QUANTITATIVE METHODS IN ECONOMICS
Vol. XIV, No. 1, 2013, pp. 276 – 282

STOCHASTIC EQUIVALENCE SCALES IN LOG-NORMAL DISTRIBUTIONS OF EXPENDITURES

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Abstract: In the paper, the properties of the stochastic equivalence scales (SES) are analysed when expenditure distributions are log-normal. The SES provides the equivalent distribution of expenditures when the population of households is heterogeneous with respect to such attributes as household size, demographic composition, etc. For log-normal expenditure distributions, the non-parametric SES deflators are proportional to the ration of geometric means in compared distributions. The statistical analysis of expenditure distributions for Poland in the years 2005-2010 shows that these deflators perform quite well.

Keywords: equivalence scales expenditure distributions, log-normal distribution

INTRODUCTION

The main purpose of this paper is to develop the formulae and estimators of nonparametric stochastic equivalence scales when expenditure distributions are log-normal. We also estimate such scales using data from Polish Household Budget Surveys for the years 2005-2010.

When households differ in all aspects other than expenditures[1], e.g., the size and the composition of the household, the age of the adults, the age of the children, the disabilities of the household members, etc., serious problems arise when making judgments and decisions to address inequality, welfare and poverty. Traditionally, equivalence scales have been used to homogenize heterogeneous household populations [Buhmann et al. 1988, Jones and O'Donnell, 1995].

[1] In this paper, we confine ourselves to expenditures; however, a significant amount of our considerations also relates to incomes.

Muellbauer (1977) defines equivalence scales as budget deflators that are used to calculate the relative amounts of money two different types of households necessary to attain the same standard of living. The distribution of the expenditures or incomes of heterogeneous household populations is adjusted by such deflators. As a result, the initial heterogeneous population becomes homogeneous. It is this adjusted distribution of expenditures in this now homogeneous population that is used to assess welfare, poverty, inequality, etc.

However, various practical advantages of equivalence scales are offset by a significant disadvantage, as the specification of an equivalence scale requires strong assumptions concerning the relationship between income and needs, and there may not be wide agreement concerning the validation of the appropriate assumptions. Furthermore, numerous severe identification issues arise in the estimation of equivalence scales (see, in particular, [Pollak and Wales, 1979, 1992], [Blundell and Lewbel, 1991], [Blackorby and Donaldson, 1993], and the surveys of [Lewbel, 1997, and Slesnick, 1998]). Moreover, there are evidences that the results of distributional comparisons are sensitive to the choice of the equivalence scale (Coulter et al. 1992a,b).

The aforementioned approaches to the problem of equivalence scales seem to be unsatisfactory. Many economists maintain that, "There is no single 'correct' equivalence scale for adjusting incomes - a range of scale relativities is both justifiable and inevitable" [Coulter et al. 1992a]. Jäntti and Danziger [2000, p.319] remark that, "There is no optimal method for deriving an equivalence scale". Indeed, without additional assumptions, there is no way of selecting the basis for choosing an equivalence scale. The *independence of base* (IB) (or *exactness of equivalence* scale) is one such assumption. Several papers have tested this assumption and have ultimately rejected it [Blundell and Lewbel, 1991, Blundell et al. 1998, Dickens et al. 1993, Pashardes, 1995, Gozalo, 1997, Pedankur, 1999].

The concept of the stochastic equivalence scale (SES)offers the method for the adjustment of expenditure distributions in heterogeneous populations [Kot, 2012]. The SES is any function that transforms the expenditure distribution of a specific group of households in such a way that the resulting distribution is stochastically indifferent from the expenditure distribution of a reference group of households. The stochastic indifference criteria are also used in developing the method of the estimation of the *SES.*

In this paper, the *SES* is applied when theoretical expenditure distributions are log-normal. The formula for such scales is developed. Then the *SES* are estimated using data from Polish Household Budget Surveys for the years 2005-2010.

The rest of the paper is organized as follows. Section THEORETICAL BACKGROUND offers a theoretical background concerning the *SES* and the formula of deflators when log-normal distribution is assumed. In Section EMPIRICAL RESULTS FOR POLAND 2005-2010, the results of estimation of deflators are presented. Section CONCLUSIONS summarize the paper.

THEORETICAL BACKGROUD

Suppose that a society is composed of heterogeneous households and each household is distinguished by two attributes: expenditures and a type. The household type may be interpreted in various ways, e.g., as an index of neediness which increases with family size.[2] We also allow for an analysis of the household types, which may not necessarily reflect household needs. We assume that there exists a given and finite number ($m+1 \geq 2$) of types of household groups that differ in many respects other than their expenditures.

We arbitrarily chose certain type of households as the 'reference group'. The expenditure distribution for this group will be called the 'reference distribution'. This distribution will be described by a positive continuous random variable Y with the distribution function $G(y)$ (abbreviated $Y \sim G(y)$). The remaining m groups of households will be called the 'evaluated groups'. The corresponding *evaluated distribution* of the expenditures will represent the set of the positive continuous random variables $X_1,\ldots,X_m$ with the distribution functions $F_1(x),\ldots,F_m(x)$, respectively. The random variables Y, $X_1,\ldots,X_m$ describe the distribution of expenditures per household.

Formally, the *SES* is defined as follows. Let $\mathbf{s}(\cdot) = [s_i(\cdot),\ldots,s_m(\cdot)]$ be a vector function for which the inverse function $\mathbf{s}^{-1}(\cdot) = [s_i^{\ -1}(\cdot),\ldots,s_m^{\ -1}(\cdot)]$ exists and is differentiable. Let the random variable $Z_i = s_i(X_i)$ with the distribution function $H_i(z)$ be the transformation of evaluated expenditure distribution X_i. Hereafter, the random variable $Z_i \sim H_i(z)$ will be called the 'transformed expenditure distribution'.
Definition 1. With the above notations, the function $\mathbf{s}(\cdot)$ will be called *the stochastic equivalence scale* (*SES*) if and only if the following equality holds:

$$\forall z>0,\ i=1,\ldots,m;\ H_i(z) = G(z). \qquad (1)$$

When the function $\mathbf{s}(\cdot)$ is the *SES*, $Z_i = s_i(X_i)$ will be called 'the equivalent expenditure distribution'.

Definition 1 of the *SES* is axiomatic in the sense that it only postulates the criterion for a function to be recognized as an *SES*. This definition does not describe how an *SES* should be constructed or its conditions of existence. In other words, *any* function $\mathbf{s}(\cdot)$ that fulfils condition (1) has to be recognized as an *SES*.

The validation of condition (1) can be verified by Kolmogorov-Smirnov *K-S* test. The *K-S* statistic can be also used as the loss function in developing estimators of parametric or nonparametric *SES*. Details of the statistical procedures of testing and estimating the *SES* are presented in Kot (2012).

[2]We follow Ebert and Moyes (2003) in associating the household type with family size for convenience. However, this framework can be extended by taking into account the vector of the household attributes comprising the number of adults, the number of children, the age of household members, etc.

The relative *SES* can be defined as follows. Let **d** = [d_i], $i = 1,\dots,m,$ be the vector of positive numbers called 'deflators' which transforms the evaluated expenditure distributions $X_1,\dots,X_m$ as follows:

$$Z_i = X_i/d_i \sim H_i(z), i = 1,\dots,m. \tag{2}$$

Definition 2. Under the above notations, the vector **d** will be called the *relative SES* if and only if the deflators $d_1,\dots,d_m$ are such that equality (1) holds.

The following corollary summarizes the properties of *SES*:

Corollary 1. Let X be the distribution of expenditures of the evaluated group of households, Y the distribution of expenditures of the reference group of households, and $Z = s(X)$.If s is the *SES*, then the following equivalent conditions hold:

- Z is stochastically indifferent to Y, or
- Social welfare in Z is exactly the same as in Y, for all von Neuman–Morgenstern utility functions, or
- Poverty in Z is exactly the same as in Y for all poverty lines, or
- Inequalities in Z are exactly the same as in Y.

[Davidson, 2008].

One may ask what kind of homogeneity *SES* provides. If an initial heterogeneous population of households consists of $m+1$ subpopulations (including reference subpopulation), then the adjustment of each m different expenditure distribution by the *SES* will give new fictitious subpopulations which are homogeneous with respect to utilitarian social welfare, inequality and poverty.

Let expenditure distributions be two-parameter log-normal $X \sim \Lambda(\mu, \sigma)$ with the density function given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, \quad x > 0 \tag{3}$$

[Kleiber, Kotz, 2003, p. 107]. The k*th* moment in this distributions is given by

$$E[X^k] = \exp\left\{k\mu + \frac{k^2\sigma^2}{2}\right\} \tag{4}$$

and the geometric mean (x_{geom}) is given by

$$x_{geom} = e^{\mu} \tag{5}$$

which coincides with the median [Kleiber, Kotz, 2003, p. 112].

Let $Y \sim \Lambda(\mu_y, \sigma_y)$ and $X \sim \Lambda(\mu_x, \sigma_x)$ denote the expenditure distributions of the reference group and the evaluated group respectively. If we adjust X by the d deflator, as in (2), then the transformed distribution $Z=X/d$ will be log-normal $\Lambda(\mu_z, \sigma_z)$,where $\mu_z = \mu_x - \log d$, and $\sigma_z = \sigma_x$.

It is easy to see that the *d* deflator will provide the *SES* if compared distributions *Z* and *Y* have the same parameters, i.e., $\mu_y = \mu_x - \ln d$ and $\sigma_y = \sigma_x$. Then the *d* deflator is given by

$$d = \exp\{\mu_x - \mu_y\} = \frac{x_{geom}}{y_{geom}}. \tag{6}$$

In other words, the deflator *d* of the relative *SES* (2) in log-normal expenditure distributions will be simply the *ratio of geometric means* (*RGM*) if the condition $\sigma_y = \sigma_x$ holds. The validation of this condition can be easily checked using standard statistical test.

It might happen, however, that statistical test rejects the aforementioned condition $\sigma_y = \sigma_x$. In order to assess how the violation of this condition affects the *d* deflator we use the fact that the *SES* implies the equality of moments (4) for all *k*. After simple algebra, we can get the following formula for an adjusted *d**

$$d^* = \frac{x_{geom}}{y_{geom}} \exp\left\{k\,\frac{\sigma_x^2 - \sigma_y^2}{2}\right\}. \tag{7}$$

Obviously, (7) coincides with (6) if $\sigma_y = \sigma_x$. Because of that $\sigma_y > \sigma_x$ in practice, the greater disparity between σ_x and σ_y, the lower *d*, for all *k*. This means that the violation of the $\sigma_y = \sigma_x$ condition leads to underestimation of the *d* deflator.

EMPIRICAL RESULTS FOR POLAND 2005-2010

We will use expenditure distributions for estimating the relative *SESs*. The monthly micro-data come from the Polish Household Budget Surveys for the years 2005-2010. The expenditures are expressed in constant 2010 year prices. The household groups are distinguished according to the number of members (household size). The households of single childless persons are chosen as the reference group.

The chi-square test rejects the null hypothesis that expenditure distributions are log-normal. This result is not uncommon in applications involving large sample sizes. In fact, all theoretical models of income or expenditure distributions have been usually rejected at conventional level of significance [McDonald, Xu, 1995].

Table 1 presents estimates of *RGM d* and *d** deflators given by eq. (6) and (7) respectively. These deflators are estimated separately for each household group. We calculate the *d** deflator assuming *k*=1.

Table 1. The estimates of *RGM* deflators for Poland 2005-2010

Year	2005		2006		2007		2008		2009		2010	
Size	d	d*	d	d*	d	d*	d	d*	d	d*	d	d*
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.617	1.609	1.604	1.609	1.672	1.668	1.699	1.693	1.667	1.650	1.678	1.676
3	1.889	1.875	1.928	1.921	2.038	2.031	2.103	2.078	2.066	2.036	2.063	2.039
4	2.049	2.019	2.075	2.056	2.239	2.215	2.292	2.246	2.245	2.185	2.220	2.172
5	2.028	1.964	2.108	2.051	2.257	2.179	2.348	2.277	2.272	2.175	2.251	2.163
6 or more	2.183	2.105	2.299	2.212	2.477	2.380	2.495	2.387	2.460	2.330	2.505	2.405

Source: own calculations using data from Polish Household Budget Surveys

An analysis of the results presented in Table 1 shows that differences between *d* and *d** are rather small and they can be neglected if one decimal place is taken into account. This means that the violation of the assumption $\sigma_y = \sigma_x$ does not seriously affect the estimates of deflators. Because of that variances σ^2 of logarithms of expenditures within each year have turned out non-homogeneous, the adjusted *d** deflator seems to be more adequate approximation of the *SES* than *d* deflator.

CONCLUSIONS

The possibility of estimating non-parametric *SES*s opens new interesting perspectives for applications when expenditures have log-normal distribution. The estimation of non-parametric deflators is very easy because it requires estimates of geometric means of compared distributions only.

Empirical results exhibit two important features. First, equivalence scale varies over time. This means that one definite form of the scale does not exists. Second, Polish equivalence scale are very flat. This is an indication of the economies of scale enjoyed by Polish households in the years 2005-2010.

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