ON APPLICATION OF NEWTON'S METHOD
TO SOLVE OPTIMIZATION PROBLEMS
IN THE CONSUMER THEORY. EXPANSION'S PATHS
AND ENGEL CURVES

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Abstract: The paper continues investigations concerned with numerical modeling of the neoclassical theory of consumption begun in our previous paper [Strasburger et al. (2009)] This time we are interested in expansion's paths and Engel curves for separable utility functions.. The used numerical procedure is based on Newton's approximation method, implemented in the MATLAB environment. Based on the prior determination of the approximate solutions of the optimum problem we determine in this paper the expansion paths and Engel curves and compare them with their standard form studied in econometrics.

Key words: neoclassical theory of consumption, numerical approximations, multidimensional Newton method, expansion path, Engel curve

INTRODUCTION

The first mention of "Engel Laws", whose formulation preceded introduction of "Engel curves", dates back to the year 1857. Thus had started an important research line of what later would be called "Econometrics". This line in short can be described as a study of dependence of consumer demand on his/her income (expenditures). The subsequent development of those studies was split into two parallel directions. The first one was an empirical determination of the shape

of what was assumed to be the functional dependence of the quantity of a particular good on total expenditures − the price of that good considered constant. The resulting curve has became to be known as Engel curve. In the second one researchers became interested in a theoretical determination of this functional relation based on assumptions regarding the formation of consumer demand and its dependence on the total income (expenditures). It is with the latter line that our paper is concerned.

In our previous paper, *On application of Newton's method to solve optimization problems in the consumer and production theories,* [Strasburger, Zembrzuski (2009)] hereafter referred to as **I,** we have dealt with the problem of determining the demand functions by means of a numerical procedure based on the Newton's approximation method. Here, extending that study further, we show that by the same approximation procedure we can derive a tractable numerical approximation of expansion paths and Engel curves from the assumed form of the utility function, and then by using the Ordinary Least Squares (OLS) method we can determine resulting functional form of those curves.

We now briefly present indispensable notions and results from the classical theory of consumer demand underlying this note and summarize essentials of the approximation method elaborated in **I**.

RESUME OF THE CONSUMER THEORY

SETTING

Following the standard microeconomical approach to the consumer theory we assume that the consumer preferences may be described by means of a utility function $u : x \in X \rightarrow u(x) \in R$ defined on the consumption set X, taken for simplicity to be the nonnegative orthant of the Euclidean space R^k of k dimensions. Here k is the number of commodities considered for the problem, while vectors $x = (x_1, \ldots, x_k)$ represent commodity bundles in such a way that the quantity of an l-th commodity is given by the number x_l, $l = 1, \ldots, k$. We let $p = (p_1, \ldots, p_k) \in R_+^k$ to denote a price vector, whose components p_l have the meaning of the amount paid in exchange for one unit of the l-th commodity. Thus the inner product $\langle x, p \rangle = \sum_{i=1}^{k} x_i p_i$ is the value of a commodity bundle x with respect to the given price system p. Further, we denote by I the total income (wealth) at the consumer 's disposal, so that the set by $D(p, I) = \{x \in X \mid \langle x, p \rangle \leq I\}$ is the budget set of the consumer. It represents the

set of all consumption bundles available to the consumer. In the case $X = R_+^k$ the budget set is the solid compact k-dimensional simplex in R_+^k.

MARSHALIAN DEMAND FUNCTION

This fundamental object of the consumer theory is obtained as the solution of the problem of maximization of the utility function within the budget set.

Roughly speaking, the solution of the following maximization problem

$$\max\{u(x) \mid x \in X\}$$

under conditions   (1)

$$x_i \geq 0, \quad \text{for } i = 1, \ldots, k, \quad \text{and} \quad \langle x, p \rangle = \sum_{i=1}^{k} x_i p_i \leq I$$

regarded in its relation to prices p and income I constitutes **the demand correspondence**.

Under suitable assumption on the utility function (monotonicity and strict quasi-concavity) and positivity of the price vector p it can be proved that the problem is solved in a unique way leading to the demand function rather than the demand correspondence in a manner described by the following theorem (cf. Barten and Böhm in [Arrow, Intriligator 1982, p. 409]).

Theorem 1 *Given a positive price vector p and a positive wealth I the maximization problem above is uniquely solved by a certain vector $x^0 \in D(p, I)$ with positive coordinates, so that:*

There exists a unique positive constant (Lagrange multiplier) λ^0 such that the vector x^0 is the unique solution of the system of equations

$$\frac{\partial u}{\partial x_i}(x) = \lambda^0 p_i, \quad \text{for } i = 1, \ldots, k,$$

$$\langle x, p \rangle = I,$$ (2)

with $k + 1$ unknowns $x_1, \ldots, x_k, \lambda$.

Unfortunately, this system of equations is in general highly nonlinear and therefore only in rare cases its solution can be given explicitly. In order to apply our approximation procedure we recast the system into vectorial formulation.

Let us denote the equations of the system (2) by $\phi_i(x_i, \lambda, p_i, I) = 0$, where $i = 1, \ldots, k + 1$, and the functions ϕ_i are given by

$$\phi_i(x, \lambda, p, I) = \frac{\partial u}{\partial x_i}(x) - \lambda p_i, \quad \text{for } i = 1, \ldots, k,$$

$$\phi_{k+1}(x, \lambda, p, I) = I - \langle x, p \rangle.$$ (3)

Introducing the vector function $\Phi : (x, \lambda, q, I) \in R_+^{k+1} \times R_+^{k+1} \to$ $\to \Phi(x, \lambda, q, I) \in R^{k+1}$ defined by

$$\Phi(x, \lambda, p, I) = \begin{pmatrix} \phi_1(x, \lambda, p, I) \\ \vdots \\ \phi_{k+1}(x, \lambda, p, I) \end{pmatrix} \tag{4}$$

enables us to write down the equations (2) in a form of one vector equation

$$\Phi(x, \lambda, p, I) = 0, \tag{5}$$

which is to be solved for the unknown (x, λ) for any given wealth I and the price system p.

Although the above theorem guarantees existence of a unique solution for any data (price system and wealth) so that we obtain an implicitly defined function $(x, \lambda) = Q(p, I)$, it says nothing about the functional form of this solution. Apart from the Lagrange multiplier λ this function represents the Marshalian Demand Function written in the vector form as $x = Q(p, I)$. Fixing the prices and varying the income I only, this function describes what is called an Expansion Curve []. Referring to a single commodity one at the time we have the demand functions for individual goods

$$x_i = Q_i(p, I), \quad i = 1, \ldots, k.$$

If the prices are absorbed into the functional form, these functions take up the form

$$x_i = q^*_i(I), \quad i = 1, \ldots, k, \tag{6}$$

which is regarded as the conventional form describing *the Engel curves,* [Deaton, Muellbauer, 1980]

THE NEWTON'S APPROXIMATION METHOD

In the formulation given above this clearly is an instance of the "Implicit Function Problem" in its general form, so we can employ the machinery of the multivariable approximation methods for obtaining and studying its solution, as indicated e.g. in [Krantz, Parks 2002]. We would like to point out that there were attempts to use approximation methods to describe the solution to the optimization problem (1), cf. e.g. [Panek E., (ed.) (2001), pp], but in our opinion they can hardly be considered satisfactory. More satisfactory method to obtain this goal, which uses the multivariable Newton's Method has been described in our preceding paper **I**, and we present briefly its basic results here.

If in the equation (5) we consider (p, I) fixed and absorb it in the functional form, the question reduces to a solution of an equation of the form $\Phi(x, \lambda) = 0$.

Denoting by $\Phi'\left(x, \lambda\right)^{-1}$ the inverse of the full derivative (Jacobian matrix) of $\Phi(x, \lambda)$, we define the recursive sequence

$$\left(x^{(n+1)}, \lambda^{(n+1)}\right) = \left(x^{(n)}, \lambda^{(n)}\right) - \Phi'\left(x^{(n)}, \lambda^{(n)}\right)^{-1} \cdot \Phi\left(x^{(n)}, \lambda^{(n)}\right) \qquad (7)$$

with an initial value $\left(x^{(0)}, \lambda^{(0)}\right)$ suitably chosen. It can be shown in general that for a large class of functions this sequence is convergent and that the limit is a solution of (5) (see e.g. [Fortuna et al (2005)]).

NUMERICAL CALCULATIONS

ALGORITHM

In **I** this method was tested for some choice of the utility functions, depending on a varied number of variables (varied number of commodities) and certain range of parameters (p, I). To be more specific, we assumed the utility functions to be separable, that is given by the following formula,

$$u(x_1, x_2, \ldots, x_k) = x_1^{a_1} + x_2^{a_2} + \ldots + x_k^{a_k}, \quad \text{where } 0 < a_l < 1. \qquad (8)$$

It is well known that those functions satisfy the above formulated assumptions, in particular the maximization problem is uniquely soluble.

The calculations were performed for a few exemplary functions with the number of commodities ranging from $k = 2$ to $k = 20$.

The discussed algorithm starts from some arbitrary initial values $\left(x^{(0)}, \lambda^{(0)}\right)$ and in following steps (see eq. 7) it seeks the optimal solution. We have found that the algorithm is numerically stable and the vector $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}\right)$ converges to the optimal solution very quickly, just in several steps of iterations in the presented examples. The results do not depend on the assumed initial values, which may or may not satisfy the budget limitations $\langle x, p \rangle \leq I$.

CHOICE OF UTILITY FUNCTION

In the present paper we extend previous considerations by studying dependence of the solution of the optimization problem (1) on the income of the consumer. This gives us the so called expansion paths and Engel curves, primarily depending on the utility function used in our optimization problem. We look at the separable utility function (8) with two commodities only, $k = 2$. For the sake of the numerical calculations we assume:

$$u(x_1, x_2) = x_1^{0,2} + x_2^{0,5}, \qquad (9)$$

and choose the prices as $p_1 = 10$ (*numeraire*) and $p_2 = 2, 5, 10$. The budget bound has the form $x_1 p_1 + x_2 p_2 \leq I$, where various values of the income I are taken into consideration.

RESULTS

The numerical calculations were performed with the use of a program written in the Matlab language (see e.g. [Pratap (2006), Zalewski, Cegieła (2002)]).

Applying the Newton's method we solve the optimization problem and determine the demand by computing values x_1 and x_2 any given value of the income I. Varying subsequently the value of I with fixed p_1 and p_2 we determine the expansion paths in this model, i.e. graphs of the demand (x_1, x_2) as a function of I, which show how the income is distributed between two commodities. The results are shown in Fig. 1 where different curves correspond to different values of $p_2 = 2, 5, \text{and} 10$ and $p_1 = 10$ kept fixed. The shape of so obtained curves depends on the assumed form of the utility function (9). In particular eliminating the parameter I we may express the expansion path in the functional form $x_1 = x_1(x_2)$, what indicates that the functions $x_1 = x_1(x_2)$ are concave and also that the second commodity is more desirable than the first one, since $x_2 > x_1$.[1] This of course is due to the fact that in the expression for the utility function the exponent of x_1 is smaller than that of x_2.

In Fig. 1 we also show points corresponding to some chosen values of the wealth, namely $I = 100$, 500 and 1000. In general, the plots confirm that the demand is an increasing function of the income, as expected from theoretical considerations. Note also, that we have tested the presented method of calculations for various input data and we have found that it is numericaly stable in a wide range of the income from less than $I = 10^{-3}$ up to more than $I = 10^9$ (not shown).

The numerical results are shown once again in Fig. 2, this time in the form of Engel's curves, i.e. x_1 and x_2 are plotted separately as the functions of the income I in the range $0 \leq I \leq 1000$. Basing on the shapes of the plots, one can draw some qualitative conclusions regarding the model. For example, the plots indicate

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[1] Strictly speaking the values of x_2 are higher than the values of $x_1 p_1/p_2$ for $x_1 > t/p_1$ and $x_2 > t/p_2$, where t is the solution of the equation $\dfrac{d}{dt}\left(\dfrac{t}{p_1}\right)^{0,2} = \dfrac{d}{dt}\left(\dfrac{t}{p_2}\right)^{0,5}$. Taking for instance $p_1 = p_2 = 10$ we obtain $t \approx 0,5$, so x_2 is higher than x_1 for $x_1, x_2 > 0,05$.

that the demand x_2 depends on I almost linearly while $x_1 = x_1(I)$ is a concave function and grows slower than x_2. Moreover the distribution of the expenditures between two goods depends on the assumed values of prices. For the price of the second commodity equal to $p_2 = 10$ (dotted lines in Fig. 2) the demand x_2 is obviously greater and x_1 is lower than for lower prices $p_2 = 5$ (dashed lines) and $p_2 = 2$ (solid lines).

Our analysis of Engel curves is purely theoretical and the results presented here are just a sample of what could be obtained from numerical solutions of the optimization problem. They are derived basing on a specific choice of a utility function (9), but a similar procedure is possible for other choices as well.

On the other hand, the empirical approach to the demand theory had produced a large number of model curves containing free parameters, which were used to describe observed patterns of demand for specific commodities, see e.g. [Tomaszewicz, Welfe (1972), Dudek (2011), Stanisz (1993)]. We list below a few examples — many more can be found in the quoted sources.

$$x(I) = \alpha_1 + \alpha_2 \cdot I \,, \tag{10}$$

$$x(I) = \alpha_1 + \alpha_2 \cdot \ln I \,, \tag{11}$$

$$x(I) = \alpha_1 + \alpha_2 / I \,, \tag{12}$$

$$x(I) = \exp\!\left(\alpha_1 + \alpha_2 \ln I + \alpha_3 \ln^2 I\right). \tag{13}$$

The parameters α_i are usually fitted through statistical analysis of the data.

Our goal now is to determine a functional form of the numerical results presented in Fig. 2. It seems to be especially interesting to know if the results of numerical simulations can be described with use of the same analytical formulae as in analyses based on empirical data for real commodities, so in the following discussion we consider the functions given by eqs. (10-13). The parameters α_i are determined with the OLS method. For simplicity we assume $p_1 = p_2 = 10$.

The results are shown in Figs. 3-6. The numerical solutions of the optimization problem are plotted with dotted lines. The points used in OLS method to determine α_i parameters are marked with circles. The evaluated parameters have in general different values for the first (Figs. 3a, 4a, 5a, 6a) and the second commodity (Figs. 3b, 4b, 5b, 6b). The solid lines are plots of the considered functions (eqs. 10-13) with the values of the parameters α_i determined.

First we consider the linear function (10). Since the dependency x_2 versus I obtained from the optimization problem is almost linear, the parameters can be chosen in such a way, that there are no visible difference between the dotted and solid line (Fig. 3b). On the other hand, the linear function can not describe properly

the dependence of x_1 on I, which is nonlinear (Fig. 3a). We note that the linear function (10) would be the proper model for both x_1 and x_2 if the two exponents in the utility function were equal: $u(x_1, x_2) = x_1^a + x_2^a$.

The best possible fits of the functions (11) and (12) are presented in Figs. 5 and 6, respectively. One can see, that none of those functions describe properly the behavior of either x_1 or x_2,
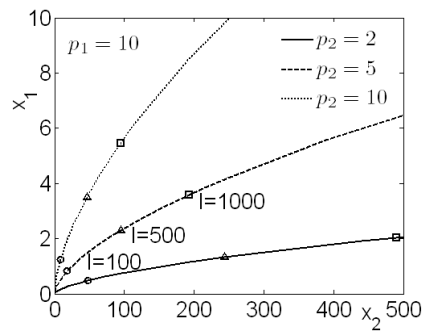
Finally we take the last of the considered functions given by the formula (13). Using the OLS method we obtain $\alpha_1 = -2,935$, $\alpha_2 = 0,708$, $\alpha_3 = -0,005$ for the first commodity (Fig. 6a) and $\alpha_1 = -2,863$, $\alpha_2 = 1,133$, $\alpha_3 = -0,009$ for the second one (Fig. 6b). In both cases the analytical functions (solid lines) agree very well with the numerical results (dotted lines). We can conclude that a proper functional form of our numerical results has been determined and it is given by the equations:

$$x_1(I) = \exp\left(-2,935 + 0,708 \cdot \ln I - 0,005 \cdot \ln^2 I\right), \tag{14}$$

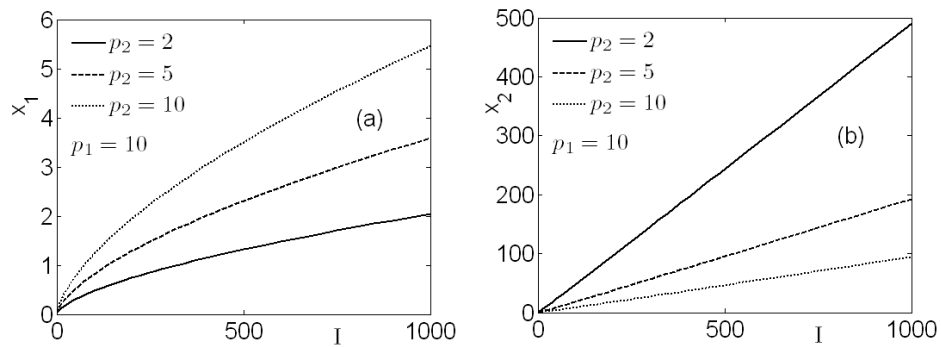$$x_2(I) = \exp\left(-2,863 + 1,133 \cdot \ln I - 0,009 \cdot \ln^2 I\right). \tag{15}$$

The obtained functions correspond to the utility function assumed in the separable form (9). However the utility functions are purely a theoretical concept and they can not be measured or determined from empirical data. Nevertheless it seems possible to obtain some conclusions about these functions from studies of empirical Engel's curves. If an empirical dependency between the demand and income for some real commodities is well described by the functions (11) or (12) then it means, that the consumers' preferences can not be described properly by the utility function in the form (9), since this formula leads to a different dependency between the demand and the wealth. On the other hand, if the empirical Engel's curves have an analytical form given by eq. (13), in such a case one can not exclude the utility function in separable form, which was assumed in our paper (9).

Figure 1. The expansion paths obtained for the prices p_1=10 and p_2=2 (solid), p_2=5 (dashed), p_2=10 (dotted line). The points corresponding to chosen values of the income, I=100 (circles), I=500 (triangles) and I=1000 (squares), are shown.

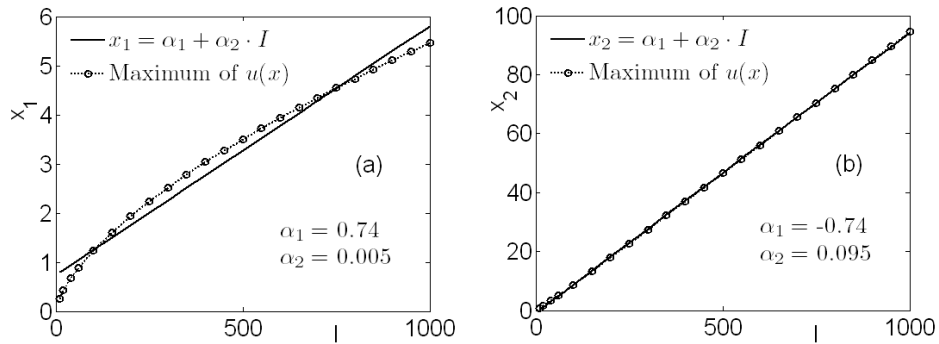Source: own preparation

Figure 2. The demand x_1 (a) and x_2 (b) presented as a functions of the income I. The prices p_1=10, p_2=2 (solid), p_2=5 (dashed) and p_2=10 (dotted line) are assumed.
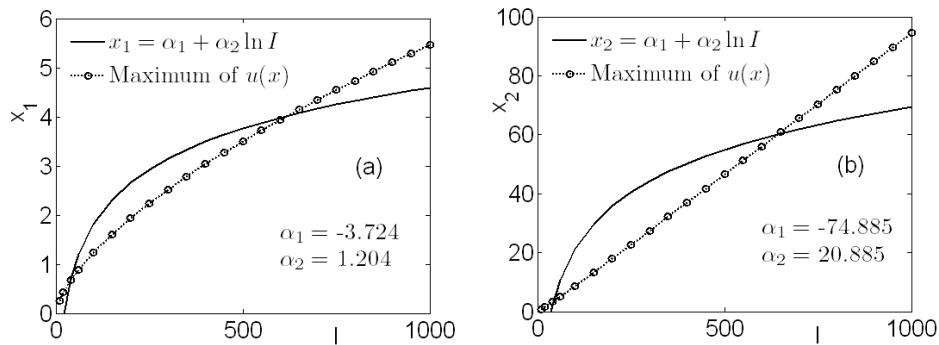

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Figure 3. The demand x_1 (a) and x_2 (b) shown as a functions of the income I. The dotted lines correspond to the solutions of the optimization problem with $p_1 = p_2 = 10$, while the solid ones are plots of the linear functions $x = \alpha_1 + \alpha_2 \cdot I$ with the parameters α_i obtained using the OLS method.
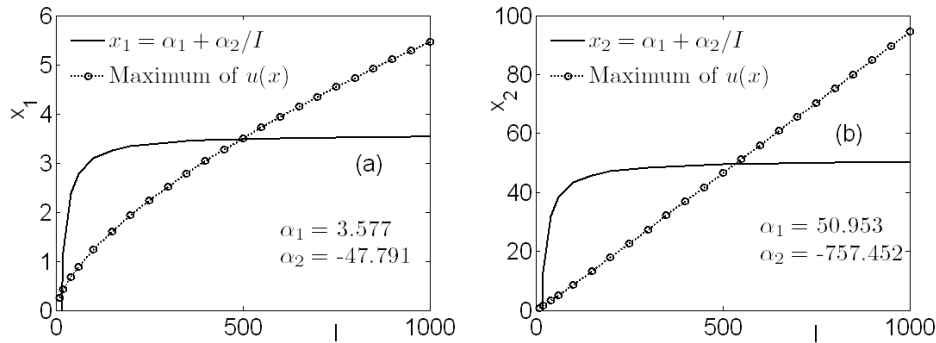

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Figure 4. The same as in Fig. 3 for the function $x = \alpha_1 + \alpha_2 \ln I$ (solid line).
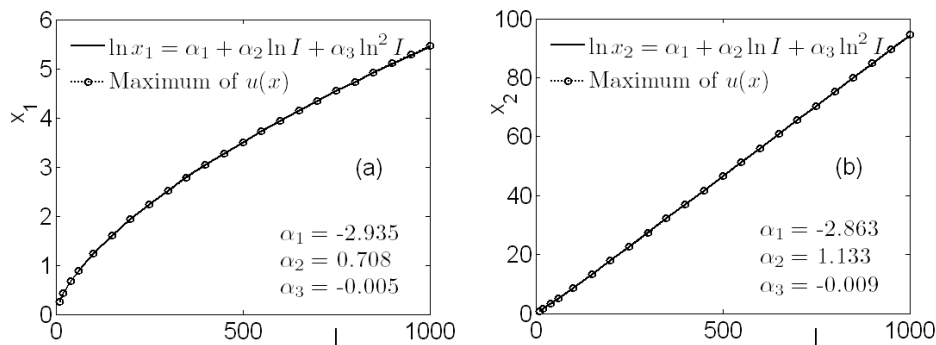

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Figure 5. The same as in Fig. 3 for the function $x = \alpha_1 + \alpha_2/I$ (solid line).

Source: own preparation

Figure 6. The same as in Fig. 3 for the function $x = \exp(\alpha_1 + \alpha_2 \ln I + \alpha_3 \ln^2 I)$ (solid line).

Source: own preparation

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