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# GAMES WITH DISTORTED INFORMATION AND SELF-VERIFICATION OF BELIEFS WITH APPLICATIONS TO FINANCIAL MARKETS

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**Abstract:** In the paper we examine discrete time dynamic games in which the global state variable changes in response to a certain function of the profile of players' decisions, called statistic, while the players form some expectations about its future values based on the history. Besides, there are also players' private state variables.
A general model is built, encompassing both games with finitely many players as well as games with infinitely many players. This model extends the class of games with distorted information considered by the author in [20], in which there were no private state variables and there were much stronger assumptions about the statistic of players' decisions considered. The notions of pre-belief distorted Nash equilibrium (pre-BDNE), self-verification and belief distorted Nash equilibrium (BDNE), defined already in [20], are applied to our wider class of games. The relations between Nash equilibria, pre-BDNE and BDNE are examined as well as the existence and properties of pre-BDNE.
A model of a financial market – a simplified stock exchange – is presented as an example. Pre-BDNE using threshold prices are proposed. One of further results in this example is potential self-verification of fundamental beliefs and beliefs in infinite speculative bubbles.

**Key words:** games with continuum of players, $n$-player dynamic games, Nash equilibrium, pre-belief distorted Nash equilibrium, subjective equilibrium, self-verification of beliefs, financial markets.

## Introduction

### Motivation

The starting point of the research on games with distorted information were two phenomena arising from the problems examined in earlier papers of the author: exploitation of common renewable resources, especially by a large groups of users, and modelling financial markets.

The problem of a common ecosystem in the context of pre-belief distorted Nash equilibria (pre-BDNE) and self-verification of beliefs was examined by the author in Wiszniewska-Matyszkiel [20]. A concept of belief distorted Nash equilibria (BDNE) encompassing both pre-BDNE and self-verification of beliefs during the game was also introduced.

An example which is especially illustrative for the class of games considered in this paper, and which in fact forced extension of the class of games considered in Wiszniewska-Matyszkiel [20], is a stock exchange.

In such an institution prices of shares are calculated by the equilibrating mechanism given only a profile of players' orders. However, players usually formulate various prognostic techniques, often ridiculed by economists. Can such a behaviour with no scientific explanation turn out to be rational? Why people still believe in things like technical analysis?

### Imperfect information, beliefs and game theory

There were various concepts taking beliefs into account, usually in games with stochastic environment or randomness caused by using mixed strategies.

The first were Bayesian equilibria, introduced by Harsanyi [7].

That approach was continued by e.g. Battigalli and Siniscalchi [5] using the concept of $\Delta$-rationalizability being an iterative procedure of eliminating type-strategy pairs in which the strategy is strictly dominated according to player's beliefs or which are contradicted by a history of play.

Another concept – subjective equilibria were introduced by Kalai and Lehrer [8] and [9] (although the very idea of subjectivity in games appeared already in Aumann [2] and [3]). They apply to repeated games. In those concepts every player maximizes his expected payoff assuming some environment response function (so, in fact, calculation of equilibrium is decomposed into separate decision making problems) and non-falsification of his assumption during the game was added.

A comparison of these absolutely different concepts to the concepts of pre-BDNE and BDNE, can be found in Wiszniewska-Matyszkiel [20].

### Games with a measure space of players

The term *games with a measure space of players* is usually perceived as a synonym of *games with infinitely many players* called also *large games*. In order to make it possible to evaluate the influence of the infinite set of players on aggregate variables, a measure is introduced on a $\sigma$-field of subsets of the set of players. However, the notion *games with a measure space of players* encompasses also games with finitely many players, where e.g. the counting measure on the power set may be considered.

Large games illustrate situations where the number of agents is large enough to make a single agent from a subset of the set of players (possibly the whole set) insignificant – *negligible* – when we consider the impact of his action on aggregate variables while joint action of a set of such negligible players is not negligible. This happens in many real situations: at competitive markets, stock exchange, or while we consider emission of greenhouse gases and similar global effects of exploitation of the common global ecosystem.

Although it is possible to construct models with countably many players illustrating the phenomenon of this negligibility, they are very inconvenient to cope with. Therefore simplest examples of large games are so called *games with continuum of players*, where players constitute a nonatomic measure space, usually unit interval with the Lebesgue measure.

The first attempts to use models with continuum of players are contained in Aumann [1] and Vind [13].

Some theoretical works on large games are Schmeidler [11], Mas-Colell [10], Balder [4] and Wiszniewska-Matyszkiel [14].

The general theory of dynamic games with continuum of players is still being developed, mainly by the author in [15] and [16].

Introducing a continuum of players instead of a finite number, however large, can change essentially properties of equilibria and the way of calculating them even if the measure of the space of players is preserved in order to make the results comparable. Such comparisons were made by the author in [17] and [18].

## Formulation of the model

A *game with distorted information* $\mathfrak{G}$ is a tuple of the following objects $((\mathbb{I},\Im,\lambda),\mathbb{T},\mathbb{X},\{\mathbb{W}_i\}_{i\in\mathbb{I}},(\mathbb{D},\mathcal{D}),\{D_i\}_{i\in\mathbb{I}},U,\phi,\{\kappa_i\}_{i\in\mathbb{I}},\{P_i\}_{i\in\mathbb{I}},\{G_i\}_{i\in\mathbb{I}},\{B_i\}_{i\in\mathbb{I}},\{r_i\}_{i\in\mathbb{I}})$, whose interpretation and properties will be defined in the sequel.

The *set of players* is denoted by $\mathbb{I}$. In order that the definitions of the paper encompassed both games with finitely many players as well as games with infinitely many players we introduce a structure on $\mathbb{I}$ consisting of a $\sigma$-field $\Im$ of its subsets and a measure $\lambda$ on it.

The game is dynamic, played over a discrete *time set* $\mathbb{T}$, without loss of generality $\mathbb{T}=\{t_0,t_0+1,\ldots,T\}$ or $\mathbb{T}=\{t_0,t_0+1,\ldots\}$, which, for uniformity of notation, will be treated as $T=+\infty$. We introduce also the symbol $\overline{\mathbb{T}}$ denoting $\{t_0,t_0+1,\ldots,T+1\}$ for finite $T$ and equal to $\mathbb{T}$ in the opposite case.

The game is played in a *global system* with the *set of states* $\mathbb{X}$. The state of the global system (*state* for short) changes over time in response to players' decisions, constituting a trajectory $X$, whose equation will be stated in the sequel. The set of all potential *trajectories* – functions $X:\overline{\mathbb{T}}\to\mathbb{X}$ – will be denoted by $\mathfrak{X}$.

Besides the global system, player $i$ has his *private state* variable with values in a *set of private states of player* $i$ denoted by $\mathbb{W}_i$. By $\mathbb{W}$ we denote a superset containing all $\mathbb{W}_i$. The vector of private state variables also changes in response to players' decisions, constituting a trajectory $W:\overline{\mathbb{T}}\to\mathbb{W}^{\mathbb{I}}$, whose equation will be stated in the sequel.

At each time $t$ given state $x$ and his private state $w_i$ player $i$ chooses a decision from his *available decision set* $D_i(t,x,w_i)\subset\mathbb{D}$ – the *set of* (potential) *actions*. These available decision sets of player $i$ constitute the *correspondence of available decision sets of player* $i$ $D_i:\mathbb{T}\times\mathbb{X}\times\mathbb{W}\multimap\mathbb{D}$, while all available decision sets constitute a *correspondence of available decision sets* $D:\mathbb{I}\times\mathbb{T}\times\mathbb{X}\times\mathbb{W}\multimap\mathbb{D}$ with nonempty values. We shall also need a $\sigma$-field of subsets of $\mathbb{D}$, denoted by $\mathcal{D}$.

For any time $t$, state $x$ and a vector of private states $w \in \mathbb{W}^{\mathbb{I}}$ we call any $\mathcal{D}$-measurable selection $\delta$ from the correspondence $i \multimap D(i,t,x,w_i)$ a *static profile available at $t$, $x$ and $w$*. The set of all static profiles available at $t$, $x$ and $w$ is denoted by $\Sigma(t,x,w)$. The union of all the sets of static profiles available at various $t$, $x$ and $w$ is denoted by $\Sigma$.

The definitions of a strategy (dynamic strategy) and a profile (dynamic profile) will appear in the sequel, since first we have to define the domains of these functions.

The influence of a static profile on the global state variable is via its statistic. Without loss of generality the same statistic is the only parameter besides player's own strategy influencing the evolution of his private state variable and the value of his payoff. Formally, a *statistic* is a function $U : \Sigma \times \mathbb{X} \overset{\text{onto}}{\to} \mathbb{U}$ for a set $\mathbb{U}$ called the *set of profile statistics* and such that $U(\delta, x) = \gamma\left(\left[\int_{\mathbb{I}} g_k(i,\delta(i),x)d\lambda(i)\right]_{k\in\mathbb{K}}\right)$ for a collection of functions $\{g_k : \mathbb{I} \times \mathbb{D} \times \mathbb{X} \to \mathbb{R}\}_{k\in\mathbb{K}}$ which are $\Im \otimes \mathcal{D}$-measurable for every $x \in \mathbb{X}$ and for every $k$ in the set of indices $\mathbb{K}$, and a function $\gamma : \mathbb{R}^{\mathbb{K}} \to \mathbb{U}$. If $\Delta : \mathbb{T} \to \Sigma$ represents choices of profiles at various time instants and $X$ is a trajectory of the global system, then by $U(\Delta, X)$ we denote the function $u : \mathbb{T} \to \mathbb{U}$ such that $u(t) = U(\Delta(t), X(t))$. The set of all such functions will be denoted by $\mathfrak{U}$.

We do not assume any kind of continuity of the function $\gamma$. In the case of modelling financial markets it is by assumption discontinuous – it is defined, as in section , as a point at which a maximum of a kind of lexicographic ordering on a subset of $\mathbb{R}^{\mathbb{K}}$ is attained.

This class of statistic functions is a generalization of the class of statistic functions used in the previous paper Wiszniewska-Matyszkiel [20], in which $\mathbb{K}$ was finite and $\gamma$ was absent.

Obviously, in the case of games with finitely many players with finite dimensional strategy set, the statistic can be the profile itself.

In a model of stock exchange an obvious candidate for such a statistic is the market price of the asset considered (in the example we shall see that another two coordinates will be useful).

Given a function $u : \mathbb{T} \to \mathbb{U}$, representing the statistics of profiles chosen at various time instants, the global system evolves according to the equation $X(t+1) = \phi(X(t), u(t))$ with the initial condition $X(t_0) = \bar{x}$. We call such a trajectory *corresponding to $u$* and denote it by $X^u$. If $u = U(\Delta, X^u)$, where $\Delta : \mathbb{T} \to \Sigma$ represents a choice of static profiles at various time instants, then, by a slight abuse of notation, we shall denote the trajectory corresponding to $u$ by $X^{\Delta}$ and call it corresponding to $\Delta$ and instead of $U(\Delta, X^{\Delta})$ we write $U(\Delta)$ – the statistic of $\Delta$.

Given functions $u : \mathbb{T} \to \mathbb{U}$ and $d : \mathbb{T} \to \mathbb{D}$, representing, correspondingly, the subsequent statistics of static profiles chosen and subsequent decisions of player $i$, the private system of player $i$ evolves according to the equation $W_i(t+1) = \kappa_i(W_i(t), d(t), X^u(t), u(t))$ with the initial condition $W_i(t_0) = \bar{w}_i$. We call such a trajectory of private system *corresponding to $d$ and $u$* and denote it by $W_i^{d,u}$. If a function $\Delta : \mathbb{T} \to \Sigma$ represents choices of profiles at various time instants, then the trajectory of private state variables $W^{\Delta}$ defined by $\left(W^{\Delta}\right)_i = W_i^{\Delta_i, U(\Delta)}$ is called *corresponding to* $\Delta$. This is another generalization of [20] in which only the global state variable was considered.

At each time $t$ and the state of the global system $x$ and the vector of players' private states $w$ players get instantaneous payoffs. The *instantaneous payoff* of player $i$ is a function $P_i : \mathbb{D} \times \mathbb{U} \times \mathbb{X} \times \mathbb{W} \to \mathbb{R} \cup \{-\infty\}$.

Besides, in the case of finite time horizon players get also *terminal payoffs* (after termination of the game) defined by the functions $G_i : \mathbb{X} \times \mathbb{W} \to \mathbb{R} \cup \{-\infty\}$. For uniformity of notation we take $G_i \equiv 0$ in the case of infinite time horizon.

Players observe some histories of the game, but not the whole profiles. At time $t$ they observe the states $X(s)$ for $s \leq t$ and the statistics $u(s)$ of chosen static profiles for time instants $s < t$. Therefore the set of histories at time $t$ equals $\mathbb{X}^{t-t_0+1} \times \mathbb{U}^{t-t_0}$. In order to simplify notation we introduce the set of all, possibly infinite, histories of the game $\mathbb{H} = \mathbb{X}^{T-t_0+2} \times \mathbb{U}^{T-t_0+1}$ and for such a history $H \in \mathbb{H}$ we denote by $H|_t$ the actual history at time $t$, while by $H(t)$ the pair $(X(t-1), u(t))$.

Given a history observed at time $t$, $H|_t$, players formulate their suppositions about future values of $u$ and $X$, depending on their decision $a$ made at time $t$. This is formalized as a multivalued *correspondence of belief of player* $i$, $B_i : \mathbb{T} \times \mathbb{D} \times \mathbb{H} \multimap \mathbb{H}$ with nonempty values. To reflect the fact that beliefs are based only on observed history, we assume that beliefs $B_i(t, a, H)$ are identical for histories $H$ with identical $H|_t$ and that for all $H' \in B_i(t, a, H)$ we have $H'|_t = H|_t$. For simplicity of some further notation we also assume that for every history $H' \in B_i(t, a, H)$ and $H'' \in \mathbb{H}$ differing from $H'$ only by $u'(t) \neq u''(t)$ we have also $H'' \in B_i(t, a, H)$, which means that the belief correspondence codes no information about the current value of $u$.

In our problem we allow players to have very compound closed loop strategies − dependent on time instant, state, private state and belief at the actual history of the game at this time instant. Formally, a (*dynamic*) *strategy of player* $i$ is a function $S_i : \mathbb{T} \times \mathbb{X} \times \mathbb{W} \times \mathbb{H} \to \mathbb{D}$ such that for each time $t$, state $x$, a private state $w_i$ and history $H$ we have $S_i(t, x, w_i, H) \in D_i(t, x, w_i)$.

Such choices of players' strategies constitute a function
$S : \mathbb{I} \times \mathbb{T} \times \mathbb{X} \times \mathbb{W} \times \mathbb{H} \to \mathbb{D}$. The set of all strategies of player $i$ will be denoted by $\mathfrak{S}_i$.

For simplicity of further notation, for a choice of strategies $S = \{S_i\}_{i \in \mathbb{I}}$ we can consider the open loop form of it $S^{OL} : \mathbb{T} \to \Sigma$, defined by
$S_i^{OL}(t) = S_i(t, X(t), W_i(t), H)$, where $H$ is the history of the game resulting from choosing $S$. It is well defined, whenever the history is well defined (note that the statistic was defined only for measurable selections from players' strategy sets, therefore the statistics at time $t$ is well defined if the function $S_{\cdot}^{OL}(t-1)$ is measurable). Therefore, we restrict the notion (*dynamic*) *profile* (*of players' strategies*) to choices of strategies such that for every $t$ the function $S_{\cdot}^{OL}(t)$ is a static profile available at $t \in \mathbb{T}$, $X^{S^{OL}}(t)$ and $W^{S^{OL}}(t)$. The set of all dynamic profiles will be denoted by $\mathbf{\Sigma}$. Since the choice of a dynamic profile $S$ determines the history of the game, we shall denote this history by $H^S$.

If the players choose a dynamic profile $S$, then the actual *payoff of player* $i$ $\Pi_i : \mathbf{\Sigma} \to \overline{\mathbb{R}}$ in the game depends only on the actions actually chosen by players at subsequent time instants, i.e. the open loop form of the profile, and it is equal to

$$\begin{aligned}\Pi_i(S) = & \textstyle\sum_{t=t_0}^{T} P_i\left(S_i^{OL}(t), U\left(S_i^{OL}(t)\right), X^{S^{OL}}(t), W_i^{S^{OL}}(t)\right) \cdot \left(\frac{1}{1+r_i}\right)^{t-t_0} + \\ & + G_i\left(X^{S^{OL}}(T+1), W_i^{S^{OL}}(T+1)\right) \cdot \left(\frac{1}{1+r_i}\right)^{T+1-t_0}\end{aligned},$$

where $r_i > 0$ is called a *discount rate.*

However, players do not know the whole profile, therefore instead of the actual payoff at each future time instant they can use in their calculations the *anticipated payoff functions* $\Pi_i^e : \mathbb{T} \times \mathbf{\Sigma} \to \overline{\mathbb{R}}$ corresponding to their beliefs at the corresponding time instants (the world "anticipated" is used in the colloquial meaning of "expected", while the world "expected" is not used in order not to cause associations with expected value with respect to some probability distribution).
This function for player $i$ is defined by

$$\begin{aligned}\Pi_i^e(t,S) = & P_i\left(S_i^{OL}(t), U\left(S_i^{OL}(t)\right), X^{S^{OL}}(t), W_i^{S^{OL}}(t)\right) + \\ & + V_i(t+1, W_i^{S^{OL}}(t+1), B_i(t, S_i^{OL}(t), H^S)) \cdot \tfrac{1}{1+r_i},\end{aligned}$$

where $V_i : \overline{\mathbb{T}} \times \mathbb{W}_i \times (\mathfrak{P}(\mathbb{H})\backslash\emptyset)) \to \overline{\mathbb{R}}$, (the *function of guaranteed anticipated value*) represents the present value of the minimal future payoff given his belief correspondence and assuming player $i$ chooses optimally in the future.

Formally, for time $t$, private state $w_i$ and belief $\mathbb{B} \in \mathfrak{P}(\mathbb{H})\backslash\emptyset$ we define

$$V_i(t, w_i, \mathbb{B}) = \inf_{H\in\mathbb{B}} v_i(t, w_i, H),$$

where the function $v_i : \overline{\mathbb{T}} \times \mathbb{W}_i \times \mathbb{H} \to \overline{\mathbb{R}}$ is the present value of the future payoff of player $i$ along a history assuming he chooses optimally in the future: for $t \in \mathbb{T}$ we define it by

$$\begin{aligned}v_i(t, w_i, (X,u)) = & \sup_{d:\mathbb{T}\to\mathbb{D}\ d(\tau)\in D_i(\tau, X(\tau), W_i(\tau))\text{ for }\tau\geq t} \\ & \textstyle\sum_{\tau=t}^{T} \frac{P_i(\tau, d(\tau), u(\tau), X(\tau), W_i(\tau))}{(1+r_i)^{\tau-t}} + \frac{G_i(X(T+1), W_i(T+1))}{(1+r_i)^{T+1-t}}\end{aligned}$$

for $W_i$ defined by $W_i(t) = w_i$ and
$W_i(\tau+1) = \kappa(W_i(\tau), d(\tau), X(\tau), u(\tau))$ for $\tau < T$;
$v_i(T+1, w_i, H) = G_i\left(X(T+1), w_i\right)$.

Note that such a definition of anticipated payoff is inspired by the Bellman equation for calculation of best responses of players to the strategies of the others. For various versions of this equation see e.g. Blackwell [6] or Stokey and Lucas [12].

We also introduce the symbol $\mathfrak{G}_{t,H,w}$ (for $H = (X,u)$), called *subgame with distorted information at* $t$, $x$ and $w$ and denoting the game with the set of players $\mathbb{I}$, the sets of their strategies $D_i(t, X(t), w_i)$ and the payoff functions $\bar{\Pi}_i^e(t, H, \cdot)$ defined by $\bar{\Pi}_i^e(t, H, \delta) = \Pi_i^e(t, S)$ for a profile $S$ such that $S(t) = \delta$ and $H^S|_t = H|_t$ (note that the dependence of $\Pi_i^e(t, \cdot)$ on the profile is restricted to its static profile at time $t$ only and the history $H|_t$, therefore the definition does not depend on the choice of specific $S$ from this class).

Nash equilibria and pre-belief distorted Nash equilibria

One of the basic concepts in game theory, Nash equilibrium, assumes that every player (almost every in case of large games with a measure space of players) chooses a strategy which maximizes his payoff given the strategies of the remaining players.

In order to simplify the notation we shall need the following abbreviation: for a profile $S$ and a dynamic strategy $d$ of player $i$ the symbol $S^{i,d}$ denotes the dynamic profile such that $S_i^{i,d} = d$ and $S_j^{i,d} = S_j$ for $j \neq i$.

**Definition 1.** A profile $S$ is a *Nash equilibrium* if for a.e. $i \in \mathbb{I}$ and for every strategy $d$ of player $i$ we have $\Pi_i(S) \geq \Pi_i(S^{i,d})$.

However, the assumption that a player knows the strategies of the remaining players or at least the statistic of these strategies which influences his payoff, is usually not fulfilled in real life situations. Moreover, even details of the other players' payoff functions or available strategy sets are sometimes not known precisely, while the other players' information at a specific situation is usually unknown. This is especially visible at financial markets.

Therefore, given their beliefs, players maximize their anticipated payoffs.

**Definition 2.** A profile $S$ is a *pre-belief distorted Nash equilibrium* (*pre-BDNE* for short) if for a.e. $i \in \mathbb{I}$, and every strategy $d$ of player $i$ and every $t \in \mathbb{T}$ we have $\Pi_i^e(t, S) \geq \Pi_i^e(t, S^{i,d})$.

If we use the notation introduced in the formulation of the model, then a profile $S$ is a pre-BDNE in $\mathfrak{G}$ if at every time $t$ the static profile $S^{OL}(t)$ is a Nash equilibrium in $\mathfrak{G}_{t,H^S,W^S(t)}$.

In order to state an existence result for games with a nonatomic space of players we have to introduce the following notation.
Let $e(\delta, x) = \left[\int_{\mathbb{I}} g_k(i, \delta(i), x) d\lambda(i)\right]_{k \in \mathbb{K}}$. The functions $\widetilde{P}_i(a, e, x, w_i)$, $\widetilde{\kappa}_i(w_i, a, x, e)$ and $\widetilde{\phi}(x, e)$ will denote $P_i(a, \gamma(e), x, w_i)$, $\kappa_i(w_i, a, x, \gamma(e))$ and $\phi(x, \gamma(e))$, respectively.

**Theorem 3. *Existence of pre-BDNE***
*Let $(\mathbb{I}, \Im, \lambda)$ be a nonatomic measure space and let $\mathbb{K}$ be finite, $\mathbb{D} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$ with the $\sigma$-fields of Borel subsets and let the function $\mathbb{I} \ni i \mapsto \bar{w}_i$ be measurable. Assume that for every $t$, $x$, $w_i$, $H$ and for almost every $i$ the following* continuity-compactness assumptions *hold: the sets $D_i(t, x, w_i)$ are compact, the functions $\widetilde{P}_i(a, e, x, w_i)$ and*
*$V_i(t, \widetilde{\kappa}_i(w_i, a, x, e), B_i(t, a, H))$ are upper semicontinuous in $(a, e)$ jointly while for every $a$ they are continuous in $e$ and for all $k$ the functions $g_k(i, a, x)$ are continuous in $a$ for $a \in D_i(t, x, w_i)$ and assume that for every $t$, $x$, $e$, $H$ the following* measurability assumptions *hold: the graph of $D_\cdot(t, x, \cdot)$ is measurable and the following functions defined on $\mathbb{I} \times \mathbb{W} \times \mathbb{D}$ are measurable $(i, w, a) \mapsto \widetilde{P}_i(a, e, x, w)$, $r_i$, $V_i(t, \widetilde{\kappa}_i(w, a, x, e), B_i(t, a, H))$, $\widetilde{\kappa}_i(w, a, x, e)$ and $g_k(i, a, x)$ for every $k$. Moreover, assume that for every $k$ and $x$ there exists an integrable function $\Gamma : \mathbb{I} \to \mathbb{R}$ such that for every $w$ and every $a \in D_i(t, x, w)$ $|g_k(i, a, x)| \leq \Gamma(i)$. Under these assumptions there exists a pre-BDNE for $B$.*

**Proof.** It is a conclusion from one of theorems on the existence of pure strategy Nash equilibria in games with continuum of players: Wiszniewska-Matyszkiel [14] theorem 3.1 or Balder [4] theorem 3.4.1 applied to the sequence of games $\mathfrak{G}_{t,H,w}$ for any history $H$ such that $H|_t$ is the actual history of the game observed at time $t$ while $w$ describes the private states of players at time $t$.

In order to apply one of those theorems we first have to prove measurability of the function $i \mapsto w_i(t)$ given a measurable initial data and a dynamic profile, which is immediate. It implies that the graph of $D.(t, x, w.)$ (in $\mathbb{I} \times \mathbb{D}$) is measurable and the following functions defined on $\mathbb{I} \times \mathbb{D}$ are measurable: $(i, a) \mapsto \widetilde{P}_i(a, e, x, w_i)$, $r_i$, $V_i(t, \widetilde{\kappa}_i(w_i, a, x, e), B_i(t, a, H))$. ■

Now we return to show some properties of pre-BDNE for a special kind of belief correspondence – the perfect foresight.

**Definition 4.** A belief correspondence $B_i$ of player $i$ is the *perfect foresight* at a profile $S$, if for every $t$, $B_i(t, S_i^{OL}(t), H^S) = \{H^S\}$.

**Theorem 5.** ***Equivalence between pre-BDNE for perfect foresight and Nash equilibria***

*Let $(\mathbb{I}, \Im, \lambda)$ be a nonatomic measure space and let $\sup_{S \in \mathbf{\Sigma}} \Pi^e(t, S)$ and $\inf_{S \in \mathbf{\Sigma}} \Pi^e(t, S)$ be finite for every $t$.*

*a) Let $\bar{S}$ be a Nash equilibrium profile. If $B$ is the perfect foresight at a profile $\bar{S}$ and the profiles $\bar{S}^{i,d}$ for a.e. $i$ and every strategy $d$ of player $i$, then for every $t$ $\bar{S}^{OL}(t) \in \operatorname{Argmax}_{a \in D_i(t, X(t), W_i(t))} \bar{\Pi}_i^e(t, H^{\bar{S}}, (\bar{S}^{OL}(t))^{i,a})$, where the symbol $\delta^{i,a}$ for a static profile $\delta$ denotes the profile $\delta$ with strategy of player $i$ changed to $a$, and $\bar{S}_i^{OL}|\{t+1, \ldots, T\}$ are consistent with the results of the player's optimizations used in the definition of $v_i$, i.e. $\bar{S}_i^{OL}$ is an element of the set*

$$\operatorname{Argmax}_{d: \mathbb{T} \to \mathbb{D}\ d(\tau) \in D_i(\tau, X(\tau), W_i(\tau))\ for\ \tau \geq t} \sum_{\tau=t}^{T} P_i(d(\tau), u(\tau), X(\tau), W_i(\tau)) \cdot \left(\frac{1}{1+r_i}\right)^{\tau - t} + G_i\left(X(T+1), W_i(T+1)\right) \cdot \left(\frac{1}{1+r_i}\right)^{T+1-t}.$$

*b) Every Nash equilibrium profile $\bar{S}$ is a pre-BDNE for a belief correspondence being the perfect foresight at $\bar{S}$ and the profile $\bar{S}^{i,d}$ for a.e. $i$ and every strategy $d$ of player $i$.*

*c) Let $\bar{S}$ be a pre-BDNE for a belief $B$. If $B$ is the perfect foresight at $\bar{S}$ and the profiles $\bar{S}^{i,d}$ for a.e. $i$ and every strategy $d$ of player $i$, then choices of players are consistent with the results of their optimizations used in definition of $v_i$.*

*d) If a profile $\bar{S}$ is a pre-BDNE for a belief $B$ being the perfect foresight at this $\bar{S}$ and $\bar{S}^{i,d}$ for a.e. player $i$ and his every strategy $d$, then it is a Nash equilibrium.*

**Proof.** The proof is similar to analogous result in [20]. Nevertheless, introduction of private state variables makes it much more complicated.

First note that if the measure $\lambda$ is nonatomic, then a choice of strategy by a single player influences neither $u$ nor $X$. It only influences player's private state variable. Therefore, instead of looking for the best response to the profiles of the remaining players' strategies it is enough to look for best responses to the statistic of this profile, equal to the statistic of the whole profile.

In all the cases we shall consider player $i$ outside the set of measure 0 of players for whom the condition of maximizing payoff or expected payoff does not hold.

a) We shall prove that along the perfect foresight path the equation for the expected payoff of player $i$ becomes the Bellman equation for optimization of the actual payoff by player $i$ and $V_i$ coincides with the value function.

Formally, given the profile of the strategies of the remaining players coinciding with $\bar{S}$, with the statistic $u$ and trajectory of the global system $X$, let us define the value function for the decision making problem of player $i$, $\widetilde{V}_i : \mathbb{T} \times \mathbb{W} \to \overline{\mathbb{R}}$.

$\widetilde{V}_i(t, w_i) =$

$= \sup_{d:\mathbb{T}\to\mathbb{D}\ d(\tau)\in D_i(\tau,X(\tau),W_i(\tau)) \text{ for } \tau\geq t} \sum_{\tau=t}^{T} P_i(d(\tau), u(\tau), X(\tau), W_i(\tau)) \cdot \left(\frac{1}{1+r_i}\right)^{\tau-t} +$

$+ G_i\left(X(T+1), W_i(T+1)\right) \cdot \left(\frac{1}{1+r_i}\right)^{T+1-t}$,

where $W_i$ is defined recursively by $W_i(t) = w_i$ and $W_i(\tau+1) = \kappa(W_i(\tau), d(\tau), X(\tau), u(\tau))$.

In the finite horizon case $\widetilde{V}_i$ fulfills the Bellman equation

$\widetilde{V}_i(t, w_i) = \sup_{a\in D_i(t,X(t),w_i)} P_i(a, u(t), X(t), w_i) + \widetilde{V}_i(t+1, \kappa_i(w_i, a, X(t), u(t))) \cdot \left(\frac{1}{1+r_i}\right)$

with the terminal condition

$\widetilde{V}_i(T+1, w_i) = G_i\left(X(T+1), w_i\right)$.

In the infinite horizon case $\widetilde{V}_i$ also fulfills the Bellman equation, but the terminal condition sufficient for the solution of the Bellman equation to be the value function is different. The simplest one is $\lim_{t\to\infty} \widetilde{V}_i(t, W_i(t)) \cdot \left(\frac{1}{1+r_i}\right)^{t-t_0} = 0$ for every admissible $W_i$ (see e.g. Blackwell [6] or Stokey and Lucas [12]). In this paper it holds by the assumption that the payoffs are bounded.

If we write the formula for $\widetilde{V}_i$ from the definition in the r.h.s. of the Bellman equation, then we get

$\widetilde{V}_i(t, w_i) = \sup_{a\in D_i(t,X(t),w_i)} P_i(a, u(t), X(t), w_i) + \left(\frac{1}{1+r_i}\right) \cdot$

$\sup_{d:\mathbb{T}\to\mathbb{D}\ d(\tau)\in D_i(\tau,X(\tau),W_i(\tau)) \text{ for } \tau\geq t+1} \sum_{\tau=t+1}^{T} P_i(d(\tau), u(\tau), X(\tau), W_i(\tau)) \cdot \left(\frac{1}{1+r_i}\right)^{\tau-t-1}$

$+ G_i\left(X(T+1), W_i(T+1)\right) \cdot \left(\frac{1}{1+r_i}\right)^{T+1-t}$ subject to

$W_i(t) = w_i$, $W_i(t+1) = \kappa(W_i(t), a, X(t), u(t))$ and $W_i(\tau+1) = \kappa(W_i(\tau), d(\tau), X(\tau), u(\tau))$ for $\tau > t$.

Note that the last supremum is equal to $\widetilde{V}_i(t+1, \kappa_i(w_i, a, X(t), u(t)))$, but also to $v_i(t+1, \kappa(w_i, a, X(t), u(t)), (X, u))$.

Since $(X, u)$ is the only history in the belief correspondence along both $\bar{S}$ and all profiles $\bar{S}^{i,d}$, it is also equal to $V_i(t+1, \kappa(w_i, a, X(t), u(t)), \{(X, u)\})$.

Therefore $\widetilde{V}_i(t, w_i) = \sup_{a\in D_i(t,X(t),w_i)} P_i(a, u(t), X(t), w_i) +$

$+ \left(\frac{1}{1+r_i}\right) \cdot V_i(t+1, \kappa(w_i, a, X(t), u(t)), B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})) =$

$= \sup_{a\in D_i(t,X(t),w_i)} \Pi_i^e(t, \bar{S}^{i,d_{t,a}})$, where by $d_{t,a}$ we denote such a strategy of player $i$ that $d(t, X(t), w_i, H^{\bar{S}}) = a$ and at any other point of the domain it coincides with $\bar{S}_i$.

Let us note that for all $t$ the set

$\mathrm{Argmax}_{d:\mathbb{T}\to\mathbb{D}\ d(\tau)\in D_i(\tau,X(\tau),W_i(\tau)) \text{ for } \tau\geq t} \sum_{\tau=t}^{T} P_i(d(\tau), u(\tau), X(\tau), W_i(\tau)) \cdot \left(\frac{1}{1+r_i}\right)^{\tau-t} +$

$+G_i\left(X(T+1),W_i(T+1)\right)\cdot\left(\frac{1}{1+r_i}\right)^{T+1-t}$ is both the set of open loop forms of strategies of player $i$ being best responses of player $i$ to the strategies of the remaining players along the profiles $\bar{S}$ and $\bar{S}^{i,d}$ and the set at which the supremum in the definition of the function $v_i$ for a history being the perfect foresight along $\bar{S}$ and $\bar{S}^{i,d}$ is attained.
We only have to show that $\bar{S}_i(t)\in\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\bar{\Pi}_i^e(t,H^{\bar{S}},\bar{S}^{OL}(t)^{i,a})$.
By the definition this set is equal to
$\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(a,u(t),X(t),w_i)+$
$+\left(\frac{1}{1+r_i}\right)\cdot V_i(t+1,W_i^{\bar{S}^{OL}}(t+1),B_i(t,\bar{S}_i^{OL}(t),H^{\bar{S}}))=$
$=\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(t,a,u(t),X(t),w_i)+\left(\frac{1}{1+r_i}\right)\cdot\widetilde{V}_i(t+1,W_i^{\bar{S}^{OL}}(t+1))$, which, by the Bellman equation, defines the value of the best response at time $t$, which contains $\bar{S}_i(t)$, since $\bar{S}$ is an equilibrium profile.

b) An immediate conclusion from a)

d) Given $\bar{S}$, we consider $\widetilde{V}_i$ defined as in the proof of a).
By the definition of pre-BDNE
$\bar{S}_i(t)\in\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\bar{\Pi}^e(t,H^{\bar{S}},\bar{S}^{OL}(t)^{i,a})=$
$=\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(a,u(t),X(t),w_i)+$
$+\left(\frac{1}{1+r_i}\right)\cdot V_i(t+1,\kappa(w_i,a,X(t),u(t)),B_i(t,\bar{S}_i^{OL}(t),H^{\bar{S}}))=$
$=\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(a,u(t),X(t),w_i)+$
$+\left(\frac{1}{1+r_i}\right)\cdot V_i(t+1,\kappa(w_i,a,X(t),u(t)),\{(X,u)\})=$
$=\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(t,a,u(t),X(t),w_i)+$
$+\left(\frac{1}{1+r_i}\right)\cdot\max_{d:\mathbb{T}\to\mathbb{D}\ d(\tau)\in D_i(\tau,X(\tau),W_i(\tau))\text{ for }\tau\geq t+1}\sum_{\tau=t+1}^{T}P_i(d(\tau),u(\tau),X(\tau),W_i(\tau))$
$\cdot\left(\frac{1}{1+r_i}\right)^{\tau-t}+G_i\left(X(T+1),W_i(T+1)\right)\cdot\left(\frac{1}{1+r_i}\right)^{T+1-t}$ (for $W_i$ defined by $W_i(t)=w_i$ and $W_i(\tau+1)=\kappa(W_i(\tau),d(\tau),X(\tau),u(\tau))$ for $\tau>t$)
$=\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(a,u(t),X(t),w_i)+\left(\frac{1}{1+r_i}\right)\cdot\widetilde{V}_i(t+1,\kappa(w_i,a,X(t),u(t)))$.

If we add the fact that
$\widetilde{V}_i(t,w_i)=\max_{a\in D_i(t,X(t),w_i)}P_i(a,u(t),X(t),w_i)+\left(\frac{1}{1+r_i}\right)\cdot\widetilde{V}_i(t+1,\kappa(w_i,a,X(t),u(t)))$, then, by the Bellman condition, the set
$\mathrm{Argmax}_{a\in D_i(t,X(t),w_i)}\ P_i(a,u(t),X(t),w_i)+\left(\frac{1}{1+r_i}\right)\cdot\widetilde{V}_i(t+1,\kappa(w_i,a,X(t),u(t)))$ represents the value of the optimal choice of player $i$ at time $t$, given $X$, $u$ and $w_i$. Since we have this property for a.e. $i$, the profile defined in this way is a Nash equilibrium.

c) By d) and a). ■

We can also prove an equivalence theorem for repeated games – dynamic games without state variables, which can be modelled as both global and private state sets being singletons.
Not surprisingly, analogous theorem from Wiszniewska-Matyszkiel [20] can be cited and the proof does not change.

**Theorem 6.** *Let $\mathfrak{G}$ be a repeated game with a belief correspondence not dependent on players' own strategies in which payoffs and anticipated payoffs are bounded for a.e. player.*

*a) If* $(\mathbb{I}, \Im, \lambda)$ *is a nonatomic measure space, then a profile* $S$ *is a pre-BDNE if and only if it is a Nash equilibrium.*

*b) Every profile* $S$ *with strategies of a.e. player being independent of histories is a pre-BDNE if and only if it is a Nash equilibrium.*

## Self-verification of beliefs and BDNE

The concept of pre-BDNE lacks a kind of guarantee that after some stage of the game players can still have the same beliefs, i.e. that beliefs are not contradicted by players' observations.

First we state what we mean by potential and perfect self-verification.

**Definition 7.** a) A collection of beliefs $\{B_i\}_{i\in\mathbb{I}}$ is *perfectly self-verifying* if for every pre-BDNE $\bar{S}$ for $B$ for a.e. $i \in \mathbb{I}$ we have $H^{\bar{S}} \in B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})$.

b) A collection of beliefs $\{B_i\}_{i\in\mathbb{I}}$ is *potentially self-verifying* if there exists a pre-BDNE $\bar{S}$ for $B$ such that for a.e. $i \in \mathbb{I}$ we have $H^{\bar{S}} \in B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})$.

c) A collection of beliefs $\{B_i\}_{i\in\mathbb{J}}$ of a set of players $\mathbb{J}$ is *perfectly self-verifying* against beliefs of the other players $\{B_i\}_{i\in\setminus\mathbb{J}}$ if for every pre-BDNE $\bar{S}$ for $\{B_i\}_{i\in\mathbb{I}}$ for a.e. $i \in \mathbb{J}$ we have $H^{\bar{S}} \in B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})$.

d) A collection of beliefs $\{B_i\}_{i\in\mathbb{J}}$ of a set of players $\mathbb{J}$ is *potentially self-verifying* against beliefs of the other players $\{B_i\}_{i\in\setminus\mathbb{J}}$ if there exists a pre-BDNE $\bar{S}$ for $\{B_i\}_{i\in\mathbb{I}}$ such that for a.e. $i \in \mathbb{J}$ we have $H^{\bar{S}} \in B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})$.

And, finally, the notion of BDNE.

**Definition 8.** a) A profile $\bar{S}$ is a *belief-distorted Nash equilibrium (BDNE) for* a collection of beliefs $B = \{B_i\}_{i\in\mathbb{I}}$ if $\bar{S}$ is a pre-BDNE for $B$ and for a.e. $i$ and every $t$ $H^{\bar{S}} \in B_i(t, \bar{S}_i^{OL}(t), H^{\bar{S}})$

b) A profile $\bar{S}$ is a *belief-distorted Nash equilibrium* (*BDNE*) if there exists a collection of beliefs $B = \{B_i\}_{i\in\mathbb{I}}$ such that $\bar{S}$ is a BDNE for $B$.

**Remark 9.** Theorems 5 and 6 remain valid with pre-BDNE replaced by BDNE. ■

## Simplified stock exchange

Here we present an example of games with distorted information – a model of stock exchange simplified in order to avoid enormous complexity inherent to real world large systems. More complex models of stock exchange with more than one asset sold, and coping with the problem of distorted information and self-verification of various prognostic techniques against another prognostic techniques, including random ones, were considered by the author in [19].

We examine a model of stock exchange with two assets: money $m$ of interest rate $r$ and a share $s$ traded at the stock exchange. This share pays a deterministic dividend $\{A_t\}_{t\in\mathbb{N}}$. Both assets are infinitely divisible. The transaction cost is linear with a rate $C > 0$. The horizon of players' optimization is $T = +\infty$.

The set of players is either the unit interval with the Lebesgue measure or a large finite set with the normalized counting measure.

Each player $i$ has an initial portfolio of assets $(\bar{m}_i, \bar{s}_i) \in \mathbb{R}^2_+$ with at least one coordinate strictly positive. Portfolios constitute private states of players $w_i = (m_i, s_i) \in \mathbb{R}^2_+$. Prices are, for simplicity of notation, positive integers.

We define the state of the global system being the price at the previous period $X(t) = p(t-1)$, with $p(t_0 - 1) > 0$ given. This auxiliary state variable is introduced, since price at time $t$, as we shall see in the sequel, depends on previous price.

At each stage players state their bid and ask prices $p_i^B, p_i^S \in \mathbb{N}$ – at which they want to buy or sell, respectively, and which constitute their decisions with the available decisions correspondence $D_i \equiv \mathbb{D} = \mathbb{N}^2$, where the first coordinate denotes $p_i^B$ while the second $p_i^S$. These, together with players' actual portfolios, constitute orders – in their orders players buy or sell as much as they can. There are no other constraints for orders. The statistic function will be three dimensional with the actual market price as the first coordinate. The market price $p$ is a function of players' decisions as follows.

Given a profile of players strategies at a fixed time instant, first the aggregate supply and demand functions $AS : \mathbb{N} \to \mathbb{R}_+$ and $AD : \mathbb{N} \to \overline{\mathbb{R}}_+$ are calculated. The *aggregate* (or *market*) *supply* at a price $q$ is defined by

$$AS(q) = \int_{\mathbb{I}} s_i(t) \cdot \mathbf{1}_{p_i^S(t) \le q} d\lambda(i),$$

while the *aggregate* (or *market*) *demand* at a price $q$ is defined by

$$\begin{aligned} AD(q) &= \int_{\mathbb{I}} \frac{m_i}{q \cdot (1+C)} \cdot \mathbf{1}_{p_i^B(t) \ge q} d\lambda(i) \text{ for } q > 0, \\ AD(0) &= +\infty, \end{aligned}$$

where the symbol $\mathbf{1}_{condition}$ denotes 1 when the condition is fulfilled and 0 otherwise. If we consider a finite set of players with the counting measure, then the integral is simply the sum.

The market mechanism considered in the paper at each time $t \ge t_0$ returns a price, called the *market price*, contained in the set $[(1-h) \cdot p(t-1), (1+h) \cdot p(t-1)] \cap \mathbb{N} \backslash \{0\}$, where $h > 0$, being a constraint for variability of prices, is large compared to $r$.

The procedure of calculating the market price operates as follows.

First we look for a strictly positive price maximizing a lexicographic order with criteria, starting from the most important:

1. *maximizing volume* i.e. the function $Vol(q) = \min(AS(q), AD(q))$;
2. *minimizing disequilibrium* i.e. the function $Dis(q) = |AS(q) - AD(q)|$;
3. *minimizing the number of shares in selling orders with price limit less then q and buying orders with price limits higher than q* i.e. the function $N(q) = (AS(q-1) - AD(q))^+ + (AD(q+1) - AS(q))^+$, where the symbol $(x)^+ = \max(0, x)$;
4. *minimizing the absolute value of the difference between the q and the previous price* i.e. $|q - p(t-1)|$.

If the resulting $q$ is not in the interval $[(1-h) \cdot p(t-1), (1+h) \cdot p(t-1)]$, then we project it on $[(1-h) \cdot p(t-1), (1+h) \cdot p(t-1)] \cap \mathbb{N} \backslash \{0\}$.

The resulting unique price constitutes the first of three coordinates of the statistic function.

The procedure is copied from the regulations of the Warsaw Stock Exchange for the single price auction system, but it is used also at other stock exchanges.

In order that the model was complete, in case when the supply at the market price is not equal to the demand, each of the orders on the excess side is reduced at the same rate.

The buying orders are multiplied by

$BR(t) = \begin{cases} 1 & \text{if } AD(p(t)) \leq AS(p(t)), \\ \frac{AS(p(t))}{AD(p(t))} & \text{otherwise,} \end{cases}$

while the selling orders by

$SR(t) = \begin{cases} 1 & \text{if } AD(p(t)) \geq AS(p(t)), \\ \frac{AD(p(t))}{AS(p(t))} & \text{otherwise.} \end{cases}$

These $BR$ and $SR$ constitute the remaining two coordinates of the statistic.

The instantaneous payoff is equal to the change of money stock exactly at the time instant considered: $P_i(t, (p_i^B, p_i^S), (p, BR, SR), x, (m_i, s_i)) =$
$= SR \cdot s_i \cdot p \cdot (1 - C) \cdot \mathbf{1}_{p_i^S \leq p} - BR \cdot m_i \cdot \mathbf{1}_{p_i^B \geq p} + A_t \cdot s_i$.

The payoff at a profile $S = \left\{(p_i^B, p_i^S) : \mathbb{T} \to \mathbb{N}^2\right\}_{i \in \mathbb{I}}$ is, therefore, equal to

$\Pi_i(S) = \sum_{t=t_0}^{\infty} \frac{SR(t) \cdot s_i(t) \cdot p(t) \cdot (1-C) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} - BR(t) \cdot m_i(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} + A_t \cdot s_i(t)}{(1+r)^t}$.

Now we can complete the definition of our game by defining the behaviour of the state variables.

The global system fulfills $X(t+1) = p(t)$, which determines $\phi$ in the obvious way, while the private state variables change as follows.

Money fulfil $m_i(t+1) = (1+r) \cdot \Big(m_i(t)(1 - BR(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)}) +$
$+ s_i(t) \cdot (SR(t) \cdot p(t) \cdot (1 - C) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} + A_t)\Big)$,

while shares $s_i(t+1) = s_i(t) - SR(t) \cdot s_i(t) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} + BR(t) \cdot \frac{m_i(t)}{(1+C) \cdot p(t)} \cdot \mathbf{1}_{p_i^B(t) \geq p(t)}$, which defines the functions $\kappa_i$.

In this game we shall consider various belief correspondences.

### Threshold prices and pre-BDNE for simplified stock exchange

For a history $H$ and a time instant $t$ we define $\bar{p}_i^S(t) \in \mathbb{N}$, called the *selling threshold price of player $i$* who has $s_i(t) > 0$, as the minimum of the prices such that for all histories in the current belief the anticipated payoff for selling at this price is not less then the anticipated payoff for not selling at all at time $t$, and $\bar{p}_i^B(t) \in \mathbb{N} \cup \{+\infty\}$, called the *buying threshold price of player $i$* who has $m_i(t) > 0$, as the maximum of the prices such that for all histories in the current belief the anticipated payoff for buying at this price is not less then the anticipated payoff for not buying at all at time $t$.

Formally, we have the following definition.

**Definition 10. Threshold prices**

Let us consider a time instant $t$, a history $H$ and player $i$ who has $w_i = W_i(t)$ resulting from some previous realization of the profile. We introduce an auxiliary anticipated payoff $\hat{\Pi}_i^e$ which does not take into account player's influence on the

current statistic of the profile $\hat{\Pi}_i^e : \mathbb{T} \times \mathbb{H} \times \mathbb{N} \times \mathbb{D} \times \mathbb{W} \to \overline{\mathbb{R}}$ by
$\hat{\Pi}_i^e(t, H, p, a, w_i) = P_i(t, a, (p, 1, 1), x, w_i) + \frac{V_i(t+1, \kappa_i(w_i, a, x, (p,1,1)), B_i(t,a,H))}{1+r}$ for some $x$. Note that in our case the functions considered are independent on $x$, so the definition is correct.

If player $i$ has $s_i(t) > 0$, then the price $\bar{p}_i^S(t) \in \mathbb{N} \cup \{+\infty\}$ defined by
$\bar{p}_i^S(t) = \min\{p_i^S \in \mathbb{N}\backslash\{0\} : \hat{\Pi}_i^e\left(t, H, p, \left(p_i^B, p_i^S\right), w_i\right) \geq \hat{\Pi}_i^e\left(t, H, p, \left(p_i^B, p+1\right), w_i\right)$ for every $p_i^B, p \in \mathbb{N}\}$ if this set is nonempty,
$\bar{p}_i^S(t) = +\infty$ otherwise,
is called the *selling threshold price of player $i$*.

If player $i$ has $m_i(t) > 0$, then the price $\bar{p}_i^B(t) \in \mathbb{N} \cup \{+\infty\}$ defined by
$\bar{p}_i^B(t) = \max\{p_i^B \in \mathbb{N} : \hat{\Pi}_i^e\left(t, H, p, \left(p_i^B, p_i^S\right), w_i\right) \geq \hat{\Pi}_i^e\left(t, H, p, \left(p-1, p_i^S\right), w_i\right)$ for every $p_i^S, p \in \mathbb{N}\}$
is called the *buying threshold price of player $i$*. This definition is correct because the set is nonempty – since the inequality holds for $p_i^B = 0$.

We are interested in the existence and properties of the threshold prices.

**Proposition 11.** *Consider a belief $B$ independent on players' own decisions and the game $\mathfrak{G}_{t,H^S,W(t)}$ with $W(t)$ such that for player $i$ $m_i(t), s_i(t) > 0$ and $\int_{\mathbb{I}\backslash\{i\}} m_j(t) d\lambda(j), \int_{\mathbb{I}\backslash\{i\}} s_j(t) d\lambda(j) > 0$.*

*a) Both $\bar{p}_i^B(t)$ and $\bar{p}_i^S(t)$ are well defined and either $\bar{p}_i^B(t) < \bar{p}_i^S(t)$ or they are both $+\infty$.*

*b) Consider profiles of the form $\delta = \left\{(p_i^B(t), p_i^S(t))\right\}_{i \in \mathbb{I}}$.*

*(i) If $p_i^S(t) < \bar{p}_i^S(t)$, $\bar{p}_i^S(t) \geq (1-h) \cdot p(t-1)$ and $p_i^S(t) \leq (1+h) \cdot p(t-1)$, then $\bar{\Pi}_i^e\left(t, H^S, \delta\right) \leq \bar{\Pi}_i^e\left(t, H^S, \delta^{i,(p_i^B(t), \bar{p}_i^S(t))}\right)$, with strict inequality for at least one $\delta$.*

*(ii) If $p_i^B(t) > \bar{p}_i^B(t)$, $\bar{p}_i^B(t) \leq (1+h) \cdot p(t-1)$ and $p_i^B(t) \geq (1-h) \cdot p(t-1)$, then $\bar{\Pi}_i^e\left(t, H^S, \delta\right) \leq \bar{\Pi}_i^e\left(t, H^S, \delta^{i,(\bar{p}_i^B(t), p_i^S(t))}\right)$, with strict inequality for at least one $\delta$.*

*c) Assume a continuum of players and profiles of the form $\delta = \left\{(p_i^B(t), p_i^S(t))\right\}_{i \in \mathbb{I}}$. For every $\delta$ we have $\bar{\Pi}_i^e\left(t, H^S, \delta\right) \leq \bar{\Pi}_i^e\left(t, H^S, \delta^{i,(p_i^B(t), \bar{p}_i^S(t))}\right)$ and
$\bar{\Pi}_i^e\left(t, H^S, \delta\right) \leq \bar{\Pi}_i^e\left(t, H^S, \delta^{i,(\bar{p}_i^B(t), p_i^S(t))}\right)$.*

**Sketch of the proof** a) First note that $V_i$ is strictly increasing in both $s_i$ and $m_i$.

Assume that the inequality does not hold at time $t$. This means that there exists a price $\bar{p} \in [\bar{p}_i^S(t), \bar{p}_i^B(t)]$.

Let player $i$ choose his threshold prices. If the market price is $\bar{p}$, then player $i$ both buys and sells at time $t$ and pays commission twice. So he can increase his anticipated payoff $\hat{\Pi}^e$ by either inreasing $p_i^S(t)$ or by decreasing $p_i^B(t)$. To prove this, we can increase the set of possible strategies allowing players to buy or sell using only a fraction of their assets and change the model respectively. Note that the actual strategies in our game can be viewed as extremal points of such an enlarged strategy set corresponding to fractions 0 and 1.

After this change the fact that buying and selling nontrivially at the same time instant decreases payoff is immediate. Because of linearity of the model, if there exists an optimal strategy, then there exists an optimal strategy consisting of extreme points.

b) (i) The inequality "$\leq$" is from definition and by the fact that choosing $p_i^S(t)$ lower than $\bar{p}_i^S(t)$ can influence price or reductions only if the resulting price is less than $\bar{p}_i^S(t)$ and resulting $SR$ is positive.
Tedious analysis of possible cases shows that a situation in which player who wants to buy shares at lower price will not gain using such a strategy.

The strict inequality holds for a profile with $p(t) = \bar{p}_i^S(t)$ and $SR(t) \neq 0$ (which is admissible);

(ii) analogously.

c) In games with continuum of players chosing a strategy by a player does not influence neither price, nor reductions.

The cases which are not covered by b) can be proven by the fact that either nothing changes or in such cases player can only decrease his payoff because of not buying or not selling at a price at which it leads to increase of anticipated payoff. ■

**Corollary 12.** *In games with continuum of players and beliefs independent of player's own choice a profile consisting of pairs of respective threshold prices is a pre-BDNE.* ■

### .1 Fundamental beliefs

In the "obvious" Nash equilibria, in which players believe that prices are equal to the fundamental value $F_t = \sum_{s=t+1}^{T} A_s \cdot (\frac{1}{1+r})^{s-t}$, prices are close to the fundamental value as follows.

First, lets us formally define what we mean by fundamental beliefs.

**Definition 13.** A belief $B_i$ is called a *fundamental belief* if for every $t$, every $H$ and every $a \in \mathbb{D}$
$B_i(t, a, H) \subset \{(X, (p, BR, SR)) : p(t) = F_t, X(t) = p(t-1) \text{ for all } t\}$.

Of course, such beliefs only make sense if for every $t$ $F_t \in \mathbb{N}$. Otherwise, we can substitute $p(t) = F_t$ by $|p(t) - F_t| \leq \frac{1}{2}$ and we shall obtain similar results with similar proofs.

Formally, we state the following proposition.

**Proposition 14.** ***Beliefs based on fundamental analysis***

*Assume that a.e. player has fundamental beliefs.*

*For every time instant $t$ such that*
$\left(\left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil\right) \cap [(1-h) \cdot p(t-1), (1+h) \cdot p(t-1)] \cap \mathbb{N} \backslash \{0\} \neq \emptyset$ *and such that the set of players with positive $m_i(t)$ and the set of players of positive $s_i(t)$ are of positive measure we have*

*a) The threshold prices are $\bar{p}_i^B(t) = \left\lfloor \frac{F_t}{1+C} \right\rfloor$ and $\bar{p}_i^S(t) = \left\lceil \frac{F_t}{1-C} \right\rceil$.*

*b) Every profile such that $p_i^B(t) \leq \left\lceil \frac{F_t}{1-C} \right\rceil$ and $p_i^S(t) \geq \left\lfloor \frac{F_t}{1+C} \right\rfloor$ for a.e. player $i$ is a pre-BDNE for fundamental beliefs.*
*At this profile $p(t) \in \left( \left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil \right)$ and there is no trade (i.e. $Vol(p(t)) = 0$).*

**Proof.** Take any history $H$ and player $i$ for whom both $m_i(t), s_i(t) > 0$.

a) We shall prove $\bar{p}_i^B(t) = \left\lfloor \frac{F_t}{1+C} \right\rfloor$ and $\bar{p}_i^S(t) = \left\lceil \frac{F_t}{1-C} \right\rceil$ are the threshold prices.

A simple comparison between trade and no trade strategies shows that for every decision $a$ $V_i(t+1, (m_i(t+1), s_i(t+1)), B_i(t, a, H)) = s_i(t+1) \cdot (1+r) \cdot F_t$ and the maximum is attained for a strategy at which player $i$ does not trade.
Therefore, in order to calculate the threshold prices we have to maximize
$SR(t) \cdot s_i(t) \cdot p(t) \cdot (1-C) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} - BR(t) \cdot m_i(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} + A_t \cdot s_i(t) +$
$+ \frac{1}{1+r} \cdot s_i(t+1) \cdot (1+r) \cdot F_t =$
$= SR(t) \cdot s_i(t) \cdot p(t) \cdot (1-C) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} - BR(t) \cdot m_i(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} + A_t \cdot s_i(t) +$
$+ \left( s_i(t) - SR(t) \cdot s_i(t) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} + BR(t) \cdot \frac{m_i(t)}{p(t) \cdot (1+C)} \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} \right) \cdot F_t =$
$= SR(t) \cdot s_i(t) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} \cdot (p(t) \cdot (1-C) - F_t) +$
$BR(t) \cdot m_i(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} \cdot \left( \frac{F_t}{p(t) \cdot (1+C)} - 1 \right) + A_t \cdot s_i(t) + s_i(t) \cdot F_t$
for $BR(t) = SR(t) = 1$.

The last two terms of the maximized function are independent of player's own decision at time $t$, therefore we can equivalently take into account maximization of the function
$f(p_i^B(t), p_i^S(t)) = 1 \cdot s_i(t) \cdot \mathbf{1}_{p_i^S(t) \leq p(t)} \cdot (p(t) \cdot (1-C) - F_t) +$
$+ \left( 1 \cdot m_i(t) \cdot \mathbf{1}_{p_i^B(t) \geq p(t)} \cdot \left( \frac{F_t}{p(t) \cdot (1+C)} - 1 \right) \right)$.

If $p(t) < \frac{F_t}{(1-C)}$, then $p(t) \cdot (1-C) - F_t < 0$, therefore the first term of $f$ is at most 0 and the maximal value is attained if player $i$ chooses a price at which he does not sell i.e. $p_i^S(t) > p(t)$. Since it holds for all $p(t) < \frac{F_t}{(1-C)}$, we get $p_i^S(t) \geq \frac{F_t}{(1-C)}$. Since prices are integers, we have $p_i^S(t) \geq \left\lceil \frac{F_t}{(1-C)} \right\rceil$.
If $p(t) \geq \frac{F_t}{(1-C)}$, then $p(t) \cdot (1-C) - F_t \geq 0$, therefore the first term of $f$ is nonnegative. It is 0 if $p_i^S(t) > p(t)$ and for $p(t) > \frac{F_t}{(1-C)}$ it is strictly positive whenever $p_i^S(t) \leq p(t)$. Therefore $\bar{p}_i^S(t) = \left\lceil \frac{F_t}{(1-C)} \right\rceil$.

An analogous reasoning applies to the second term and it proves that $\bar{p}_i^B(t) = \left\lfloor \frac{F_t}{(1+C)} \right\rfloor$.

b) We have to prove that such a strategy of player $i$ is a best response to the strategies of the remaining players.

Assume that all players besides $i$ choose $\left( \left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil \right)$. If player $i$ also chooses this strategy, then he does not trade at time $t$, and his anticipated payoff is $s_i(t) \cdot (A_t + F_t)$. If he chooses any strategy with $p_i^B(t) < \left\lceil \frac{F_t}{1-C} \right\rceil$ and $p_i^S(t) > \left\lfloor \frac{F_t}{1+C} \right\rfloor$, he does not change his payoff since there is still no trade. If he chooses any strategy with $p_i^S(t) \leq \left\lfloor \frac{F_t}{1+C} \right\rfloor$ then the first term of $f$ is negative and if he chooses $p_i^B(t) \geq \left\lceil \frac{F_t}{1-C} \right\rceil$, then the second term of $f$ is negative, while the other term is, as we have just proven, at most 0, therefore he only decreases his payoff.

We shall calculate the price at the profile at which players choose $\left( \left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil \right)$: the market price $p(t)$ will be the price within the interval $\left( \left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil \right)$ closest to $p(t-1)$ if this interval has a nonempty intersection with $[(1-h) \cdot p(t-1), (1+h) \cdot p(t-1)] \cap \mathbb{N} \backslash \{0\}$. For this price there is no trade. ■

## .2 Speculative bubbles

However, we are interested in more counter-intuitive pre-BDNE, which correspond to speculative bubbles. They usually reflect strong trends in technical analysis. In the sequel we shall prove that belief in strong trends may be self verifying.

**Proposition 15.** ***Beliefs in strong trends***

*Assume $B$ is such that for every $t$ at which a nonnegligible set of players $i$ has $m_i(t) > 0$, and such that for every action $a$ of a.e. player $i$ with positive $m_i(t)$ or $s_i(t)$, every history $H$ and every history in $B_i(t, a, H)$,*

*1. there exists $\bar{\tau} > t$ such that $\frac{p(\bar{\tau}) \cdot (1-C)}{(1+r)^{\bar{\tau}-t}} > (\lfloor p(t-1) \cdot (1+h) \rfloor + 1) \cdot (1+C)$ and $SR(\bar{\tau}) = \{1\}$ and*

*2. for every $t < \tau \leq \bar{\tau}$ $p(\tau) \geq p(\tau-1) \cdot (1+r)$.*

*Assume also that for this $B$ the anticipated payoff is always finite.*

*Then*

*a) The threshold prices fulfil $\bar{p}_i^B(t), \bar{p}_i^S(t) > \lfloor (1+h) \cdot p(t-1) \rfloor$.*

*b) Every profile such that $p_i^B(t) \geq \lfloor (1+h) \cdot p(t-1) \rfloor$ and $p_i^S(t) > \lfloor (1+h) \cdot p(t-1) \rfloor$ is a pre-BDNE for $B$.*
*At this pre-BDNE $p(t) = \lfloor (1+h) \cdot p(t-1) \rfloor$, $BR(t) = 0$ and $SR(t) = 1$ for every $t$.*

*c) At every pre-BDNE we have for every $t$ and a.e. $i$ with positive $s_i(t)$ $p_i^S(t) > p(t)$ and there is no trade.*

*d) In the case of continuum of players at every pre-BDNE we have also for every $t$*
*$p(t) = \min \left( \lfloor (1+h) \cdot p(t-1) \rfloor, \operatorname{essinf}_{s_i(t)>0} p_i^S(t) - 1 \right)$, $BR(t) = 0$, $SR(t) = 1$.*

**Proof.** a) The threshold prices for these beliefs fulfil $\bar{p}^B(t), \bar{p}_i^S(t) > \lfloor (1+h) \cdot p(t-1) \rfloor$. It is enough to prove this for $\bar{p}^B(t)$, since the second inequality results from the first one by proposition 11.

We prove this by comparing a non-trade strategy with a strategy at which player $i$ buys at the market price at time $t$ and he sells at time $\bar{\tau} > t$ for which $\inf_{(X,(p,BR,SR)) \in B_i(t,a,H): SR(\bar{\tau})=1} \frac{p(\bar{\tau})}{(1+r)^{\bar{\tau}-t}}$ is maximal. This maximum is attained, since otherwise the anticipated payoff is infinite, and it constitutes the optimal $d$ in the definition of $v_i$.
We get the required inequality, whatever are $p(t)$, $SR(t)$, $BR(t)$ and the other price limit at time $t$.

b) If we consider such a profile, then any strategy with $p_i^B(t) \geq \lfloor (1+h) \cdot p(t-1) \rfloor$ and $p_i^S(t) > \lfloor (1+h) \cdot p(t-1) \rfloor$ maximizes the anticipated payoff of $i$.

For such a profile of strategies the market mechanism returns the price
$\lfloor (1+h)\cdot p(t-1)\rfloor$, $BR(t)=0$ and $SR(t)=1$ for every $t$.

c) Assume that $p_i^S(t) \le \lfloor (1+h)\cdot p(t-1)\rfloor$ for $i$ in a subset of positive measure of the set of players for which $s_i(t)>0$.

Then if $p(t) \in \left[p_i^S(t), \lfloor (1+h)\cdot p(t-1)\rfloor\right]$ and $SR(t)>0$, player $i$ can increase his anticipated payoff by changing $p_i^S(t)$ to $\lfloor (1+h)\cdot p(t-1)\rfloor + 1$.

Let us check, whether $SR(t)=0$ is possible. In such a case $AS(p(t))>0$ while $AD(p(t))=0$, which implies that for almost every $i$ for whom $m_i(t)>0$ we have $p_i^B(t)<p(t)$. Let us consider any player with positive $m_i(t)$ from this set. By changing his $p_i^B(t)$ to $p(t)$, player $i$ does not change the price above $\lfloor (1+h)\cdot p(t-1)\rfloor$ (since $p(t)\le \lfloor (1+h)\cdot p(t-1)\rfloor$) and he does not decrease $BR(t)$ to 0 (since $AS(p)>0$ for $p\ge p(t)$), so he can buy at a price below his threshold buying price, and, consequently, he increases his anticipated payoff, which contradicts the fact that the profile was a pre-BDNE.

Therefore, in such a case $p(t)<p_i^S(t)$ for a.e. player, which implies no trade.

d) Let us consider a profile being a pre-BDNE and a time instant $t$. By c) we know that $p(t)<p_i^S(t)$ for a.e. $i$ such that $s_i(t)>0$.

First we shall prove that in the continuum of players case a subset of $i$ for whom $m_i(t)>0$ of positive measure has $p_i^B(t)\ge p(t)$.

Assume the converse. We know that $p(t)\le \lfloor (1+h)\cdot p(t-1)\rfloor < \bar{p}_i^B(t)$. So if $BR(t)>0$, then, by nonatomicity of the space of players, every player $i$ for whom $m_i(t)>0$ and $p_i^B(t)<p(t)$ can increase his payoff by choosing some $p_i^B(t)\ge p(t)$ (and he does not affect nor $p$ nor $BR$). This contradicts the pre-BDNE condition.

By c) we have that $AS(p(t))=0$. Therefore we have $SR(t)=1$. We have just excluded $BR(t)>0$.

So now let us consider $BR(t)=0$, which means that $AD(p(t))>0$, i.e. for a set of players of positive measure with positive $m_i(t)$ we have $p_i^B(t)\ge p(t)$.

We shall prove that this implies
$p(t)=\min\left(\lfloor (1+h)\cdot p(t-1)\rfloor, \operatorname{essinf}_{s_i(t)>0} p_i^S(t)-1\right)$.

Indeed, either $Vol(q)=0$ for every $q$ or it is positive for some $q$ – by the form of $p_i^S$ it is obtained for some $q>\lfloor (1+h)\cdot p(t-1)\rfloor$.

In the latter case $Vol(p(t))=0$ and $p(t)$ is obtained from a value greater than $\lfloor (1+h)\cdot p(t-1)\rfloor$ by projection on the set
$[\lceil (1-h)\cdot p(t-1)\rceil, \lfloor (1+h)\cdot p(t-1)\rfloor]\cap(\mathbb{N}\setminus\{0\})$, since otherwise the equilibrating mechanism chooses $q$ maximizing $Vol(q)$. In such a case $p(t)=\lfloor (1+h)\cdot p(t-1)\rfloor$.

If $Vol\equiv 0$ and $p(t)<\lfloor (1+h)\cdot p(t-1)\rfloor$,
then $\operatorname{essinf}_{s_i(t)>0} p_i^S(t) > \operatorname{esssup}_{m_i(t)>0} p_i^B(t)$.
Let us assume that for $i$ in a subset of positive measure of the set of players with positive $m_i(t)$ we have $p_i^B(t)<\lfloor (1+h)\cdot p(t-1)\rfloor$.

As we have proven, $BR(t)=0$, $SR(t)=1$, $AD(p(t))>0$ and $AS(p(t))=0$.

Let us check what is the market price in such a situation.

The second condition restricting the price interval is minimization of disequilibrium. Let us assume that $\bar{p}<\operatorname{essinf}_{s_i(t)>0} p_i^S(t)-1$ is the market price. By the condition of minimization of disequilibrium and the facts that $AD(q)$ is a nonincreasing function of $q$ and that in our case disequilibrium for $q<\operatorname{essinf}_{s_i(t)>0} p_i^S(t)$ is equal to

$AD(q)$, we get the result that also every price $q$ in the interval $\left[\bar{p}, \operatorname{essinf}_{s_i(t)>0} p_i^S(t) - 1\right]$ minimizes $Dis(q)$.

The function $N(q)$ is either always zero at this set or it attains its minimum over this set at $\operatorname{essinf}_{s_i(t)>0} p_i^S(t) - 1$. In the latter case the market price cannot be less than $\operatorname{essinf}_{s_i(t)>0} p_i^S(t) - 1$. In the former case we have $AD(\bar{p}) = 0$, which contradicts our assumption.

Thus we have proven that $p(t) \geq \min\left(\lfloor (1+h) \cdot p(t-1) \rfloor, \operatorname{essinf}_{s_i(t)>0} p_i^S(t) - 1\right)$, $AS(p(t)) = 0$ while $AD(p(t)) > 0$.
Therefore, $SR(t) = 1$ and $BR(t) = 0$ at every pre-BDNE.
On the other hand, we have $p(t) < \operatorname{essinf}_{s_i(t)>0} p_i^S(t)$ and $p(t) \leq \lfloor (1+h) \cdot p(t-1) \rfloor$, which ends the proof. ■

Note that we made no comparison with fundamental values – the prices can grow because of players beliefs, whatever the fundamental values are, even if we have no uncertainty about fundamental values – speculative bubbles may happen even at a world with no external uncertainty.

This pre-BDNE profile is not a Nash equilibrium, since any deviating player who decides to sell his shares at any time $t$ such that $p(t) > \left\lceil \frac{F_t}{1+C} \right\rceil$ and $SR(t) > 0$ increases his payoff. We shall check whether it can be a BDNE.

### Simplified stock exchange and self-verification of beliefs

Now we screen the beliefs considered before to check self-verification and look for BDNE.

We can expect that fundamental beliefs of investors are self-verifying in a word without uncertainty. This is not obvious for the beliefs in strong trends.

**Proposition 16.** ***Self-verification***
*a) If for every $t$ $F_{t+1} \subset \left[\left\lceil (1-h) \cdot \left\lceil \frac{F_t}{1-C} \right\rceil \right\rceil, \left\lfloor (1+h) \cdot \left\lfloor \frac{F_t}{1+C} \right\rfloor \right\rfloor\right]$, $\left[\left\lceil (1-h) \cdot \left\lceil \frac{F_t}{1-C} \right\rceil \right\rceil, \left\lfloor (1+h) \cdot \left\lfloor \frac{F_t}{1+C} \right\rfloor \right\rfloor\right] \cap \mathbb{N} \backslash \{0\} \neq \emptyset$ and $F_t \in \left(\left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil\right) \cap \mathbb{N} \backslash \{0\}$, then fundamental beliefs that contain a history with $SR(s) = BR(s) = 1$ for every $s > t$ are potentially self-verifying and the profile defined by $p_i^S(t) = F_t + 1$ and $p_i^B(t) = F_t - 1$ is a BDNE for these beliefs.*

*b) There exist beliefs in strong trends as defined in proposition 15 which are potentially self-verifying and the profile consisting of the corresponding threshold prices is a BDNE for these beliefs.*

**Proof.** a) In our case the threshold strategies, as we proved in proposition 14a), are $(\bar{p}_i^B(t), \bar{p}_i^S(t)) = \left(\left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil\right)$

As we have proven in proposition 14b), the profile fulfilling $\bar{S}_i^{OL}(t) = \left(\left\lfloor \frac{F_t}{1+C} \right\rfloor, \left\lceil \frac{F_t}{1-C} \right\rceil\right)$ is a pre-BDNE.

However, the profile $S_i^{OL}(t) = (F_t - 1, F_t + 1)$ admits no trade and is also a pre-BDNE, since any individual deviation resulting in trade decreases the anticipated

payoff of the deviating player, which we prove in the same way as we have proven proposition 14b).
At this pre-BDNE the price is $F_t$, since it is the only price in the interval $(F_t - 1, F_t + 1)$. The reductions $SR(t) = BR(t) = 1$, since $AS(p(t)) = AD(p(t)) = 0$. Therefore, $H^S$ belongs to $B_i(t, S_i^{OL}(t), H^S)$.

b) An example of such belief is
$B_i(t, a, H) = \{(X, (p, BR, SR)) : p(\tau) \geq \varepsilon \cdot p(\tau - 1), SR(\tau) = 1, BR(\tau) = 0,$
$X(\tau) = p(\tau - 1)$ for every $t < \tau < \bar{\tau}$ and $(X, (p, BR, SR))|_t = H|_t\}$ for $\varepsilon \in [1+r, 1+h]$ such that for every $n \geq 1$ $\lfloor (1 + h) \cdot n \rfloor \geq \varepsilon \cdot n$ (such an $\varepsilon$ exists since $h$ is large compared to $r$) and $\bar{\tau} - t$ large enough.

Obviously, for every $t$ we have $p(t) = \lfloor (1 + h) \cdot p(t - 1) \rfloor \geq \varepsilon \cdot p(t - 1)$, $SR(t) = 1$ and $BR(t) = 0$ for the pre-BDNE defined in proposition 15b), which is, therefore, a BDNE. ■

Note that, although we have potential self-verification and the profiles considered are BDNE, the stock exchange, in fact, cannot operate. The problem of self-verification in a more compound but less formal model of a stock exchange was considered by the author in [19], including also self-verification of various beliefs against beliefs of other players, which included also dependence on a random factor. In this paper some beliefs have the property of approximately perfect self-verification against a small group of players with random beliefs if the measure of the set of players with the beliefs under consideration was large enough. However, there were also classes of beliefs that were self-falsifying in such a case – they caused changes of prices with signs opposite to the anticipated ones.

## Conclusions

In this paper new notions of equilibria in deterministic dynamic games with distorted information – *pre-belief distorted Nash equilibrium* (*pre-BDNE*) and *belief distorted Nash equilibrium* (*BDNE*) – together with concepts of self-verification of beliefs, first defined in Wiszniewska-Matyszkiel [20], were extended to a wider class of games that can encompass also models of financial markets with complicated market clearing conditions and private state variables of players besides the global state variable. These notions are especially applicable to dynamic games but they can be applied also to repeated games. In one stage games each of these concepts of equilibria is equivalent to Nash equilibrium. In games with a continuum of players also in this extended class of games the set of pre-BDNE for the perfect foresight is equal to the set of BDNE for these beliefs and to the set of Nash equilibria.

Existence and equivalence theorems were extended to the games considered in this paper.

The theoretical results were also illustrated by an example of a simplified model of a stock exchange. There were also some results proven that apply only to this model, among others no trade properties of pre-BDNE for fundamental beliefs and beliefs in strong trends, as well as potential self-verification of these beliefs. Among others, this proves that some, even very counterintuitive techniques of foreseeing prices of shares can be regarded as rational.

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