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STRATEGIC SUBSTITUTES AND COMPLEMENTS IN COURNOT OLIGOPOLY WITH PRODUCT DIFFERENTIATION

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Abstract: We consider Cournot oligopoly with differentiated product. We develop respective sufficient conditions on the inverse demand and cost function that make the oligopoly a game of strategic substitutes when goods are substitutes and a game of strategic complementarities when goods are complements. The scope of this result is illustrated by examples.
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Introduction

Cournot's model of oligopolistic competition (1838) represents the starting point of formalized economic theory and game theory. Nowadays it continues to be widely used in a number of applications in economic theory. The reason for this is its tractability, when product homogeneity can be assumed, and the fact that its properties are well established. In this paper we follow the line of literature dealing with the monotonicity of reaction correspondences, which is important in particular for studying problems of equilibrium existence and comparative statics.

Increasing reaction correspondences characterizes games with strategic complementarities, while games of strategic substitutes have downward sloping best replies. This expresses the strategic relationship among actions in this kind of games. Strategic complementarities covers situations when an increase in one player action leads to an increase in other players' marginal payoffs. Strategic substitutes describe the opposite situation, when an increase in one player action causes a decrease in other players' marginal payoffs. The former kind of games always possesses pure strategy Nash equilibria. Moreover, the set of pure strategy Nash equilibria, correlated equilibria and rationalizable strategies have identical bounds (see, Milgrom and Roberts, 1990, and Milgrom and Shannon, 1994). The latter kind of games, in the case of $2-$player game, can be converted into the game of strategic complementarities by reordering of one player's action set (Milgrom and Roberts, 1990).

To establish properties of a Cournot oligopoly with heterogeneous products several simplifying assumptions have been imposed. For instance, inverse demand function have an aggregative form with respect to other players' strategies (cf. Dubey et al., 2005). Hoernig (2003) assumes that, departing from the situation when firms produce the same quantities, when one firm deviates by raising its output, other firms adjust their outputs to the level allowing for avoiding an increase in market price. He conducts comparative statics in case of market entry and increasing number of firms, extending the results of Amir and Lambson (2000) for homogeneous products.

In this paper we provide general conditions for a Cournot oligopoly with product differentiation to have monotonic reaction correspondences. These results are a generalization of the results of Amir (1996) for homogeneous products to the case of differentiated product. We give various sufficient conditions for downward and upward sloping reaction correspondences. They allow for identifying increasing best responses even in case of inverse demand being submodular, and similarly, decreasing best responses in case of supermodular inverse demand. Examples illustrating the scope of applicability of these results are provided.

This approach gives a significant value added to the problem of equilibria existence and comparative statics. The standard approach demands profit function to be quasi-concave in own quantities. This is quite restrictive, particularly because nonconcavities in costs are not uncommon and very convex demand functions cannot be ruled out (Vives, 1999). The lack of quasi-concavity of payoffs causes discontinuities in the best response correspondences of firms and makes possible the nonexistence of equilibrium. Our approach covers games with monotonic reaction correspondences and does not rely on the regularity condition.

The paper is organized as follows. Next section contains an brief overview of relevant notions from supermodular optimization and games. Section 3 presents main results. In Section 4 they are discussed. All proofs are placed in Appendix.

Supermodular games

We introduce in this section a summary of all relevant notions and results from lattice theory, supermodular optimization and supermodular games useful in the remainder. We present them in context of real decision parameter spaces, since this is sufficient for our needs.

A function $F : X \times Y \to R$ is *supermodular* if, for all $x_1 \geq x_2, y_1 \geq y_2$

$$F(x_1, y_1) - F(x_2, y_1) \geq F(x_1, y_2) - F(x_2, y_2). \tag{1}$$

A function F is *submodular* whenever $-F$ is supermodular. For twice continuously differentiable functions this notion has a useful differential characterization, namely supermodularity (submodularity) is equivalent to cross-partial derivative being positive (negative).

We say that F is *log-supermodular* (*log-submodular*) if logarithm of F is supermodular (*submodular).*

Topkis (1978) showed the following theorem on monotone optimization, central to our approach.

Theorem 1. *If F is upper semi-continuous and supermodular (submodular), then the maximal and minimal selections of*

$$\arg\max_{x\in X}\{F(x,y), y\in Y\}$$

are non-decreasing (non-increasing).

If F is strictly supermodular (submodular), then this theorem holds for every selection of $\arg\max_{x\in X}\{F(x,y), y\in Y\}$.

The property of supermodularity is of cardinal nature, in the sense that it is not preserved by monotonic transformation. Below we provide a definition of another notion, of ordinal nature, which can be treated as a generalization of supermodularity.

A function F has the (*dual*) *single-crossing property* in (x,y) if, for all $x_1 \geq x_2, y_1 \geq y_2$

$$F(x_2,y_1) - F(x_2,y_2)(\leq) \geq 0 \Rightarrow F(x_1,y_1) - F(x_1,y_2)(\leq) \geq 0. \tag{2}$$

Strict (dual) single crossing property is defined by the implication with strict inequality on the right hand side.

Obviously, (1) implies (2), but the converse does not hold.

Milgrom and Shannon (1994) generalized the result of Topkis for functions possessing the single crossing property.

Theorem 2. *If F is upper semi-continuous and satisfies the (dual) single crossing property, then the maximal and minimal selections of*

$$\arg\max_{x\in X}\{F(x,y), y\in Y\}$$

are non-decreasing (non-increasing).

If F has strict (dual) single crossing property, then this theorem holds for every selection of $\arg\max_{x\in X}\{F(x,y), y\in Y\}$.

The single crossing property has no differential characterization like supermodularity. Milgrom and Shannon (1994) showed that this property can be tested using the Spence-Mirrlees condition defined in the following theorem.

Theorem 3. *Let $F : R^3 \to R$ be continuously differentiable and $F_2(a,b,s) \neq 0$.*[1] *$F(a,h(a),s)$ satisfies the single-crossing property in (a,s) for all functions $h : R \to R$ if and only if*

$$\frac{F_1(a,b,s)}{|F_2(a,b,s)|} \tag{3}$$

[1]Subscripts denote partial derivatives with respect to the certain variable, e.g. here $F_2(a,b,s) = \frac{\partial F(a,b,s)}{\partial b}$. We keep this notation throughout the paper.

is increasing in s.

In applications, verifying (3) leads to conclusion that $F(a, h(a), s)$ satisfies the single crossing property in (a, s) for suitable choice of function h (which is often an identity function, see Milgrom and Shannon, 1994, and Amir, 2005).

A game with compact real action spaces is *supermodular* (*submodular*) if each payoff function is supermodular (submodular) and upper semi continuous in own actions.

Replacing supermodularity by the single crossing property enables to define a broader class of games: a game is *ordinally supermodular* (*submodular*) when its action space is compact, the payoff functions have the (dual) single crossing property and are upper semi continuous in own actions.

Supermodularity of the payoff functions can be interpreted as a complementarity between the players' actions, namely an increase some players' actions causes an increase in the marginal payoff of the others. Hence, this kind of games are also called *games with strategic complementarities*. Submodularity expresses the opposite phenomenon - substitutability of the players actions: increasing some players actions causes a decrease in the marginal payoff of the others, thus this kind of games are also called *games of strategic substitutabilities*.

Corollary 4. *Every $n-$player game of strategic complementarities has pure strategy Nash equilibrium.*

There is no equivalent corollary for $n-$player games of strategic substitutes. Only for the case of two players the existence of equilibrium is guaranteed.

Corollary 5. *Every $2-$player game of strategic substitutes has pure strategy Nash equilibrium.*

Conditions and examples

Consider a Cournot oligopoly game, when n firms decide simultaneously about the products quantity. Let $x_i \in X_i$ be a production quantity of firm i and $x_{-i} \in X_{-i}$ represents vector of production quantities of the other $n-1$ firms. The products are heterogeneous. Denote $P^i(x_i, x_{-i})$, $i = 1, .., n$, a system of inverse demand functions describing the market. Each of the functions is twice continuously differentiable and $P^i_j(x_i, x_{-i}) = P^j_i(x_j, x_{-j})$, $i \neq j$. Then the profit of firm i is given by

$$\Pi^i(x_i, x_{-i}) = x_i P^i(x_i, x_{-i}) - C^i(x_i) \tag{4}$$

where $C^i(\cdot)$ is a differentiable cost function. Assume that $P^i(x_i, x_{-i})$ is decreasing in own action, and $C^i(\cdot)$ is increasing.

We say that goods i and j are (strict) *substitutes*, if demand for i (strictly) rises with the increase in the price of j. Goods i and j are (strict) *complements,* if demand for i (strictly) goes down with the increase of the price of j. It can be translated in terms of inverse demand function, so that P^i is (strictly) decreasing in x_j, if goods i and j are substitutes, and the converse holds if goods i and j are complements.

Define reaction (best response) correspondence of firm i as

$$r_i(x_{-i}) = \arg\max_{x_i \in X_i} \{\Pi^i(x_i, x_{-i}), x_{-i} \in X_{-i}\}.$$

From Theorem 1 the game has increasing reaction correspondences if $\Pi^i, i = 1, .., n$, is supermodular or $\Pi^i_{ij} \geq 0, \forall j \neq i$, hence if each firm's revenue is supermodular (see also Novshek, 1985). For firm i it is equivalent to the following condition:

$$P^i_j(x_i, x_{-i}) - x_i P^i_{ij}(x_i, x_{-i}) \geq 0, \forall j \neq i.$$

But from Theorem 2 it follows that even weaker conditions can secure the monotonicity of the reaction correspondences. It is enough for the payoff function to satisfy the single crossing property.

Vives (1999) formulated independent conditions on demand and cost function in a Cournot oligopoly with product differentiation to be an ordinally supermodular game. We give them in the form of a theorem.

Theorem 6. *Assume that for* $i = 1, .., n$

1. $P^i(\cdot)$ *is log-supermodular.*
2. C^i *is strictly increasing.*
3. *Goods* i *and* j *are strict complements* $\forall j \neq i$.

Then the Cournot oligopoly, with profits given by (4) is an ordinally supermodular game.

The proof (not given by Vives, 1999), presented in Appendix, does not require differentiability of the inverse demand nor the cost function.

An analogous theorem can be formulated to provide conditions on a Cournot duopoly to have decreasing reaction correspondences.

Theorem 7. *Assume that for* $i = 1, .., n$

1. $P^i(\cdot)$ *is log-submodular.*
2. C^i *is strictly increasing.*
3. *Goods* i *and* j *are strict substitutes* $\forall j \neq i$.

Then the Cournot oligopoly, with profits given by (4) is an ordinally submodular game.

To provide some intuition of the scope of duality between these theorems for a duopoly case we can use Milgrom and Roberts (1990) action reordering argument. In case of 2−players game, changing order of the action space of one of the players converts a submodular game into a supermodular game. Take a game Γ, the Cournot duopoly and consider situation of player 1 with $P^1(x_1, x_2)$ log-submodular and x_1, x_2 substitute goods. Reordering of the player's 1 action space creates a new game $\widehat{\Gamma}$.

This is the Cournot duopoly with $P^1(\widehat{x_1}, x_2)$. Observe that whenever $P^1(\widehat{x_1}, x_2)$ is positive, it is log-supermodular, since $P^1P^1_{12} - P^1_1P^1_2 \leq 0$ and $\widehat{x_1}' = -1$ implies that $\widehat{x_1}'\left(P^1P^1_{12} - P^1_1P^1_2\right) \geq 0$ for every $x_1 \in X_1$ and $x_2 \in X_2$. Moreover, $\widehat{x_1}$ and x_2 are complements now, since relation between them changed to the reverse.[2] Therefore, there is complete duality between these two results. This kind of duality does not work if there is more than two players in the game.

The theorems given above work for any increasing cost function. But considering a specific cost function in relation with the inverse demand we may relax the assumption of log-supermodularity of the demand. It is possible to formulate a more general condition for a specific cost function, such that it guarantees increasing best response correspondence. We provide it in the theorem below. The differentiability of $P^i(x_i, x_{-i})$ and $C^i(x_i)$ is assumed here .

Theorem 8. *Assume that* $\forall j \neq i$

$$P^i_{ij}(x_i, x_{-i})P^i(x_i, x_{-i}) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i}) \geq P^i_{ij}(x_i, x_{-i})C^{i\prime}(x_i). \tag{5}$$

Then the Cournot oligopoly, with profits given by (4) is an ordinally supermodular game.

Condition (5) can be reformulated into

$$P^i_{ij}(x_i, x_{-i})(P^i(x_i, x_{-i}) - C^{i\prime}(x_i)) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i}) \geq 0. \tag{6}$$

It is straightforward that this condition is met, when the assumptions of Theorem 6 are satisfied. Moreover, for a suitable choice of C^i, this condition can be satisfied for a number of inverted demand functions, which do not satisfy log-supermodularity, as long as goods i and j remain complements. In particular, there are log-submodular functions giving rise strategic complementarity in the Cournot model. An illustration of this possibility is provided in Example 10.

When we assume fixed marginal costs, condition (5) can be interpreted as log-super-modularity of net-of-cost inverse demand functions $P^i(x_i, x_{-i}) - c^i$, $i = 1, 2$. Moreover, this condition says that the firm's i perceived net-of-cost inverse demand elasticity is increasing in firm's j output. Indeed, elasticity of net-of-cost inverse demand is given by

$$\begin{aligned}\varepsilon(x_i, x_j) &\triangleq \frac{\partial\left(P^i(x_i, x_{-i}) - c^i\right)}{\partial x_i}\frac{x_i}{P^i(x_i, x_{-i}) - c^i}\\ &= P^i_i(x_i, x_{-i})\frac{x_i}{P^i(x_i, x_{-i}) - c^i}.\end{aligned}$$

It is increasing in x_j whenever its derivative with respect to x_j is positive:

$$\frac{\partial\varepsilon(x_i, x_j)}{\partial x_j} = x_i\frac{P^i_{ij}(x_i, x_{-i})\left(P^i(x_i, x_{-i}) - c^i\right) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i})}{\left(P^i(x_i, x_{-i}) - c^i\right)^2}.$$

It holds if and only if (6) holds.

[2] Now, when price of good 2 increases, the demand for $\widehat{x_1}$ goes up.

Corollary 9. *Assume that marginal cost is constant, denoted c^i and $P^i(x_i, x_{-i}) - c^i$ is log-supermodular, $i = 1, .., n$. Then the Cournot oligopoly, with profits given by (4) is an ordinally supermodular game.*

We provide now an example illustrating the scope of application of Theorem 8 in the duopoly case. Our generalization enables to capture even situations when inverse demand, being submodular and/or log-submodular, gives rise, for specific cost functions, to an ordinally supermodular game.

Example 10. Consider a 2−player game. Let

$$P^1(x_1, x_2) = 1 + \frac{1}{(x_1+1)^2} + (x_2+1)\exp(-x_1).$$

It is easily verified that

$$\begin{aligned} P^1_1(x_1, x_2) &= \frac{-2}{(x_1+1)^3} - (x_2+1)\exp(-x_1) < 0 \\ P^1_2(x_1, x_2) &= \exp(-x_1) > 0, \end{aligned}$$

Hence, the goods are complements. Also

$$P^1_{12}(x_1, x_2) = -\exp(-x_1) < 0$$

and moreover

$$\left(\ln P^1(x_1, x_2)\right)_{12} = \frac{-\left(x_1^2 + 3x_1 + 4\right) x_1 \left(x_1+1\right) e^{-x_1}}{\left(1 + \left(x_1+1\right)^2 \left(\left(x_2+1\right) e^{-x_1} + 1\right)\right)^2} < 0.$$

Hence, the inverse demand is strictly submodular and strictly log-submodular, and thus cannot be log-supermodular. Therefore, the conditions of Theorem 6 are not satisfied. Now take the following cost function: $C^1(x_1) = 2x_1$ and check the condition (6):

$$\begin{aligned} & P^1_{12}(x_1, x_2)(P^1(x_1, x_2) - C^{1\prime}(x_1)) - P^1_1(x_1, x_2)P^1_2(x_1, x_2) \\ = \; & e^{-x_1}\left(\left(x_2+1\right) e^{-x_1} + 2x_1 + \frac{2}{\left(x_1+1\right)^3}\right) > 0. \end{aligned}$$

It is satisfied for all positive x_1 and x_2. The verification that not all cost function satisfy this condition is left to the reader.

Concavity of firm's 1 profit can be easily verified, hence we use first order condition to find the reaction curve.

$$\Pi^1_1(x_1, x_2) = -\frac{\left(x_1 - 1\right)\left(x_1+1\right)^3 \left(x_2+1\right) e^{-x_1} + x_1 \left(x_1^2 + 3x_1 + 4\right)}{\left(x_1+1\right)^3} = 0.$$

It cannot be solved explicitly for x_1, thus we present a numerical plot in Figure 1.

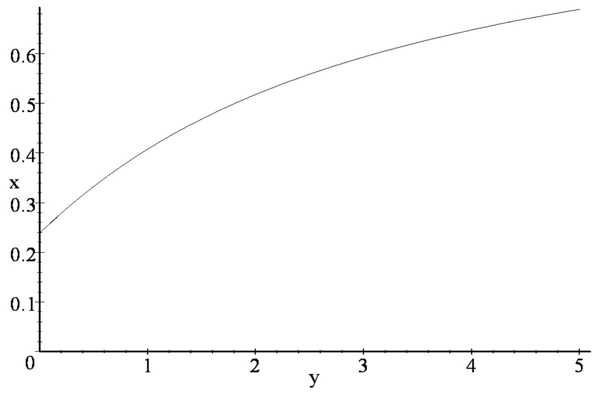

Figure 1. Reaction function of firm 1 in Example 10 is upward sloping.

Again, the dual theorem guaranteeing decreasing best responses can be formulated, and is given below.

Theorem 11. *Assume that* $\forall j \neq i$

$$P^i_{ij}(x_i, x_{-i})P^i(x_i, x_{-i}) - P^i_i(x_i, x_{-i})P^i_j(x_i, x_{-i}) \leq P^i_{ij}(x_i, x_{-i})C^{i\prime}(x_i). \quad (7)$$

Then the Cournot oligopoly, with profits given by (4) is an ordinally submodular game.

Similarly, in the case of duopoly, one can imagine this condition as derived from the reordering approach.

This condition covers all the cases, which satisfy conditions of Theorem 7 and, moreover, for specific cost functions, inverted demands, which are not log-submodular. In fact even log-supermodular inverted demands may lead to an ordinally submodular Cournot game, an illustration is provided in Example 13.

As before, we formulate a corollary for the case of constant marginal costs.

Corollary 12. *Assume that marginal cost is constant, denoted* c^i *and* $P^i(x_i, x_{-i}) - c^i$ *is log-submodular,* $i = 1, .., n$. *Then the Cournot oligopoly, with profits given by (4) is an ordinally submodular game.*

Log-submodularity of net-of-cost inverted demand can be interpreted as its elasticity being decreasing in x_j.

Usefulness of the Theorem 11 in the duopoly case is illustrated by the example below. Again, even supermodular and log-supermodular inverse demand function can give rise to an ordinally submodular game for some cost functions.

Example 13. Consider a $2-$player game and take

$$P^1(x_1, x_2) = \frac{e^{-x_1}}{(x_2 + 1)} + \frac{1}{x_1 + 1}.$$

It is easy to verify that

$$\begin{aligned} P_1^1(x_1,x_2) &= -\frac{e^{-x_1}}{(x_2+1)} - \frac{1}{(x_1+1)^2} < 0, \\ P_2^1(x_1,x_2) &= -\frac{e^{-x_1}}{(x_2+1)^2} < 0, \\ P_{12}^1(x_1,x_2) &= \frac{e^{-x_1}}{(x_2+1)^2} > 0, \end{aligned}$$

and

$$\left(\ln P^1(x_1,x_2)\right)_{12} = e^{-x_1}\frac{x_1}{(e^{-x_1}x_1 + e^{-x_1} + x_2 + 1)^2} > 0.$$

Hence, the products are substitutes and inverse demand is log-supermodular. Therefore the game does not satisfy the conditions of Theorem 7.

Take now the following cost function: $C^1(x_1) = \ln(x_1+1)$ and check condition (7),

$$\begin{aligned} P_{12}^1(x_1,x_2)(P^1(x_1,x_2) - C^{1\prime}(x_1)) - P_1^1(x_1,x_2)P_2^1(x_1,x_2) &= \\ -e^{-x_1}\frac{(x_2+1)(x_1+1)^2\ln(x_1+1) + (x_1+1)^2 e^{-x_1} + x_2 + 1}{(x_2+1)^3(x_1+1)^2} &< 0. \end{aligned}$$

It is satisfied for all positive x_1 and x_2. Therefore, the game is ordinally submodular.

From the first order condition of profit maximization we find best response curve. We verified that $\Pi^1(x_1,x_{12})$ is locally concave on the best response function. It is given by an implicit formula

$$e^{-x_1}x_1^2 - e^{-x_1}x_1 - e^{-x_1} + e^{-x_1}x_1^3 + x_2x_1 + x_1 = 0.$$

Since it cannot be explicitly solved for x_1, we present a numerical plot in Figure 2.

Discussion

Games of strategic complementarities have Nash equilibria in pure strategies. Hence, when the proper conditions are satisfied for a Cournot oligopoly with differentiated products, one can be sure that equilibria exist, moreover, in terms of strategies, there exist largest and smallest equilibria.

For games of strategic substitutes there is no equivalent result. We may only guarantee the existence of pure strategy Nash equilibria in 2−player game. For $n > 2$ one can distinguished games, in which the strategies of all opponents can be aggregated into one number. For this kind of games the existence of the pure strategy Nash equilibrium was shown by Dubey et al. (2005), see also Novshek (1985). Unfortunately, not all reasonable inverse demand functions have this property (cf. Hoernig, 2003).

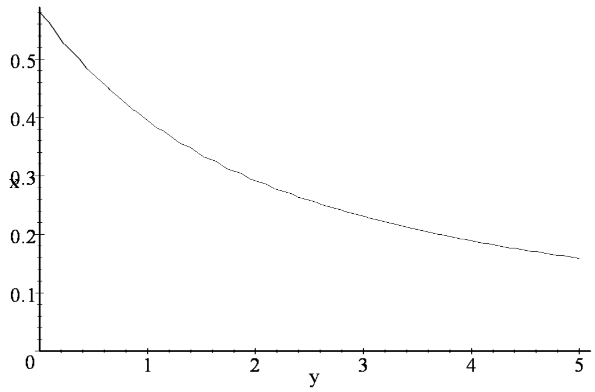

Figure 2. Downward sloping reaction function of firm 1 in Example 13.

Appendix

Proof. (of theorem 6) Consider firm i. Take $x_i^1 > x_i^2, x_{-i}^1 > x_{-i}^2$. From assumption 1 we have:

$$\ln P^i(x_i^1, x_{-i}^1) - \ln P^i(x_i^2, x_{-i}^1) \geq \ln P^i(x_i^1, x_{-i}^2) - \ln P^i(x_i^2, x_{-i}^2).$$

This implies that

$$\frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} \geq \frac{P^i(x_i^1, x_{-i}^2)}{P^i(x_i^2, x_{-i}^2)}$$

or

$$P^i(x_i^2, x_{-i}^2)\frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} \geq P^i(x_i^1, x_{-i}^2). \tag{8}$$

To prove that the game with log-supermodular inverse demand and increasing cost function is ordinally supermodular, it is enough to show that $\Pi^i(x_i, x_{-i})$ has the single crossing property. To this end we start from assuming that

$$\Pi^i(x_i^1, x_{-i}^2) \geq \Pi^i(x_i^2, x_{-i}^2).$$

Then

$$x_i^1 P^i(x_i^1, x_{-i}^2) - C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^2) - C^i(x_i^2).$$

We can replace $P^1(x_i^1, x_{-i}^2)$ from (8)

$$x_i^1 P^i(x_i^2, x_{-i}^2)\frac{P^i(x_i^1, x_{-i}^1)}{P^i(x_i^2, x_{-i}^1)} - C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^2) - C^i(x_i^2).$$

Multiplying by $\frac{P^i(x_i^2,x_{-i}^1)}{P^i(x_i^2,x_{-i}^2)}$ we get

$$x_i^1 P^i(x_i^1, x_{-i}^1) - \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} C^i(x_i^1) \geq x_i^2 P^i(x_i^2, x_{-i}^1) - \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} C^i(x_i^2).$$

$$\begin{aligned} x_i^1 P^i(x_i^1, x_{-i}^1) - x_i^2 P^i(x_i^2, x_{-i}^1) &\geq \frac{P^i(x_i^2, x_{-i}^1)}{P^i(x_i^2, x_{-i}^2)} (C^i(x_i^1) - C^i(x_i^2)) \\ &> C^i(x_i^1) - C^i(x_i^2). \end{aligned}$$

Since $P^i(x_i^2, x_{-i}^1) > P^i(x_i^2, x_{-i}^2)$ from assumption 2 and $C^i(x_i^1) > C^i(x_i^2)$ from assumption 3, it follows that

$$x_i^1 P^i(x_i^2, x_{-i}^1) - C^i(x_i^1) > x_i^2 P^i(x_i^1, x_{-i}^1) - C^i(x_i^2).$$

Hence

$$\Pi^i(x_i^1, x_{-i}^1) > \Pi^i(x_i^2, x_{-i}^1),$$

which means that (2) is satisfied. ■

Proof. (of Theorem 7) is analogous to the previous one. ■

Proof. (of Theorem 8) Consider firm i. Let the $F(a,b,s) = bP^i(a,s) - C^i(b)$. Then $F_1(a,b,s) = bP_1^i(a,s)$, $F_2(a,b,s) = P^i(a,s) - C^{i\prime}(b)$ and

$$\begin{aligned} \frac{F_1(a,b,s)}{|F_2(a,b,s)|} &= \frac{bP_1^i(a,s)}{|P^i(a,s) - C^{i\prime}(b)|} \\ & \quad \frac{\partial F_1(a,b,s)/\,|F_2(a,b,s)|}{\partial s} \\ &= b\frac{P_{12}^i(a,s)\left|P^i(a,s) - C^{i\prime}(b)\right| - P_1^i(a,s)P_2^i(a,s)}{\left(P^i(a,s) - C^{i\prime}(b)\right)^2}. \end{aligned}$$

Then, condition (3) holds when

$$P_{12}^i(a,s)\left|P^i(a,s) - C^{i\prime}(b)\right| - P_1^i(a,s)P_2^i(a,s) \geq 0.$$

Sufficient condition for this is (5). ■

Proof. (of Corollary 9) Consider firm i. Take $x_i^1 > x_i^2, x_{-i}^1 > x_{-i}^2$. Form assumption 2 we have:

$$\begin{aligned} & \ln(P^i(x_i^1, x_{-i}^1) - c^i) - \ln(P^i(x_i^2, x_{-i}^1) - c^i) \\ \geq\ & \ln(P^i(x_i^1, x_{-i}^2) - c^i) - \ln(P^i(x_i^2, x_{-i}^2) - c^i). \end{aligned}$$

This implies that

$$\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i} \geq \frac{P^i(x_i^1, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^2) - c^i}$$

and

$$(P^i(x_i^2, x_{-i}^2) - c^i)\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i} \geq P^i(x_i^1, x_{-i}^2) - c^i. \tag{9}$$

To prove that the game characterized by log-supermodular net-of cost inverse demand and constant marginal cost is ordinally supermodular, it is enough to show that $\Pi^i(x_i, x_{-i})$ has the single crossing property. To this end we start from assuming that

$$\Pi^i(x_i^1, x_{-i}^2) \geq \Pi^i(x_i^2, x_{-i}^2). \tag{10}$$

Then

$$x_i^1(P^i(x_i^1, x_{-i}^2) - c^i) \geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i).$$

We can replace $P^i(x_i^1, x_{-i}^2) - c^i$ from (9)

$$\begin{aligned} x_i^1((P^i(x_i^2, x_{-i}^2) - c^i)\frac{P^i(x_i^1, x_{-i}^1) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}) &\geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i). \\ x_i^1(\big(P^i(x_i^1, x_{-i}^1) - c^i\big)\frac{P^i(x_i^2, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}) &\geq x_i^2(P^i(x_i^2, x_{-i}^2) - c^i). \end{aligned}$$

Dividing by $\frac{P^i(x_i^2, x_{-i}^2) - c^i}{P^i(x_i^2, x_{-i}^1) - c^i}$ we get

$$x_i^1(P^i(x_i^1, x_{-i}^1) - c^i) \geq x_i^2(P^i(x_i^2, x_{-i}^1) - c^i),$$

which means that (10) implies

$$\Pi^i(x_i^1, x_{-i}^1) \geq \Pi^i(x_i^2, x_{-i}^1),$$

which concludes the proof. ■

Proof. (of Theorem 11) is analogous to the proof of Theorem 8. ■

Proof. (of Corollary 12) is analogous to the proof of Corollary 9. ■

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