{i,n}^{\beta}$, $v_{i,n}^{\star} := v_{i,n}^{\beta}$, and $\pi^{(n)} := \left(\pi_1^{(n)}, \pi_2^{(n)}\right) := \left(\pi_1^{(n),\beta}, \pi_2^{(n),\beta}\right)$ when $\beta = 1$.

By the construction above and Bellman equations in the dynamic programming in the finite horizon game (see [5, 6] or [15]), it follows that $\pi^{(n),\beta}$ is a Nash equilibrium in the n - step β-discounted game.

Main Theorem 7. *For each $n \in N$ and $i = 1, 2$ hold*

$$\lim_{\beta \to 1} f_{i,n}^{\beta}(\cdot) = f_{i,n}^{\star}(\cdot), \tag{6}$$

and

$$\lim_{\beta \to 1} v_{i,n}^{\beta}(\cdot) = v_{i,n}^{\star}(\cdot). \tag{7}$$

Both convergences are uniform on S (i.e. in sup - norm on S).

Proof. Clearly, the hypothesis is true for $n = 1$. Suppose that for some $n \in N$ the hipothesis is satisfied i.e. if $\beta \to 1$ then hold

$$f_{i,n}^{\beta}(\cdot) \to f_{i,n}^{\star}(\cdot)$$

and

$$v_{i,n}^{\beta}(\cdot) \to v_{i,n}^{\star}(\cdot)$$

uniformely on S. By Bellman equations (see [5, 15] or [15]) and conditions **C2**, we conclude that $n + 1$ step game is on the form $G(s, \eta^{\beta})$ with

$$\eta_{i,l}^{\beta}(s) := \beta \int_S v_{i,n}^{\beta} \lambda_l(ds'|s). \tag{8}$$

Let $\eta_{i,l}(\cdot) := \eta_{i,l}^{1}(\cdot)$. By induction hipothesis we obtain $\sup_{s \in S} \left| v_{i,n}^{\beta}(s) - v_{i,n}^{\star}(s) \right| \to 0$, as

$n \to \infty$. Hence

$$\begin{aligned}\sup_{s\in S}\left|\eta^{\beta}_{i,l}(s)-\eta_{i,l}(s)\right| &\leq \sup_{s\in S}\left\{\int_S \left|v^{\beta}_{i,n}(s')-v^{\star}_{i,n}(s')\right| \lambda_l(ds'|s)\right\}\\ &+ (1-\beta)\sup_{s\in S}\left\{\int_S v^{\star}_{i,n}(s')\lambda_l(ds'|s)\right\} \qquad (9)\\ &\leq \sup_{s\in S}\left|v^{\beta}_{i,n}(s)-v^{\star}_{i,n}(s)\right| + (1-\beta)n\,||u_i||_{\infty} \to 0 \quad \text{as } \beta\to 1.\end{aligned}$$

The thesis for $n+1$ follows directly from (10) and Lemma 4. Hence uniform convergences in (6) and (7) hold. ■

Remark. If we additionaly assumed that the capacity functions $a_i(\cdot)$ are Lipshitz - continous with a constant 1 and that no measures $\lambda_l(\cdot|s)$ depends on s (i.e. $\lambda_l(\cdot|s) = \lambda_l(\cdot)$ for each l), then the transition probability would be a special case of that from Amir (1996). Then we immediately would obtain that the Nash equilibria are Lipshitz continous with a constant 1, and uniform continouity in Main Theorem 7 and further in Main Theorem 15 would be satisfied immediately.

Lemma 8. For arbitrary $n \in N$ let

$$\psi_i^{(n),\beta} := (\phi_i^{(n),\beta}, \phi_i^{(n-1),\beta}, ..., \phi_i^{(1),\beta})$$

be a certain collection of Markov strategies for player i depending on $\beta \in (0,1]$. Moreover, assume that there exists a limit

$$\psi_i^{(n)} := \lim_{\beta\to 1}\psi_i^{(n),\beta}. \qquad (10)$$

If the convergence in (10) is uniform, then

$$\lim_{\beta\to 1}\left(\sup_{s\in S}\left|\gamma^{\beta}_{i,n}(\psi_1^{(n),\beta},\psi_2^{(n),\beta})(s)-\gamma^{\beta}_{i,n}(\psi_1^{(n)},\psi_2^{(n)})(s)\right|\right) = 0. \qquad (11)$$

Remark. Since Markov strategy for player i in n - step game can be treaten as n - element vector from the space F_i^n, uniform convergence of Markov strategy means uniform convergence of each component.

Proof. Clearly for $n = 1$ the hypothesis is true. Suppose that (11) holds for some n, and this convergence is uniform. Note that by Bellman equations for finite horizon game (see [5, 6] or [15]) we have

$$\left|\gamma^{\beta}_{i,n+1}\left(\psi_1^{(n+1),\beta},\psi_2^{(n+1),\beta}\right)(s)-\gamma^{\beta}_{i,n+1}\left(\psi_1^{(n+1)},\psi_2^{(n+1)}\right)(s)\right| \leq$$

$$\left|u_i(\phi_i^{(n+1),\beta}(s))-u_i(\phi_i^{(n+1)}(s))\right| + \beta\Delta_\beta(s), \qquad (12)$$

where

$$\Delta_\beta(s) := \left|\omega_1^\beta(s) - \omega_2^\beta(s)\right|,$$

$$\omega_1^\beta(s) = \int_S \gamma_{i,n}^\beta \left(\psi_1^{(n),\beta}, \psi_2^{(n),\beta}\right)(s')q(ds'|s, \phi_1^{(n+1),\beta}(s), \phi_2^{(n+1),\beta}(s))$$

and

$$\omega_2^\beta(s) = \int_S \gamma_{i,n}^\beta \left(\psi_1^{(n)}, \psi_2^{(n)}\right)(s')q(ds'|s, \phi_1^{(n+1)}(s), \phi_2^{(n+1)}(s)).$$

By uniform convergence in (10) and uniform continuity of the function u_i we know that the first part of (12) uniformely converges to 0. It is sufficient to prove that $\left|\left|\Delta^\beta(\cdot)\right|\right|_\infty$ converge do 0, when $\beta \to 1$. Note that by condition **C2**, we obtain

$$\omega_1^\beta(s) = \sum_{l=1}^L \left(\int_S \gamma_{i,n}^\beta \left(\psi_1^{(n),\beta}, \psi_2^{(n),\beta}\right)(s')\lambda_l(ds'|s)g_l\left(s - \phi_1^{(n+1),\beta}(s) - \phi_2^{(n+1),\beta}(s))\right)\right),$$

and

$$\omega_2^\beta(s) = \sum_{l=1}^L \left(\int_S \gamma_{i,n}^\beta \left(\psi_1^{(n)}, \psi_2^{(n)}\right)(s')\lambda_l(ds'|s)g_l\left(s - \phi_1^{(n+1)}(s) - \phi_2^{(n+1)}(s))\right)\right)$$

Hence we have

$$\begin{aligned}
\sup_{s\in S}\left|\Delta^\beta(s)\right| &\leq \sup_{s\in S}\left|\gamma_{i,n}^\beta\left(\psi_1^{(n),\beta}, \psi_2^{(n),\beta}\right)(s) - \gamma_{i,n}^\beta\left(\psi_1^{(n)}, \psi_2^{(n)}\right)(s)\right| \\
&+ \sup_{s\in S}\left|\sum_{l=1}^L \int_S \gamma_{i,n}^\beta\left(\psi_1^{(n)}, \psi_2^{(n)}\right)(s')\lambda_l(ds'|s)\left|\tilde{g}_l^\beta(s) - \tilde{g}_l(s)\right|\right| \\
&\leq \sup_{s\in S}\left|\gamma_{i,n}^\beta\left(\psi_1^{(n),\beta}, \psi_2^{(n),\beta}\right)(s) - \gamma_{i,n}^\beta\left(\psi_1^{(n)}, \psi_2^{(n)}\right)(s)\right| \\
&+ n\left|\left|u_i\right|\right|_\infty \sum_{l=1}^L \sup_{s\in S}\left|\tilde{g}_l^\beta(s) - \tilde{g}_l(s)\right|
\end{aligned}$$

with

$$\tilde{g}_l^\beta(s) := g_l\left(s - \phi_1^{(n+1),\beta}(s) - \phi_2^{(n+1),\beta}(s))\right)$$

and

$$\tilde{g}_l(s) := g_l\left(s - \phi_1^{(n+1)}(s) - \phi_2^{(n+1)}(s))\right).$$

Clearly, by induction hypothesis it is sufficient to show that $\tilde{g}_l^\beta(\cdot) \to \tilde{g}_l(\cdot)$ uniformly on S. But it is also clear, because of uniform continouity of g_l and uniform convergence of ϕ_i^β. ■

Denote $\mathcal{M}_i^n$ as a set of all Markow strategies for player i in n step game.

Lemma 9. Let

$$\mathcal{T}^{\beta}_{i,n} := \sup_{\psi^{(n)}\in\mathcal{M}^n_i}\sup_{s\in S}\left|\gamma^{\beta}_{i,n}(\pi^{(n,\beta)}_{-i},\psi^{(n)})(s)-\gamma^{\beta}_{i,n}(\pi^{(n)}_{-i},\psi^{(n)}))(s)\right|.$$

Then we have

$$\lim_{\beta\to 1}\mathcal{T}^{\beta}_{i,n}=0.$$

Proof. Denote $\psi^{(n)} := (\phi^{(n+1)},\phi^{(n)},...,\phi^{(1)})$. We prove this theorem by induction. Clearly this hipothesis is satisfied for $n=1$. Suppose that the thesis of the theorem is satisfied for some $n\in N$. We have

$$\begin{aligned}
\mathcal{T}^{\beta}_{i,n+1} &= \sup_{\psi^{(n)}\in\mathcal{M}^n_i}\sup_{s\in S}\left|\gamma^{\beta}_{i,n+1}(\pi^{(n+1,\beta)}_{-i},\psi^{(n+1)})(s)-\gamma^{\beta}_{i,n+1}(\pi^{(n+1)}_{-i},\psi^{(n+1)}))(s)\right| \\
&= \beta\sup_{\psi^{(n)}\in\mathcal{M}^n_i}\sup_{s\in S}\left|\sum_{l=1}^{L}\int_S\gamma^{\beta}_{i,n}(\pi^{(n,\beta)}_{-i},\psi^{(n)})(s')\lambda_l(ds'|s)g_l\left(s-f^{\beta}_{i,n+1}(s)-\phi^{(n+1)}(s)\right)\right. \\
&\quad\left.-\sum_{l=1}^{L}\int_S\gamma^{\beta}_{i,n}(\pi^{(n)}_{-i},\psi^{(n)})(s')\lambda_l(ds'|s)g_l\left(s-f^{\star}_{i,n+1}(s)-\phi^{(n+1)}(s)\right)\right| \\
&\le \mathcal{T}^{\beta}_{i,n} \\
&+ n\,||u||_{\infty}\sum_{l=1}^{L}\sup_{\phi_i\in F_i}\sup_{s\in S}\left|g_l\left(s-f^{\beta}_{i,n+1}(s)-\phi_i(s)\right)-g_l\left(s-f^{\star}_{i,n+1}(s)-\phi_i(s)\right)\right|
\end{aligned}$$

By indyction hypothesis $\mathcal{T}^{\beta}_{i,n}\to 0$ when $\beta\to 1$. By Main Theorem 7 and uniform continouity of $g_l(\cdot)$, the second term of the right side of the inequality above tends to 0 when $\beta\to 1$.

■

Main Theorem 10. *For arbitrary ε there exist a constant β_0 such that if $\beta>\beta_0$, the profile $\left(\pi^{(n)}_1,\pi^{(n)}_2\right)$ is ε- equilibrium in the β - discounted n - step game.*

Proof. Fix $n\in N$. From Main Theorem 7 it follows that, for each $n\in N$ $\pi^{(n),\beta}$ is uniformly convergent to $\pi^{(n)}$ (when $\beta\to 1$). Let $j\neq i$ and $(i,j=1,2)$. Denote σ_i as an arbitrary Markov strategy for player i. Let ϵ be also arbitrary. From Main Theorem 7 and Lemma 8 we conclude the existence of β_1, such that for $\beta>\beta_1$ holds

$$\gamma^{\beta}_{i,n}\left(\pi^{(n)}\right)(s)\ge\gamma^{\beta}_{i,n}\left(\pi^{(n),\beta}\right)(s)-\frac{\epsilon}{2}. \tag{13}$$

Since $\pi^{(n),\beta}$ is the Nash equilibrium in n step game we obtain

$$\gamma^{\beta}_{i,n}\left(\pi^{(n),\beta}\right)(s)\ge\gamma^{\beta}_{i,n}\left(\pi^{(n),\beta}_{-i},\sigma_i\right)(s). \tag{14}$$

By Lemma 9 we conclude the existing a constant $\beta_2>\beta_1$ such that for each $\beta>\beta_2$ we have

$$\gamma^{\beta}_{i,n}\left(\pi^{(n),\beta}_{-i},\sigma_i\right)(s)\ge\gamma^{\beta}_{i,n}\left(\pi^{(n)}_{-i},\sigma_i\right)(s)-\frac{\epsilon}{2}. \tag{15}$$

Combining (13), (14) and (15) we obtain

$$\gamma_{i,n}^{\beta}\left(\pi^{(n)}\right)(s) \geq \gamma_{i,n}^{\beta}\left(\pi_{-i}^{(n)}, \sigma_i\right)(s) - \epsilon.$$

■

Asymptotic Nash equilibria in infinite step model

For player i let $\mathcal{S}_i$ be a set of all stationary strategies. Let $\mathcal{S} := \mathcal{S}_1 \times \mathcal{S}_2$. Note that there is one to one correspondence between $\mathcal{S}_i$ and the set of the functions F_i. Hence if the stationary multi - strategy ψ we can describe as $\psi = (\phi, \phi, ...)$ for some borel function $\phi \in F$, we denote $\gamma_i^{\beta}(\phi)(s) := \gamma_i^{\beta}(\psi)(s)$. Similarly we can define $\gamma_{i,n}^{\beta}(\phi)(s) := \gamma_{i,n}^{\beta}(\psi)(s)$, when the profile ψ is used in n step model.

Lemma 11. Let $\psi := (\psi_1, \psi_2) \in \mathcal{S}$ be arbitrary. Let $\psi = (\phi, \phi, ...)$. Then

$$\sup_{\phi\in\mathcal{S}} \sup_{\beta\in(0,1]} \sup_{s\in S} \left|\gamma_{i,n}^{\beta}(\phi)(s) - \gamma_i^{\beta}(\phi)(s)\right| \to 0, \quad \text{when} \quad n \to \infty. \tag{16}$$

Moreover, for each stationary strategy ϕ holds

$$\gamma_i^{\beta}(\phi)(s) \leq ||u||_{\infty} \frac{C}{1-C} \tag{17}$$

with

$$C = \int_S \left(\sum_{l=1}^{L} g_l(s')\right) \mu(ds').$$

Proof. Let $\phi \in F$. Define $(s_0, s_1, ..., s_t, ...)$ as a sequence of the states generated by the stationary strategy profile ϕ. Let $s_0 = s$. It is easy to see that

$$\gamma_i^{\beta}(\phi)(s) = \sum_{t=1}^{\infty} E_s^{\phi}\left(u_i\left(\phi_i(s_t)\right)\beta^{t-1}\right). \tag{18}$$

From the assumption **C2** for $t > 1$ holds

$$\begin{aligned}
z_t^{\beta,\phi}(s) &:= E_s^{\phi}\left(u_i(\phi_i(s_t))\beta^{t-1}\right) \\
&= \sum_{l=1}^{L} \beta^{t-1} E_s^{\phi}\left(\int_S u_i(\phi_i(s_{t-1}))\lambda_l(ds'|s_{t-1}) g_l\left(s_{t-1} - \phi_1(s_{t-1}) - \phi_2(s_{t-1})\right)\right) \\
&\leq ||u||_{\infty} \sum_{l=1}^{L} E_s^{\phi}\left(g_l(s_{t-1})\right) = ||u||_{\infty}\, h_{t-1}^{\beta}(s),
\end{aligned} \tag{19}$$

where

$$h_t^{\beta}(s) := E_s^{\phi}\left(\sum_{l=1}^{L} g_l(s_t)\right)$$

for $t > 1$ and

$$h_0^\beta(s) := \sum_{l=1}^{L} g_l(s).$$

To prove this lemma we just need to show that the series $z_t^{\beta,\phi}(s)$ are uniformely convergent in (β, ϕ, s). Clearly $0 \le h_0^\beta(\cdot) \le 1$. Let $t > 0$ From assumption **C2** we have

$$\begin{aligned} h_t^\beta(s) &= \sum_{l=1}^{L} E_s^\phi \left(\int_S \sum_{l=1}^{L} g_l(s')\lambda_l(ds'|s_{t-1}) g_l\left(s_{t-1} - \phi_1(s_{t-1}) - \phi_2(s_{t-1})\right)\right) \\ &\le \int_S \left(\sum_{l=1}^{L} g_l(s')\right) \mu(ds') E_s^\psi \left(\sum_{l=1}^{L} g_l(s_{t-1})\right) \\ &= C h_{t-1}^\beta(s). \end{aligned}$$

Hence we have

$$h_t^\beta(s) \le C * h_{t-1}^\beta(s) \le ... \le C^{t-1} h_1^\beta(s) \le C^t, \tag{20}$$

By Assumption **C2** $0 < C < 1$. Obviously C is a constant independent on s, ϕ and β. Combining (19) and (20) we obtain

$$z_t^{\beta,\phi}(s) \le ||u||_\infty h_{t-1}^\beta(s) \le ||u||_\infty C^{t-1}. \tag{21}$$

Hence by Weierstrass criterion the series $z_t^{\beta,\phi}(s)$ are uniformly convergent in (s, β, ϕ), which complete the proof that the condition in (16) is satisfied. Condition (17) follows from (18), (19), (20) and (21). ■

Corollary 12. *By Lemma 11 is easy to see that $\gamma_i(\pi)(s) < \infty$ for arbitrary profile π. Moreover, repeating the reasoning in the proof of Lemma 11 we can obtain (17) for arbitrary profile π.*

Lemma 13. For arbitrary $\beta \in (0, 1]$ let $\left(\psi_1^\beta, \psi_2^\beta\right) \in \mathcal{S}$ be stationary multi - strategy. If for player i holds

$$\lim_{\beta\to 1} \sup_{s\in S} \left|\psi_i^\beta(s) - \psi_i(s)\right| = 0, \tag{22}$$

then

$$\lim_{\beta\to 1} \sup_{s\in S} \left|\gamma_i^\beta\left(\psi_1^\beta, \psi_2^\beta\right)(s) - \gamma_i^\beta\left(\psi_1, \psi_2\right)(s)\right| = 0.$$

Proof. Let $\epsilon > 0$ be arbitrary. By Lemma 11 there exists n_0 such that for $n > n_0$ we have

$$\left|\gamma_i^\beta\left(\psi_1^\beta, \psi_2^\beta\right)(s) - \gamma_i^\beta\left(\psi_1, \psi_2\right)(s)\right| \le \left|\gamma_{i,n}^\beta\left(\psi_1^\beta, \psi_2^\beta\right)(s) - \gamma_{i,n}^\beta\left(\psi_1, \psi_2\right)(s)\right| + \epsilon. \tag{23}$$

Fix $n > n_0$. By (22) and Lemma 8 if we take a limit with $\beta \to 1$, we obtain

$$\lim_{\beta \to 1} \left(\sup_{s \in S} \left| \gamma_{i,n}^{\beta} \left(\psi_1^{\beta}, \psi_2^{\beta} \right)(s) - \gamma_{i,n}^{\beta} \left(\psi_1, \psi_2 \right)(s) \right| \right) = 0.$$

Hence and by (23) we have

$$\limsup_{\beta \to 1} \left(\sup_{s \in S} \left| \gamma_i^{\beta} \left(\psi_1^{\beta}, \psi_2^{\beta} \right)(s) - \gamma_i^{\beta} \left(\psi_1, \psi_2 \right)(s) \right| \right) \leq \epsilon.$$

Since ϵ is arbitrary this proof is complete. ■

By the main results in [14] we can conclude uniform convergence of equilibria payoffs in the finite horizon β - discounted game to the stationary equilibrium payoff in the infinite horizon β - discounted game, when the horizon tends to infinity. Moreover, we also know pointwise convergence of Nash equilibria adequates to these equilibria payoffs. This theorem bellow shows that convergence of Nash euilibria is uniform as well.

Theorem 14. For arbitrary $i = 1, 2$, and $\beta \in (0, 1)$ there exist a limits

$$f_i^{\beta}(s) := \lim_{n \to \infty} f_{i,n}^{\beta}(s), \tag{24}$$

and

$$v_i^{\beta}(s) := \lim_{n \to \infty} v_{i,n}^{\beta}(s), \tag{25}$$

Moreover, these convergences above are uniform on S (i.e. in sup-norm).
Further, the stationary multi-strategy $\pi^{\beta} := (\pi_1^{\beta}, \pi_2^{\beta})$, $\pi_i^{\beta} := \left(f_i^{\beta}, f_i^{\beta}, \dots \right)$ is a Nash equilibrium in β - discounted infinie horizon game and v_i^{β} is equilibrium payoff adequate to $f^{\beta} := \left(f_1^{\beta}, f_2^{\beta} \right)$.

Proof. Fix β. In [14] it is prooven uniform covergence $\lim_{n \to \infty} v_{i,n}^{\beta}(\cdot) = v_i^{\beta}(\cdot)$. We also know that pointwise convergence in (24) holds, and $f^{\beta} := \left(f_1^{\beta}, f_2^{\beta} \right)$ is Nash equilibrium and $v_i^{\beta}(s) = \gamma_i \left(f^{\beta} \right)(s)$. We just need to show that the convergence in (24) is uniform.

From Bellman equations for finite horizon game ([5, 6] and [15]) we conclude that $n + 1$ horizon β - discounted game can be described as $G(s, \eta^{\beta,n})$ with

$$\eta_{i,l}^{\beta,n}(s) := \beta \sum_{l=1}^{L} \int_S v_{i,n}^{\beta}(s') \lambda_l(ds'|s).$$

and infinite horizon game reduce to $G(s,\eta^\beta)$ with

$$\eta^\beta_{i,l}(s) := \beta\sum_{l=1}^{L}\int_S v_i^\beta(s')\lambda_l(ds'|s).$$

Note that

$$\begin{aligned}\sup_{s\in S}\left|\eta^{\beta,n}_{i,l}(s)-\eta^\beta_{i,l}(s)\right| &\leq \sum_{l=1}^{L}\int_S\left|v^\beta_{i,n}(s')-v^\beta_i(s')\right|\lambda_l(ds'|s)\\ &\leq \sup_{s\in S}\left|v^\beta_{i,n}(s')-v^\beta_i(s')\right|\to 0\quad\text{as } \beta\to 1.\end{aligned}$$

Hence and by Lemma 4 we obtain uniform convergence in (24). ■

Remark. Problem of convergence of Nash equilibria and adequates equilibria payoffs with horizon tending to ∞ is also solved for symmetric m - person games in [2]. Hence, repeating the reasoning in the proof of Theorem 14 we would also obtain the same results for m - person symmetric game.

Main Theorem 15. *For $i=1,2$ hold*

$$\lim_{\beta\to 1} f_i^\beta(\cdot) = f_i^\star(\cdot) \tag{26}$$

and

$$\lim_{\beta\to 1} v_i^\beta(\cdot) = v_i^\star(\cdot). \tag{27}$$

and these convergences above are uniform on S (i.e. in sup-norm on S). Moreover, $f^\star := (f_1^\star, f_2^\star)$ is a Nash equilibrium in the undiscounted game and $v_i^\star$ is an equilibrium payoff adequates to $f^\star$.

Proof. First we show that the function $\beta\to v^\beta_{i,n}(s)$ is nondecreasing. For $n=1$ this hipothesis is clear. Assume that for some $n\in N$ the function $\beta\to v^\beta_{i,n}(s)$ is nondecreasing. Consider the game $G(s,\eta^{\beta,n})$ with

$$\eta^{\beta,n}_{i,l}(s) := \beta\int_S v^\beta_{i,n}(s')\lambda_l(ds'|s).$$

Clearly $\eta^{\beta,n}$ is nondecreasing in β. Note that by Bellman equations for finite horizon game ([5, 6] or [15]) $\left(f^\beta_{1,n+1}(s), f^\beta_{2,n+1}(s)\right) = NE\,G(s,\eta^{\beta,n})$. Hence and by Lemma 6 we immediately obtain that $v^\beta_{i,n+1}$ is nondecreasing in β as well.

Since by Theorem 14 $v_i^\beta(s) = \lim_{n\to\infty} v^\beta_{i,n}(s)$, hence $v_i^\beta(s)$ is nondecreasing in β as a limit of nondecreasing functions. By Lemma 11 $v_i^\beta(s)\leq ||u||\,\frac{C}{1-C}$. Hence there exists a limit, say $v_i^\star(s) := \lim_{\beta\to 1} v_i^\beta(s)$. We show that $v_i^\star(\cdot)$ is a payoff equilibrium in undiscounted infinite horizon game. By Bellman equations ([5, 6] or [15]) for $j \neq i$ we have

$$\begin{aligned} v_i^\beta(s) &= u_i(f_i^\beta(s)) + \beta \sum_{l=1}^{L} \int_S v_i^\beta(s')\lambda_l(ds'|s) g_l\left(s - f_1^\beta(s) - f_2^\beta(s)\right) \\ &\geq u_i(x_i) + \beta \sum_{l=1}^{L} \int_S v_i^\beta(s')\lambda_l(ds'|s) g_l\left(s - f_j^\beta(s) - x_i\right). \end{aligned} \tag{28}$$

Fix $s \in S$. Suppose that $x^* := (x_1^*, x_2^*)$ is a cumulation point of $f^\beta(s) := \left(f_1^\beta(s), f_2^\beta(s)\right)$. Suppose that $\beta_k \to 1$ is such sequence for which $\lim_{k\to 1} f^{\beta_k} = x^*$. Let $(x_1, x_2) \in X(s)$ be arbitrary. Now, if we put $\beta := \beta_k$ in (28), and take a limit with $k \to \infty$, then we obtain

$$\begin{aligned} v_i^\star(s) &= u_i(x_i^*(s)) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s) g_l\left(s - x_1^*(s) - x_2^*(s)\right) \\ &\geq u_i(x_i) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s) g_l\left(s - x_j^*(s) - x_i\right). \end{aligned} \tag{29}$$

Hence we obtain that $(x_1^*(s), x_2^*(s)) = NE\Gamma(s, v_1^\star, v_2^\star, 1)$. Proposition 1 in [3] guarantes that this definition is well. Hence for each $s \in S$, there exists a limit, say $f_i^\star(s) := \lim_{\beta\to 1} f_i^\beta(s)$. By Corrolary 12 and then by Bellman equations (see [5, 6] or [15]) we conclude that $f^\star$ is a Nash equilibrium in undiscounted infinite horizon game, and $v_i^\star(s) = \gamma_i\left(f^\star\right)(s)$.

Now we show that $f_i^\star(\cdot)$ is continous. Note that by Assumption **C2** the function $s \to \int_S v_i^\star(s')\lambda_l(ds'|s)$ is continous. Let $s_n \to s_0$. By (29) we have

$$\begin{aligned} & u_i(f_i^\star(s_n)) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s_n) g_l\left(s_n - f_1^\star(s_n) - f_2^\star(s_n)\right) \\ & \quad \geq u_i(x_i) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s_n) g_l\left(s_n - f_j^\star(s_n) - x_i\right). \end{aligned} \tag{30}$$

Let $x^0 := (x_1^0, x_2^0)$ be a cumulation point of the sequence $f^\star(s_n) := (f_1^\star(s_n), f_2^\star(s_n))$. Let s_k be a subsequence of the sequence s_n (again we denote s_k instead s_{n_k} for simplify the notation), such that $x^0 = \lim_{k\to\infty} f^\star(s_k)$. Hence, if we put $s_n := s_k$ in (30) and take a limit with $k \to \infty$ we have

$$u_i(x_i^0) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s_0) g_l\left(s_0 - x_1^0 - x_2^0\right)$$
$$\geq u_i(x_i) + \sum_{l=1}^{L} \int_S v_i^\star(s')\lambda_l(ds'|s_0) g_l\left(s_0 - x_j^0(s) - x_i\right).$$

It means that $(x_1^0, x_2^0) = NE\Gamma(s_0, v_1^\star, v_2^\star, 1)$. Hence and by uniquness of Nash equilibrium in $\Gamma(s_0, v_1^\star, v_2^\star, 1)$ we obtain that the cumulation point of the sequence $(f_1^\star(s_n), f_2^\star(s_n))$ is unique and is equal $(f_1^\star(s_0), f_2^\star(s_0))$. Hence we obtain continouity of $f_i^\star(\cdot)$ and hence also $v_i^\star(\cdot)$. Hence, because $\beta \to v_i^\beta(s)$ is monotone and $v_i^\star(\cdot)$ is continous as well, by Dini Theorem it follows that $v_i^\beta(\cdot) \to v_i^\star(\cdot)$ as $\beta \to 1$ uniformely which ends proof of the uniform convergence in (27) . Consider a game $G(s, \eta^\beta)$ with

$$\eta_{i,l}^\beta := \beta \int_S v_i^\beta(s')\lambda_l(ds'|s),$$

and game $G(s, \eta)$ with

$$\eta_{i,l} := \int_S v_i^\star(s')\lambda_l(ds'|s).$$

Clearly by Bellman equations (see [5, 6] or [15]) $f^\beta(s) = NE\, G(s, \eta^\beta)$ and $f^\star(s) = NE\, G(s, \eta)$. We show that

$$\eta_{i,l}^\beta(\cdot) \to \eta_{i,l}(\cdot)\ (\beta \to 1).$$

uniformly on S. By Lemma 11 we know that $v_i^\star(\cdot) \leq \frac{C}{1-C}\,||u_i||_\infty$. Hence we have

$$\begin{aligned}
\sup_{s\in S}\left|\eta_{i,l}^\beta(s) - \eta_{i,l}(s)\right| \quad &\leq \quad \sup_{s\in S}\left\{\int_S \left|v_i^\beta(s') - v_i^\star(s')\right| \lambda_l(ds'|s)\right\} \\
&+ \quad (1-\beta)\sup_{s\in S}\left\{\int_S v_i^\star(s')\lambda_l(ds'|s)\right\} \\
&\leq \quad \sup_{s\in S}\left|v_i^\beta(s) - v_i^\star(s)\right| + (1-\beta)\frac{C}{1-C}\,||u_i||_\infty \to 0 \quad \text{as } \beta \to 1.
\end{aligned}$$

Hence and by Lemma 4 we obtain that $f_i^\beta(\cdot) \to f_i^\star(\cdot)$ uniformely on S, hence the part (26) is proven. ■

For $i = 1, 2$ define $\pi_i^\star := (f_i^\star, f_i^\star, \ldots)$.

Main Theorem 16. *For arbitrary ε and there exist β_0 such that if $\beta > \beta_0$, stationary multi - strategy $(\pi_1^\star, \pi_2^\star)$ is ε- equilibrium in β - discounted infinite horizon game.*

Proof. Fix arbitrary $\varepsilon > 0$ and $\sigma_i \in \mathcal{S}_\rangle$. By Lemma 13 and Main Theorem 15 we conclude existence of β_1 such that for $\beta > \beta_1$ we have

$$\gamma_i^\beta(f^\star)(s) \quad \geq \quad \gamma_i^\beta(f^\beta)(s) - \frac{\epsilon}{3}. \tag{31}$$

Since f^β is a Nash equilibrium in the β discounted game, hence we obtain

$$\gamma_i^\beta(f^\beta)(s) \quad \geq \quad \gamma_i^\beta(f_{-i}^\beta, \sigma_i)(s), \tag{32}$$

By Lemma 11 there exists n_0 such that for all $n > n_0$ holds

$$\gamma_i^\beta(f_{-i}^\beta, \sigma_i)(s) \quad \geq \quad \gamma_{i,n}^\beta(f_{-i}^\beta, \sigma_i)(s) - \frac{\epsilon}{3}. \tag{33}$$

From Lemma 8 and Main Theorem 15 there exists $\beta_2 > \beta_1$ such that for $\beta > \beta_2$ we have

$$\gamma_{i,n}^\beta(f_{-i}^\beta, \sigma_i)(s) \quad \geq \quad \gamma_{i,n}^\beta(f_{-i}^*, \sigma_i)(s) - \frac{\epsilon}{3}. \tag{34}$$

Combining (31), (32), (33), (34) we obtain

$$\gamma_i^\beta(f^\star)(s) \quad \geq \quad \gamma_{i,n}^\beta(f_{-i}^\star, \sigma_i)(s) - \epsilon, \tag{35}$$

for arbitrary $\beta > \beta_2$ and $n > n_0$. By Lemma 9 if we take a limit with $n \to \infty$ in (35) we obtain

$$\gamma_i^\beta(f^\star)(s) \quad \geq \quad \gamma_i^\beta(f_{-i}^\star, \sigma_i)(s) - \epsilon. \tag{36}$$

To see that (36) is satisfied when σ_i is arbitrary strategy, we first note that for each β, there exists a stationary optimal policy on $f_i^\star$ (say σ_i^β) in the β - discounted game. If we put $\sigma_i := \sigma_i^\beta$ in (36) we immediately obtain (36) with arbitrary σ_i. To complete the proof we can take $\beta_0 := \beta_2$. ∎

Example 17. Let us consider two person game in which $S = [0, 1]$, $a_1(s) = a_2(s) = s/2$, $u_1(s) = u_2(s) = \sqrt{s}$, and the transition probability is of the form

$$q\,(\cdot\,|s, x\,) = \sqrt{s - x_1 - x_2}\lambda(\cdot) + \left(1 - \sqrt{s - x_1 - x_2}\right)\delta_0(\cdot),$$

where λ a uniform distribution on $[0, 1]$. By [2] and Theorem 2 [3] there exists an unique Nash equilibrium in the β discounted finite horizon game. We see that

$$f_{1,1}^\beta(s) = f_{2,1}^\beta(s) = s/2,$$

and

$$v_{1,1}^\beta(s) = v_{2,1}^\beta(s) = \sqrt{s/2}.$$

For $i = 1, 2$ and $n \geq 1$ we obtain $\pi_i^{(n),\beta} = f_{i,n}^\beta$ for

$$f_{i,n}^\beta(s) = \frac{s}{2 + c_n^2},$$

and equilibria functions

$$v_{i,n}^{\beta}(s) = \frac{1 + \beta c_n^2}{\sqrt{c_n^2 + 2}} \sqrt{s},$$

when c_n is define in the following way: $c_1 = 0$ and for $n \geq 1$

$$c_{n+1} = \frac{2}{3} \frac{1 + \beta c_n^2}{\sqrt{c_n^2 + 2}}.$$

If we take a limit $n \to \infty$, we obtain

$$c^{\beta} = \sqrt{\frac{\sqrt{(18 - 8\beta)^2 + 16(9 + 4\beta^2)} - (18 - 8\beta)}{2(18 - 8\beta)}}.$$

From Theorem 14 we immediately conclude that

$$f_i^{\beta}(s) = \frac{s}{2 + \left(c^{\beta}\right)^2},$$

and $\pi^{\beta} := \left(f_1^{\beta}, f_2^{\beta}\right)$ is a stationary Nash equilibrium in the β - discounted infinite horizon game and

$$v_i^{\beta}(s) = \frac{1 + \beta \left(c^{\beta}\right)^2}{\sqrt{\left(c^{\beta}\right)^2 + 2}} \sqrt{s}$$

is the equilibrium function. By Main Theorem 15, taking a limit $\beta \to 1$ we obtain

$$f_i^{\star}(s) = \frac{20}{30 + \sqrt{192}} s,$$

and

$$v_i^{\star}(s) = \frac{\sqrt{192} + 10}{\sqrt{20\sqrt{192} + 600}} \sqrt{s}.$$

By Main Theorem 16 it follows that the stationary strategy $\pi^{\star} = (f_1^{\star}, f_2^{\star})$ is a Nash equilibrium in the undiscounted stochastic game with limiting average criterion. Moreover, this strategy is ε equilibrium in β - discounted infinite horizon game for sufficiently large β.

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