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ON INCENTIVE COMPATIBLE DESIGNS OF FORECASTING CONTRACTS

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Abstract: In the paper the optimal design of forecasting contracts in principal-agent setting is investigated. It is assumed that the principal pays the agent (the forecaster) based on an announced forecast and an event that materializes next. Such a contract is called incentive compatible if the agent maximizes her payoff when she announces her true beliefs. This paper relaxes the assumption present in earlier works on this subject that agent's beliefs are deterministic by allowing them to be random (i.e. stemming from estimation). It is shown that for binary or nominal events the principal can learn only expected values of agent's predictions in an incentive compatible way independent of agent's signal space. Additionally it is proven that incentive compatible payment schemes give the agent a strictly positive incentive to improve the precision of her estimates.

Key words: contract design, forecasting, scoring rules

Introduction

The demand for reliable forecasts is omnipresent in private, business and academic applications. People watch weather forecasts, listen to GDP growth predictions given by central banks and are lively interested in expected rate of ozone layer depletion. The process of forecast preparation is often difficult and requires much effort and skill. Therefore it is not unusual that someone who demands a forecast relies on expert help. Proffessional approach to such *delegated* activity calls for probabilistic assessments of future events[1]. For example central bank economists are interested not only in point estimate of inflation, but also the distribution of the estimator. In everyday life people prefer to know the probability of rain next day — not only a simple

[1]Gneiting and Raftery (2007) give a review of literature in weather and macroeconomic domains focusing on probabilistic forecasting.

statement of the most probable weather state. An interesting practice of probabilistic forecasting is employed by Gartner Inc., an IT market monitoring company, who assigns probabilities to its predictions of future trends.

The widely spread practice of relying on external forecasts rises a question how to evaluate them. Such assessment can be simply used as a measure of forecaster performance. It can also be a part of a contract between the forecaster and the side that demands the forecast. Further the latter understanding will be taken in order to fix the terminology used. Following the standard conventions in economic literature, see for example Salanié (2005), we will call the forecaster an agent and the forecast demanding side a principal. We assume that the principal contracts the agent to produce a forecast. The payment scheme in this contract is based on the forecast given and the event that later occurs. Later, following the literature (Hendrickson and Buehler, 1971), agent's payoff function will be called a *scoring rule*. It is taken that neither the principal nor the agent can influence the probability distribution of the event. We will say that the scoring rule is *proper* if the agent maximizes her expected profit when she reveals her true beliefs about the probability of the forecasted event[2]. In economic literature, see Salanié (2005), proper scoring rules are said to implement an *incentive compatible* contract.

The problem of evaluation of forecasts has been studied since the work of Brier (1950), who considered a binary event prediction in weather forecasting applications and proposed one proper scoring rule for such a case. Later line of research focused on classification of proper scoring rules for binary (Schervish, 1989), nominal (Savage, 1971) and continuous (Gneiting and Raftery, 2007) random variables.

The implicit assumption that was made in the forecasting contracts literature was that the true expectations of the forecaster are certain. For example, in binary event case, that she has crisp beliefs about the probability of the event. McCarthy (1956) has considered an experimentation procedure on agent's side that would lead to a change in her beliefs but still the a priori and a posteriori beliefs of the agent were non-random. However, in real applications a far more common situation is that the true agent's expectations are represented as estimators. For example when predicting the share of votes on some candidate in elections a pool-making company gets its imprecise estimate by questioning a random sample of voters[3].

The objective of this research is to extend the existing results on proper scoring rules by adding the possibility that agent's beliefs are given as a random variable. A focus is put on the classical binary event case, as it allows for most clear presentation of the analysis. An extension of the results to nominal events is presented.

The paper is organized as follows. First a formal model of a binary event forecasting contract under non-random (classical) and random expectations assumption are presented and outline the standard results obtained in the literature for the former case are given. In the next section the properties of random beliefs model in binary case are investigated. Finally it is shown how the results can be extended to nominal variables. The paper is finished by concluding remarks.

[2]A formal definition of a proper scoring rule is given in section .

[3]Moreover, in practice such companies often quote the confidence of their predictions.

The binary event forecasting contract model

In this section we first introduce the standard model of binary forecasting contract and outline its properties. Next it is shown how it can be extended to agent's random beliefs.

Let $x \in \{0; 1\}$ be the event that is predicted. For example $x = 1$ can be associated with rain and $x = 0$ with no-rain forecast[4]. It is important to distinguish between agent's belief on $\Pr(x = 1)$, which will be denoted q and the value announced by her to the principal, further denoted p.

A scoring rule is a pair of mappings $S_x : [0; 1] \to \mathbf{R}$ taking announced probability p as an argument and returning agent's payoff. Notice that the mappings are indexed by the event x. The agent obtains payoff $S_1(p)$ when event $x = 1$ occurs and $S_0(p)$ when $x = 0$ happens. We assume that S_x can be set by the principal. Moreover we take that the agent wants to maximize her expected payoff, denoted $R(p)$, subject to her beliefs[5]:

$$R(p) = qS_1(p) + (1 - q)S_0(p) \to \max \tag{1}$$

Using this assumption we can define proper scoring rules.

Definition 1. A scoring rule S_x is proper if and only if

$$\forall q \in [0; 1] \forall [0; 1] \ni x \neq q : R(q) > R(x)$$

and S_x is bounded from above and real valued (except possibly that $S_1(0)$ and $S_0(1)$ can be equal to $-\infty$).

Under this condition Gneiting and Raftery (2007) give a convenient condition characterizing the class of proper scoring rules[6].

Theorem 2 (Gneiting and Raftery, 2007)**.** *Every proper scoring rule is of the form*

$$\begin{aligned} S_1(p) &= G(p) + (1 - p)G'(p) \\ S_0(p) &= G(p) - pG'(p) \end{aligned} \tag{2}$$

where $G : [0; 1] \to \mathbf{R}$ *is a bounded and strictly convex function and* $G'(p)$ *is a subgradient of* G *at point* p*, for all* $p \in [0; 1]$.

Notice that:

$$R(p) = q(G(p) + (1 - p)G'(p)) + (1 - q)(G(p) - pG'(p)) = G(p) + (q - p)G'(p)$$

so $G(q) = R(q)$.

[4]The rain/no rain probability prediction example was originally used by Brier (1950) in his pioneering research on scoring rules.

[5]This follows the standard economics assumption of agent's rationality following expectation maximization principle, see Mas-Collel et al. (1995).

[6]Gneiting and Raftery (2007) use the term *regular strictly* proper scoring rule.

Example 3. By putting $G(p) = -p(1-p)$ we get $S_1(p) = -(1-p)^2$ and $S_0(p) = -p^2$. It is a scoring rule originally proposed by Brier (1950). It is illustrated on Figure 1. Such scoring function always gives the agent negative payouts. However it can be seen from Theorem 2 that transforming $G(p)$ to $G(p) + \alpha$, where $\alpha \in \mathbf{R}$, keeps it convex while it does not affect the shape of $S_x(p)$. Therefore the principal can adjust the payouts to agent's reservation level if it exists[7]. Also notice that Brier score is a symmetric scoring function, that is $S_1(p) = S_0(1-p)$, so it does not depend on labeling of event realizations. However in general scoring rules do not have to be symmetric. Asymmetric scoring functions could be preferred by the principal when she assigns different values to $x = 1$ and $x = 0$.

If we take Shannon's entropy[8] $G(p) = p\ln(p)+(1-p)\ln(1-p)$ as a second example then $S_1(p) = \ln(p)$ and $S_0(p) = \ln(1-p)$. Note that in this case the payoff is $-\infty$ if the event is assigned probability 0 and it happens.

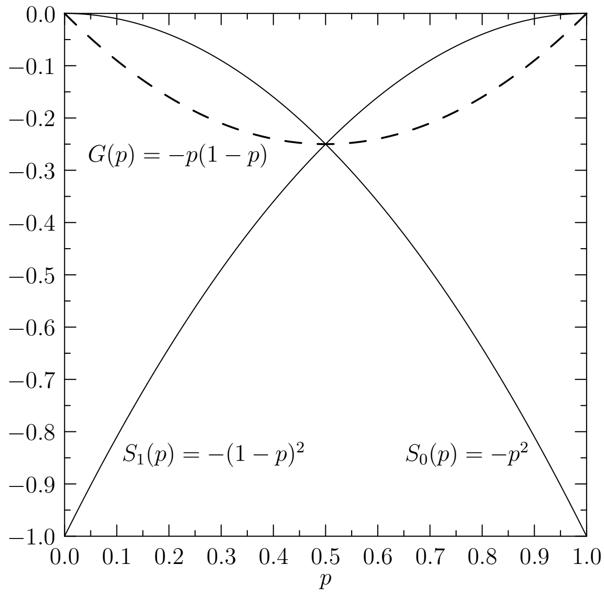

Figure 1. Brier scoring functions $S_x(p)$ and expected score $G(p)$.

Now assume that the agent does not know q exactly and Q is a random variable representing agent's uncertainty about true value of q. Denote cumulative distribution function of Q by $F : [0; 1] \to [0; 1]$.

Let us denote $\mathcal{S}$ the set of signals that can be sent by the agent. In the simplest case if $\mathcal{S} = [0; 1]$ the agent is asked to report a fraction, but in general the principal

[7]The reservation level is understood as minimal payout that the agent requires to receive.

[8]Shannon and Weaver (1949) first introduced it as a measure of amount of information contained in a message. In our case the message is what the agent tells the principal.

might want to have something else reported. For example she might ask the agent to reveal F. Based on this we define *extended scoring rules* as $S_x^E : \mathcal{S} \rightarrow \mathbf{R}$, that are natural generalization of scoring rules. Notice that in the case when $\mathcal{S} = [0;1]$ extended scoring rule reduces to scoring rule.

The calculation of agent's expected payoff takes into consideration the uncertainty of Q as follows:

$$\forall s \in \mathcal{S} : R^E(s) = \int_0^1 qS_1^E(s) + (1-q)S_0^E(s)dF(q) \tag{3}$$

In the next section the consequences of introducing uncertainty of q to agent's beliefs will be analyzed.

Uncertain expectation for binary event properties

Under uncertainty of q several natural questions arise. Firstly it will be investigated how the agent will behave when faced with classical proper scoring rules. Next it will be shown what kind of information the principal can extract from the agent by appropriately setting an extended scoring rule. Lastly a question if extended scoring rules can give an incentive to the agent to put extra effort to improve the precision of estimation of q will be answered.

Expectation revelation principle

In this subsection we will show that when the agent has uncertain beliefs about q the only information that the principal can obtain from her in an incentive compatible way is an expected value of Q — the phenomenon will be called *expectation revelation principle*. First it is shown that it is possible to construct a scoring rule under which the agent will reveal $\mathrm{E}(Q)$.

Theorem 4. *If $\mathcal{S} = [0;1]$ and $S_x^E(s)$ is a proper scoring rule then $R^E(s)$ is maximized for $s = E(q)$.*

Proof. Notice that:

$$R^E(s) = \int_0^1 qS_1^E(s) + (1-q)S_0^E(s)dF(q) = \mathrm{E}(Q)S_1^E(s) + (1-\mathrm{E}(Q)) - S_0^E(s)$$

so $R^E(s)$ equals $R(s)$ for $q = \mathrm{E}(Q)$. Therefore by Theorem 2 $R^E(s)$ is maximized for $s = \mathrm{E}(Q)$. ■

The above theorem shows that by using proper scoring rules the principal will learn the expected value of Q. Using this one can notice that all results in existing literature concerning binary forecasting contracts can be directly extended to uncertain estimation of q by assumption that the agent reveals her expected value of Q.

In general the principal might be interested in other statistics of the random variable Q than expected value. In general we can denote the statistics as a function of cumulative distribution function of Q: $\chi : [0;1]^2 \rightarrow \mathcal{S}$. This could be for example a median of the distribution — then $\chi(F) = F^{-1}(0.5)$ or the distribution function — then $\chi(F) = F$. Analogously to to the proper scoring rules we define χ-proper extended scoring rule.

Definition 5. An extended scoring rule S_x^E is χ-proper if and only if for all cumulative distribution functions F:

$$\forall \mathcal{S} \ni s \neq \chi(F) : R^E(\chi(F)) > R^E(s)$$

and S_x^E is bounded.

The following theorem shows that the principal is unable to retrieve any significantly different information from the agent than expected value of Q.

Theorem 6. *If S_x^E is χ-proper extended scoring rule then there exists such mapping $h(\cdot)$ that $\chi(F) = h(E(Q))$.*

Proof. Assume that the converse is true and such χ-proper extended scoring rule exists. Then there exist such random variables Q_1 and Q_2 with cumulative distribution functions F_1 and F_2 for which:

$$\mathrm{E}(Q_1) = \mathrm{E}(Q_2) \wedge \chi(F_1) \neq \chi(F_2).$$

The above condition reads that χ is not a function of expected value of random variable it transforms.

But then using equation 3 and definition of χ-proper extended scoring rule we get:

$$\begin{cases} \quad \mathrm{E}(Q_1)(S_1^E(\chi(F_1)) - S_0^E(\chi(F_1))) + S_0^E(\chi(F_1)) > \\ > \mathrm{E}(Q_1)(S_1^E(\chi(F_2)) - S_0^E(\chi(F_2))) + S_0^E(\chi(F_2)) \\ \quad \mathrm{E}(Q_2)(S_1^E(\chi(F_1)) - S_0^E(\chi(F_1))) + S_0^E(\chi(F_1)) < \\ < \mathrm{E}(Q_2)(S_1^E(\chi(F_2)) - S_0^E(\chi(F_2))) + S_0^E(\chi(F_2)) \end{cases}$$

However, by assumption $\mathrm{E}(Q_1) = \mathrm{E}(Q_2)$ so the above equations are contradicting. ■

The above result shows two important properties of the forecasting contract problem. Firstly - the only information the principal can get from the agent is expected value of Q under any assumption on the structure of agent's signal. Secondly - we know that $\mathrm{E}(Q)$ is reported under classical proper scoring rules. Therefore it is obsolete to consider extended scoring rules — all the information that can be obtained by the principal from the agent can be extracted in the standard framework.

Agent's effort optimization

In this section it will be analyzed how can proper scoring rules incentivize the agent to improve the precision of her initial estimation of Q. For this assume that the agent can gather some information i coming frome the information space $\mathcal{I}$. The expected realization of gathered information i is a priori random and conditional on agent's beliefs on the distribution of Q. It is assumed that, conditional on Q, the agent can assign a probability measure to $\mathcal{I}$ and define a random variable I representing the predicted distribution of received information. We assume that after the information i is gathered the agent updates her beliefs on the distribution of Q. We will denote the updated distribution conditional on realized i by Q_i.

Example 7. Assume that initial distribution of Q is given by Beta distribution with parameters α and β (it will be denoted $B(\alpha, \beta)$)[9]. We know that $\mathrm{E}(Q) = \alpha/(\alpha+\beta)$. Now we take, that the experiment that can be made is one random draw of x from the population. We have $\mathcal{I} = \{0; 1\}$ and conditional on agent's beliefs $\Pr(I = 1) = \alpha/(\alpha+\beta)$. Note that beta distribution is a conjugate prior to Bernoulli distribution. Using standard results on Bayesian updating, see DeGroot (2004), posterior belefs of the agent will be:

$$\begin{array}{llll} Q_0 \sim B(\alpha, \beta+1) & \text{for} & i = 0 & \text{with probability} \quad \beta/(\alpha+\beta) \\ Q_1 \sim B(\alpha+1, \beta) & \text{for} & i = 1 & \text{with probability} \quad \alpha/(\alpha+\beta). \end{array} \tag{4}$$

Continuing the analysis we have from the Theorem 2 that a priori agent's expected profit under proper scoring rule is equal to $G(\mathrm{E}(Q))$. Having seen i she will report $\mathrm{E}(Q_i)$, further denoted X_i. A priori agent's expected profit, conditional on the decision of running the experiment, is equal to $\mathrm{E}(G(X_I))$ (notice that this expectation is taken over I). Also we can notice that $\mathrm{E}(X_I) = \mathrm{E}(Q)$, as a priori the expected beliefs must be equal. The following example follows Example 7 to illustrate this property:

Example 8. Following equations 4 we can calculate that:

$$\begin{array}{l} X_0 = \alpha/(\alpha+\beta+1) \\ X_1 = (\alpha+1)/(\alpha+\beta+1). \end{array}$$

Using this we have:

$$\mathrm{E}(X_I) = \frac{\beta}{\alpha+\beta}\alpha/(\alpha+\beta+1) + \frac{\alpha}{\alpha+\beta}(\alpha+1)/(\alpha+\beta+1) = \alpha/(\alpha+\beta) = \mathrm{E}(Q).$$

So the agent does not expect to change her expectations.

Returning to the main line of reasoning it can be seen that the change in expected profit of the agent when she decides to run the experiment is equal to:

$$V(Q, \mathcal{I}) = \mathrm{E}(G(X_I)) - G(\mathrm{E}(Q)). \tag{5}$$

It can be shown that this change is always positive:

Theorem 9. *If X_I is not certain then $V(Q, \mathcal{I}) > 0$.*

Proof. It is enough to show that $G(\mathrm{E}(Q)) < \mathrm{E}(G(X_I))$. However, $G(\mathrm{E}(Q)) = G(\mathrm{E}(X_I))$, so using the convexity of G and Jensen's inequality we get the result. ■

[9] The probability distribution function is given as $f(x, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, see Gelman et al. (2004).

This means that the agent is given the incentive to improve the precision of the estimate of Q. Of course she will have to compare $V(Q,\mathcal{I})$ with her cost of running the experiment.

As a special case we can assume that the experiment gives perfect information on the value of q. In such a situation we have that cumulative distribution functions of X_I and Q are equal. Under this the expected value of perfect information (EVPI) is equal to $\mathrm{E}(G(Q)) - G(\mathrm{E}(Q))$. The above results are illustrated with the example using Brier score (see Example 3):

Example 10. Under Brier score we have $G(q) = -q(1-q)$ therefore:

$$\mathrm{E}(G(X_I)) = \mathrm{E}(-X_I(1-X_I)) = \mathrm{E}(X_I^2) - \mathrm{E}(X_I) = \mathrm{D}^2(X_I) + \mathrm{E}^2(X_I) - \mathrm{E}(X_I).$$

Remembering that $\mathrm{E}(X_I) = \mathrm{E}(Q)$ it follows that:

$$\begin{aligned} V(Q,\mathcal{I}) &= \mathrm{E}(G(X_I)) - G(\mathrm{E}(Q)) = \\ &= \left(\mathrm{D}^2(X_I) + \mathrm{E}^2(Q) - \mathrm{E}(Q)\right) - \left(\mathrm{E}^2(Q) - \mathrm{E}(Q)\right) = \mathrm{D}^2(X_I). \end{aligned}$$

In particular $\mathrm{EVPI}(Q) = \mathrm{D}^2(Q)$.

Summing up the results of this section it has been shown that the only information that the principal can get from the agent in an incentive compatible way is $\mathrm{E}(Q)$ and allowing the agent to signal other information than her probability beliefs does not change the result. Finally — the agent gets a positive incentive to improve the precision of the estimate of Q. In the next section it will be shown how to extend this result to nominal events.

Nominal event forecasting contracts

In the nominal event case we assume that $x \in \{1,\dots,n\}$ and $\mathbf{N} \ni n > 2$. That is — there is a finite number of possible outcomes. In certain beliefs case by $\mathbf{q} = (q_1,\dots,q_n)$ we will denote agent's expectations and by $\mathbf{p} = (p_1,\dots,p_n)$ announced probabilities. In this case every scoring rule S_x takes $\mathbf{p}$ as an argument. Agent's expected payoff can be calculated as $R(\mathbf{p}) = \sum_{i=1}^n q_i S_i(\mathbf{p})$. The Definition 1 of proper scoring rule is extended naturally by requiring that the strict maximum of $R(\mathbf{p})$ is attained in $\mathbf{q}$.

For uncertain beliefs we define $\mathbf{Q} = (Q_1,\dots,Q_n)$ as a random vector representing beliefs of the agent. The definition of S_x^E remains unchanged (as it relates to arbitrary set $\mathcal{S}$) and we can calculate expected payoff $R^E(s) = \mathrm{E}\left(\sum_{i=1}^n Q_i S_i^E(s)\right)$. The definition of χ-proper extended scoring rule is also unchanged (except for that now $\chi : [0;1]^{n+1} \to \mathcal{S}$). In nominal case it can be shown that the binary results can be naturally extended:

Theorem 11. *If $\mathcal{S} = [0;1]^n$ and $S_x^E(s)$ is a proper scoring rule then $R^E(s)$ is maximized for $s = E(\mathbf{q})$.*

Proof. Using the linear separability of expected value operator the definition of $R^E(s)$ in nominal case can be rewritten as $R^E(s) = \sum_{i=1}^n \mathrm{E}(Q_i) S_i^E(s)$. And because $S_x^E(s)$ is a proper scoring rule we get that in optimum $s = \mathrm{E}(\mathbf{q})$. ∎

Theorem 12. *If S_x^E is χ-proper extended scoring rule then there exists such mapping $h(\cdot)$ that $\chi(F) = h(E(\mathbf{Q}))$.*

Proof. Assume that the converse is true and such χ-proper extended scoring rule exists. Then there exist such random vectors $\mathbf{T}$ and $\mathbf{U}$ with n-variate cumulative distribution functions F_1 and F_2 for which: $\mathrm{E}(\mathbf{T}) = \mathrm{E}(\mathbf{U}) \wedge \chi(F_1) \neq \chi(F_2)$. But S_x^E χ-proper extended scoring rule so:

$$\begin{array}{c} \sum_{i=1}^n \mathrm{E}(T_i) S_i^E(\chi(F_1)) > \sum_{i=1}^n \mathrm{E}(T_i) S_i^E(\chi(F_2)) \\ \sum_{i=1}^n \mathrm{E}(U_i) S_i^E(\chi(F_1)) < \sum_{i=1}^n \mathrm{E}(U_i) S_i^E(\chi(F_2)) \end{array}$$

A contradiction, so it is impossible that S_x^E is χ-proper. ■

Now we will show that Theorem 9 also holds for nominal case. For this a result first noted by McCarthy (1956) is needed:

Theorem 13 (McCarthy, 1956)**.** *If S_x is a proper scoring rule for nominal x then $R(\mathbf{p})$ is strictly convex.*

And it can be seen that the proof of Theorem 9 remains unchanged for nominal event.

Summary

In the paper a forecasting contract design with uncertain forecaster's (agent's) beliefs was considered. It was assumed that such contract is incentive compatible if it is strictly optimal for the agent to reveal her true beliefs.

It was shown that in binary and nominal case the principal can learn only expected values of agent's predictions on event probabilities in an incentive compatible way. Payment schemes having this property, called proper scoring rules, can rely only on asking the agent to reveal her expectation. Additionally it was shown that proper scoring rules give the agent strictly positive incentive to improve the precision of her estimates.

The consequences of changing the assumption that the agent maximizes her expected payoff can be investigated in further work. Here one remark will be only made. Changing this assumption to maximization of *expected utility* (see for example Mas-Collel et al., 1995) does not change the results if we use a natural assumption that the utility function is strictly increasing. Then it is reversible and any proper (or χ-proper) scoring rule under expected utility maximization assumption must be equivalent to some proper scoring rule under expected payoff maximization.

It it also worth to investigate the problem on designing such incentive schemes that would allow the principal to learn other statistics of agent's beliefs than expected value. However, from the results presented in the paper, it can be seen that in order to achieve this goal such systems must allow for some additional information, for example: repeated measurement or comparison of several forecasters.

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