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THE APPLICATION OF NASH GAME FOR DETERMINE OF CONTROLLING MULTIDIMENSIONAL OBJECTS

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Abstract: The article presents the approach of decentralization of calculations related to the synthesis of controlling through division of an optimalization problem into lesser dimensionality subproblems. Applied was differential Nash game for objects of serial structure. The main idea of the method is to assign individual functionals of quality to subobjects and conduct the synthesis of regulators in a sequential way, beginning with the first subobject. Thanks to the sequential structure of the object it is possible to obtain a solution by decentralized optimizations.

Keywords: complex systems, dynamics equations, differential game, functionals of quality, Riccati equation, decentralized calculation of regulator gain

INTRODUCTION

The synthesis of complex systems controlling is a complicated task. Technically, it can be made in two ways. The first, a natural one, requires constructing decentralized control system consisting of local regulators, possibly equipped with interaction compensators. However, compensation is sensitive to differences between a model and an object. In the case of greater factors it can even lead to instability. No method that would in a decentralized way allow to construct a stable system with coupling for a system of arbitrary structure has been so far created [Golnaraghi and Kuo 2010], [Kaczorek 1999], [Han 2009] . All successful results so far eventuate from intuition of a designer who is familiar with the process [Kociemba et al. 2013] , [Wenwu et al. 2013].

The second way bases on designing a regulator for the entire system using classic method, i.e. with the use of location of poles or basing on solving linear-quadratic problem. In practice, when dimensionality exceeds 10, it turns problematic.

Limiting speculations to linear structure only, a method of decentralized designing of a feedback system is presented. The system is stable and thus more efficient than local regulators with interaction compensator and control system created with the method of coupling disorders. It was initially proposed by Özgüner and Perkins [Kwater 1987]. It bases on assigning individual quality indicators to subobjects and conducting synthesis of regulators in a sequential manner, beginning with the first subobject. Thanks to cascade structure of an object, solution is made by decentralized optimization and therefore the issue of dimensionality is made irrelevant.

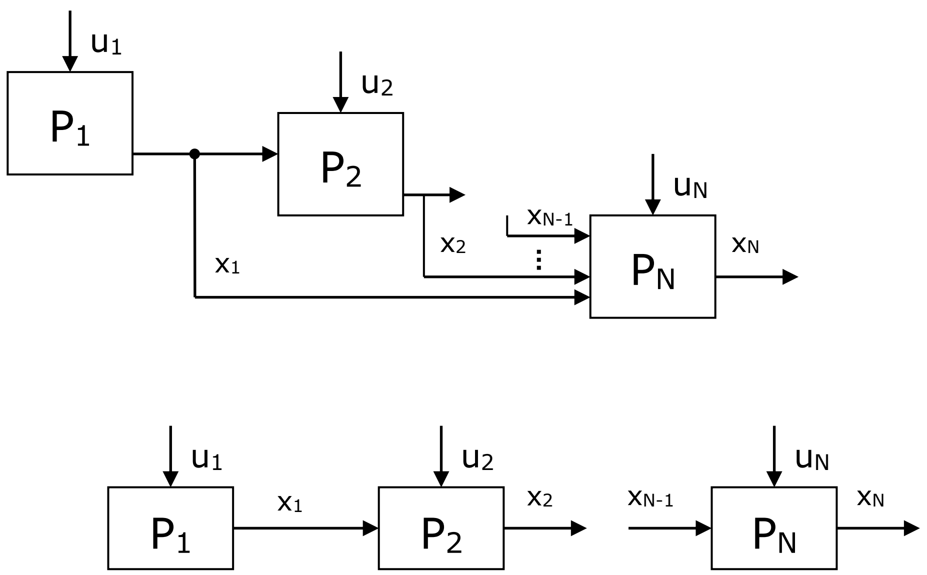
DYNAMIC OBJECTS OF A LUMPED PARAMETER SYSTEM

A large proportion of composite objects is composed of objects whose each subsequent subobject is linked to the preceding ones. (Figure1).

Figure 1. Objects of: a) cascade structure, b) linear structure

a)

b)



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Considered is a linear stationary cascade object, composed of N subobjects. Dynamics of i-subobject is expressed in the following equation:

$$\frac{d}{dt}\mathbf{x_i}(t) = \mathbf{A_i x_i}(t) + \mathbf{A_{i,i-1} x_{i-1}}(t) + \mathbf{B_i u_i}(t), \qquad (1)$$
$$\mathbf{x_i}(t_0) = \mathbf{x_0},$$

where: $\mathbf{x_i}$, $\mathbf{u_i}$, $\mathbf{A_i}$, $\mathbf{B_i}$ – vectors and matrices of state and control of i-subobject.

$\mathbf{A_{i,\,i+1}}$ is a matrix accommodating links between neighboring subobjects.

Collective mathematical model of a cascade object is:

$$\frac{d}{dt}\tilde{\mathbf{x}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(t), \qquad (2)$$
$$\tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0,$$

where: $\tilde{\mathbf{x}}(t) = col[\mathbf{x_1},...,\mathbf{x_{i-1}},\mathbf{x_i},...,\mathbf{x_N}]$, $\tilde{\mathbf{u}}(t) = col[\mathbf{u_1},...,\mathbf{u_{i-1}},\mathbf{u_i},...,\mathbf{u_N}]$,

$\mathbf{B} = diag[\mathbf{B_1},...,\mathbf{B_{i-1}},\mathbf{B_i},...,\mathbf{B_N}]$ **-** Block diagonal matrix. Matrix of state $\mathbf{A}$ is:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A_1} & \phi\cdots & & 0 \\ \mathbf{A_{21}} & \mathbf{A_2}\ddots & & \vdots \\ 0 & \mathbf{A_{i,i-1}} & \mathbf{A_i} & \\ \vdots & & \ddots & . \\ 0 & \cdots & \mathbf{A_{N,N-1}} & \mathbf{A_N} \end{bmatrix} \qquad (3)$$

Submatrices on diagonal in (3) concern particular subobjects, whereas the rest presents interaction between them.

Models of type (2) present quite a wide range of technological processes. Typical examples include rolling of metal sheets, reactors in a cascade of distillation columns and multizone pusher-type furnace. For the majority, division into subobjects is a natural consequence of mechanical construction, locally dominating chemical reactions or methods of controlling. Non-technological processes constitute a considerable group of objects of linear structure. In the group one may include a biologically contaminated river and transport system consisting of a stream of vehicles, where subobjects are segments between bigger tributaries and distances between crossroads respectively.

TASK OF CONTROLLING

One may consider the issue of a feedback system in reference to model (2). The effective synthesis of a multi-parameter regulator when objects's dimensionality exceeds 10 poses a complex computational problem. Local regulators designed individually for subobjects regardless of mutual connections may, in turn, not provide sufficient quality of regulation of the entire system.

Solving the problem of the synthesis of regulators of state in large systems may be conducted in multiple ways. One of the solutions is to assign individual quality functional to each subobject. Assuming that the first subobject operates under the control of a regulator, one approaches the second subobject and determined controlling for a group of subobjects (the first one with a regulator and the other subobject without one). In a similar manner, designed are subsequent regulators up until the end. As a result, obtained is a group of local regulators with additional cross-coupling from preceding subobjects. Such technique makes it possible to conduct calculations of all regulator gains in a decentralized way, while preserving the accurate succession. Firstly, based on solutions of Riccati equations for subobjects defined are local gains. Further on, cross-coupling gains are defined, basing on certain bilinear equations parallel to matrix Lapunow equation. The whole procedure employs lower dimentionality equations adequate to subobjects, which allows to avoid computational problems for an object treated as inseparable body.

Such natural conduct is justified on the grounds of game theory.

One may assume that controls $\mathbf{u_i}(t)$ in equations (1) are created by N partners (players) in a differential game. Every player interacts with an object through "their" control $\mathbf{u_i}(t)$ in order to minimise given functional of quality J_i assigned to objects:

$$J_i = \int_T [\mathbf{x_i}(t)^T \mathbf{Q_i} \mathbf{x_i}(t) + \mathbf{u_i}(t)^T \mathbf{R_i} \mathbf{u}_i(t)]dt \tag{4}$$

Due to occurring interactions, the consequences of one player's decision in general influence the achievement of goals of other players. Hence, J_i depends on $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_N}$. Nash equilibrium is defined by $\mathbf{u_i}^*$ control that assures the inequality:

$$J_i(\mathbf{u_1^*},\ldots,\mathbf{u_{i-1}^*},\mathbf{u_i^*},\mathbf{u_{i+1}^*},\ldots,\mathbf{u_N^*}) \le J_i(\mathbf{u_1^*},\ldots,\mathbf{u_{i-1}^*},\mathbf{u_i},\mathbf{u_{i+1}^*},\ldots,\mathbf{u_N^*}) \;,\; \forall \mathbf{u_i} \in U_i \;,$$

$$i=1,2,\ldots,N, \tag{5}$$

where U_i is a space of allowable controls for i-player.

Assumption that none of the partners takes dominant position protects players from one-sided withdrawal from declared $\mathbf{u_i}^*$ control. Such partner can only lose.

It should be noted that despite the occurrence of only local variables of state $\mathbf{x_i}$ in the functional (4), it also depends on controls of preceding subobjects, due to the interactions expressed in matrix $\mathbf{A_{i,\,i-1}}$.

One may also assume that subobjects are asymptotically stable, i.e. matrices $\mathbf{A_i}$ have eigenvalues in the left half-plane. From the cascade structures, (4) satisfies the inequalities:

$$J_i(\mathbf{u_1^*,...,u_i^*}) \le J_1^*(\mathbf{u_1^*,...,u_{i-1}^*,u_i}) \text{ for } i=1,...,N. \tag{6}$$

Hence, one obtains formula for optimal control of an object of cascade structure:

$$\mathbf{u_i}(t) = -\sum_{j=1}^{i} \mathbf{G_{i\,j}} \mathbf{x_j}(t), \qquad i=1,\dots,N. \tag{7a}$$

Gains are:

$$\mathbf{G_i} = \mathbf{G_{ii}} = \mathbf{R_i^{-1} B_i^T P_{ii}^i} \tag{7b}$$

$$\mathbf{G_{ij}} = \mathbf{R_i^{-1} B_i^T P_{ij}^i},$$

where $\mathbf{P_{ii}^i}$ is a solution of differential Riccati equation

$$\mathbf{P_{ii}^i A_i} + \mathbf{A_i^T P_{ii}^i} - \mathbf{P_{ii}^i B_i R_i^{-1} B_i^T P_{ii}^i} + \mathbf{Q}_i = \frac{d\mathbf{P_{ii}^i}}{dt} \tag{8}$$

whereas $\mathbf{P_{ii}^i}, i > j$ is a solution of bilinear equation

$$\mathbf{P_{ij}^i A_i^*} + \mathbf{A_i^{*\,T} P_{ij}^i} - \sum_{k=j+1}^{i-1} \mathbf{P_{ik}^i A_{kj}^*} - \mathbf{P_{ii}^i A_{ij}} = \frac{d\mathbf{P_{ij}^i}}{dt}$$

$$\mathbf{A_{kj}^*} = \mathbf{A_{kj}} - \mathbf{B_k R_k^{-1} B_k^T P_{kj}^k} \tag{9}$$

$$\mathbf{A_i^*} = \mathbf{A_i} - \mathbf{B_i R_i^{-1} B_i^T P_{ii}^i}$$

$$(\mathbf{A_{kj}} = 0 \quad dla \quad k \supset j+1).$$

Equation (9) is similar to matrix Lapunow equation.

The same results may be obtained as a consequence of sequential (heuristic) procedure which is based on determining optimal control for subsequent subobjects, beginning with the first one. Control of *i*- subobject is obtained when assuming that all preceding, i.e. $1,\ldots,i-1$ have their controls determined.
Considering the abovementioned procedure in details, one subsequently gets:

Stage 1. Optimal control of the first subobject is expressed in a formula:

$$\mathbf{u}_1^*(t) = -\mathbf{R}_1^{-1}\mathbf{B}_1^{\mathbf{T}}\mathbf{P}_{11}^{1}\mathbf{x}_1(t),$$

where $\mathbf{P}_{11}^{1}$ is a solution of Riccati equation (8), here inafter local. Having $\mathbf{u}_1^*$ we close feedback loop.

Stage 2. Considered is a system consisting of a first subobject (controlled optimally) and the second. Following dependencies are in operation:

$$\frac{d}{dt}\tilde{\mathbf{x}}_2(t) = \tilde{\mathbf{A}}_2\tilde{\mathbf{x}}_2(t) + \tilde{\mathbf{B}}_2\mathbf{u}_2(t); \quad \tilde{\mathbf{x}}_2(t_0) = \tilde{\mathbf{x}}_{20},$$

$$\text{where}: \tilde{\mathbf{x}}_2 \quad col[x_1, x_2], \quad \tilde{\mathbf{B}}_2 = col[\phi, \mathbf{B}_2],$$

$$\tilde{\mathbf{A}}_2 = \begin{bmatrix} \mathbf{A}_1^* & \phi \\ \mathbf{A}_{21} & \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{A}_1^* = \mathbf{A}_1 - \mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{B}_1^{\mathbf{T}}\mathbf{P}_{11}^{1}$$

and

$$J_2 = \int_T [\mathbf{x}_2(t)^T\tilde{\mathbf{Q}}_2\tilde{\mathbf{x}}_2(t) + \mathbf{u}_2(t)^T\mathbf{R}_2\mathbf{u}_2(t)]dt, \quad \tilde{\mathbf{Q}}_2 = \begin{bmatrix} \phi & \phi \\ \phi & \mathbf{Q}_2 \end{bmatrix}.$$

Optimal control is:

$$\mathbf{u}_2^*(t) = -\mathbf{R}_2^{-1}\tilde{\mathbf{B}}_2^{\mathbf{T}}\mathbf{P}^2\tilde{\mathbf{x}}_2(t),$$

where $\mathbf{P}^2 = \begin{bmatrix} \mathbf{P}_{11}^{2} & \mathbf{P}_{21}^{2} \\ \mathbf{P}_{21}^{2} & \mathbf{P}_{22}^{2} \end{bmatrix}$. Due to the matrix $\tilde{\mathbf{B}}_2$, to determine $\mathbf{u}_2^*$ it is enough to determine $\mathbf{P}_{22}^{2}$ from the equation (8), and $\mathbf{P}_{21}^{2}$ from (9).

Stage 3 and subsequent stages. For the rest of subobjects, optimalization problems are defined in formulas:

$$\frac{d}{dt}\begin{vmatrix}\tilde{\mathbf{x}}_{\mathbf{i-1}}\\ \mathbf{x_i}\end{vmatrix}=\begin{vmatrix}\tilde{\mathbf{A}}_{\mathbf{i-1}} & \phi\\ \mathbf{A_{i,i-1}} & \mathbf{A_i}\end{vmatrix}\begin{vmatrix}\tilde{\mathbf{x}}_{\mathbf{i-1}}\\ \mathbf{x_i}\end{vmatrix}+\begin{vmatrix}\phi\\ \mathbf{B_i}\end{vmatrix}\mathbf{u_i},$$

$$J_i=\int_T\left\{\left[\tilde{\mathbf{x}}^{\mathbf{T}}_{\mathbf{i-1}},\mathbf{x_i^T}\right]\begin{vmatrix}\phi & \phi\\ \phi & \mathbf{Q_i}\end{vmatrix}\begin{vmatrix}\tilde{\mathbf{x}}_{\mathbf{i-1}}\\ \mathbf{x_i}\end{vmatrix}+\mathbf{u_i^T}\mathbf{R_i}\mathbf{u}_i\right\}dt, \tag{10}$$

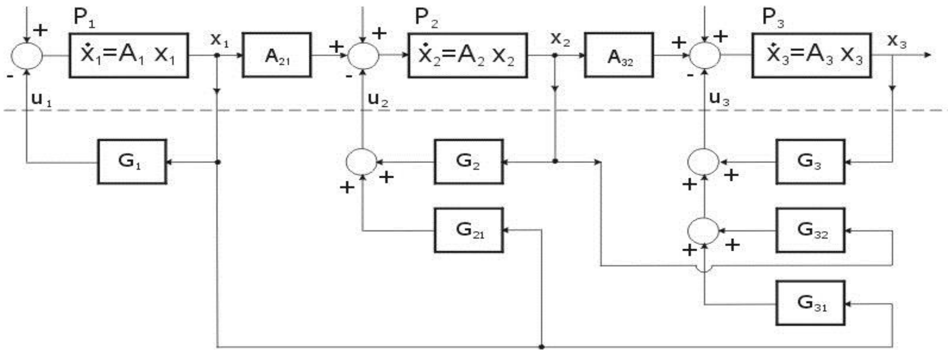
$$i=3,\dots,N,$$

where $\tilde{\mathbf{x}}_{\mathbf{i-1}}=col[\mathbf{x_1},\dots,\mathbf{x_{i-1}}]$. $\tilde{\mathbf{A}}_{\mathbf{i-1}}$ is a triangular matrix obtained as a result of closing feedback for subobjects stage by stage. Finally, control of i-subobject is in a form:

$$\mathbf{u_i^*}(t)=-\mathbf{R_i^{-1}}\mathbf{B_i^T}\sum_{j=1}^{i}\mathbf{P_{ij}^i}\mathbf{x_j}(t) \tag{11}$$

One should note the importance of the order in which equations are solved. Firstly, one should solve all local Riccati equations (8). Subsequently, basing on $\mathbf{P_{ii}^i}$ one should solve bilinear (9), starting with $j=i-1$ and ending with $j=1$. Such decentralized manner allows to avoid computational problems related to dimensionality. To emphasize such possibility of synthesis, regulator is called sequential. For an object composed of three subobjects, the structure of a regulator is presented in Figure 2.

Figure 2. Cascade of three subobjects with a system of regulators determined in a equential manner



Source: own work

As is apparent, despite local gains $\mathbf{G_i}$ there are gains $\mathbf{G_{ij}}$ resulting from the interaction among subobjects. Controlling i-subobject is generated basing on own state and states of preceding subobjects.

CONCLUSION

In the presented approach received the composed regulator, which consists of the local regulators and the regulators that use the feedback from the previous subobjects. The feedback is the result of interaction between subobjects obtained. Parameters of local regulators are determined on the basis of individual Riccati equations o for subobjects. Cross-coupling gains are determined by the bilinear equations solutions in a form similar to the Lapunov equation. As a result of this approach received is a decentralized method of determining gains in a regulation system, i.e. so called system of sequence regulators.

Qualities of the presented method:

- Realization of a regulator requires knowledge of vector of state only of the preceding subobjects.
- Gains may be determined in a sequential manner, solving problems of small dimensionality.
- Local feedback are calculated basing on solution of Riccati equation, and cross basing on bilinear equations.

The presented method leads to obtaining structure adequate to classic linear-quadratic regulator. The way of synthesis of a regulator for lumped parameters systems can be naturally extended to systems of linear structure expressed in partial equations of transport type.

The method may be also adapted to other objects of cascade or linear structure. The advantage of the method are relatively simple calculation algorithms.

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