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THE METHOD OF SUPPORTING DECISIONS UNDER RISK BASED ON MULTIOBJECTIVE OPTIMIZATION

Andrzej Łodziński
Department of Econometrics and Statistics
Warsaw University of Life Sciences – SGGW
e–mail: andrzej_lodzinski@sggw.pl

Abstract: The method of supporting decisions under risk was presented in this paper. Making decision under risk takes place when a result of a given decision is not explicit and depends on the condition of the environment. A decision-making process based on multiobjective optimization has been presented in this paper. Methods of multiobjective optimization do not give one unique solution, but a whole set of them. A decision making relies on interactive conducting of the decision making process. Selection of given decision is made by way of solving a problem with parameters defining aspirations of a decision maker and the evaluation of obtained results. A decision maker defines a parameter, for which a solution is indicated. Then he or she evaluates the received solution by either accepting or rejecting it. In the second case a decision maker provides a new parameter value and a problem is solved again for the new parameter.

Keywords: multiobjective optimization, symmetrically efficient solution, scalarization function, supporting a decision-making process

INTRODUCTION

The method of supporting decisions under risk based on multiobjective optimization has been presented in this paper. Making decisions under risk takes place when the results of activities undertaken by a decision maker are uncertain due to the likelihood of occurring unexpected circumstances or factors that disturb these unexpected circumstances or disturbing factors - conditions of the environment - are called the scenarios. The higher dispersal of results, the higher level of uncertainty. Simultaneously, each scenario explicitly defines completion of results for individual decisions.

For example, the enterprise, which launches its new product on the market must take into account numerous uncertainties, starting from costs of developmental works, sales volume or possible competition reactions. In the case of a decision made under risk it is possible to estimate probabilities, with which these uncertain results take place, e.g. an enterprise can rely on forecasts of experts for market research.

Making decisions under risk is modeled with the help of a special task of multiobjective optimization. It is a task with non–descendingly ordered functions. Methods of multiobjective optimization do not give one unique solution, but a whole set of them. The method of supporting decision is based on interactive conducting of the process of making decisions that is, selection of given decision is made by way of solving a problem with controlling parameters defining aspirations of a decision maker and the evaluation of obtained results. A decision maker defines a parameter, for which a solution is indicated. Then he or she evaluates the received solution by either accepting or rejecting it. In the second case a decision maker provides a new parameter value and a problem is solved again for the new parameter.

MODELLING OF A DECISION SITUATION UNDER RISK

The problem of selecting a decision under risk should be modelled by implementing these scenarios to a problem of selecting decisions, which represent possible conditions of the environment. Probability distribution is provided for the scenarios. If we assume that the probability of occurring individual scenarios are rational numbers, then it is possible to lead to the situation by multiple repetition of appropriate scenarios, in which probability of occurring each scenario is the same. The number of occurring of a definite scenario refers to the probability that is assigned to it. Definite scenarios $S_i, i = 1,...,m$ correspond to realizations of mark functions $f_i(x), i = 1,...,m$. A higher value of a mark function is preferred for each scenario.

We consider a situation, in which for each decision $x \in X_0$ there is one of m possible results $f_1(x),..., f_m(x)$. Probabilities of these results are the same and amount to $p = \frac{1}{m}$.

The problem of making decisions under risk is modeled as a special task of multiobjective optimization:

$$\max_x \{ (f_1(x),..., f_m(x)) : \quad x \in X_0 \} \tag{1}$$

where: $x \in X_0$ – decision that belongs to the set of admissible decisions, $X_0 \subset R^n$,
$S_i, i = 1,...,m$ - scenarios (environment conditions),
$f = (f_1,..., f_m)$ - vector function, which allocates for each vector of decision variables $x \in X_0$ mark vector $y = f(x)$; individual coordinates $y_i = f_i(x),\ \ i = 1,...,m$ – represent scalar mark functions - result of a decision x when a scenario $S_i, i = 1,...,m$ takes place,
X_0 - set of feasible decisions.

It is a task for multiobjective optimization put into equally probable scenarios. The results are equally probable - each coordinate of a mark function has the same significance.

Vector function $y = f(x)$ allocates for each vector of decision variables x mark vector $y \in Y_0$, which measures decision quality x from the point of view of the defined set of quality indicators $y_1,..., y_m$. The image of feasible set X_0 for functions f constitutes the set of achievable mark vectors Y_0.

The task (1) is based on finding such admissible decision $x \in X_0$, for whose m marks it assumes the best values. This task is considered in relation to the marks, that is the following task is examined:

$$\max_x \{(y_1,..., y_m): \quad y \in Y_0\} \tag{2}$$

where: $x \in X$ – vector of decision variables,
$y = (y_1,..., y_m)$ - vector quality indicator, individual coordinates $y_i = f_i(x),\ \ i = 1,...,m$ represent individual scalar criteria,
Y_0 - set of admissible mark vectors.

Mark vector $y = (y_1,..., y_m)$ in the multiobjective problem (2) represents a decision result x in the form of a vector with m equally probable $p = \dfrac{1}{m}$ coordinates $y_i,\ i = 1,..,m$.

SYMMETRICALLY EFFICIENT SOLUTION

Making decisions under risk is modeled as a special task of multiobjective optimization with the relation of reference that meets anonymity property. The results, which differ by ordering of coordinates, are not differentiated. The solution to the problem of the selection decision is the decision of symmetrically efficient. It is an efficient decision, which meets an additional domain - anonymity property of preference relation.

Nondominated results (Pareto-optimal) are defined in the following way:

$$\hat{Y}_0 = \{\hat{y} \in Y_0 : (\hat{y} + \tilde{D}) \cap Y_0 = \emptyset) \} \tag{3}$$

where: $\tilde{D} = D \setminus \{0\}$ – positive cone without a top. The following can be assumed as a positive cone $\tilde{D} = R_{+}^{m}$.

Appropriate feasible decisions are defined within a decision area. A decision $\hat{x} \in X_0$ becomes an efficient decision (Pareto-optimal), if a mark vector corresponding to it $\hat{y} = f(\hat{x})$ is an nondominated vector [12].

In the multiobjective problem (1), which serves to make decisions under risk with a given set of mark functions, value distribution obtained by these functions is only important, whereas it is not important which value a given function has assumed. The results, which differ by ordering, are not differentiated. This requirement is formed as anonymity (neutrality) domain of preference relation.

Then this relation is called an anonymous relation, when for each mark vector $y = (y_1, y_2, ..., y_m) \in R^m$ for any permutation P of a set $\{1,...,m\}$ the following domain takes place:

$$(y_{P(1)}, y_{P(2)}, ..., y_{P(m)}) \approx (y_1, y_2, ..., y_m) . \tag{4}$$

Mark vectors that have the same coordinates, but in a different order, are identified. Preference relations that meet additional anonymity condition are called anonymous preference relation.

Nondominated vector that meets anonymity property is called a symmetrically nondominated vector. The set of symmetrically nondominated vectors is marked with $\hat{Y}_{0S}$. Symmetrically efficient decisions are defined within a decision area. A decision $\hat{x} \in X_0$ becomes a symmetrically efficient decision, if a mark vector corresponding to it $\hat{y} = f(\hat{x})$ is a symmetrically nondominated vector. The set of symmetrically efficient decisions is marked with $\hat{X}_{0S}$.

A relation of symmetric dominance can be expresses as an inequality relation for mark vectors, whose coordinates are ordered in the non-descending order. This relation can be presented with the use of transformation $T : R^k \rightarrow R^m$ that non-descendingly orders coordinates of mark orders that is a vector $T(y)$ is a vector with non-descendingly ordered coordinates of a vector y that is $T(y) = (T_1(y), T_2(y),..., T_m(y))$, where $T_1(y) \le T_2(y) \le ... \le T_m(y)$ and there is permutation P of the set $\{1,..., m\}$ so that $T_i(y) = y_{P(i)}$ for $i = 1,.., m$. The relation of symmetric dominance $\ge_a$ is an ordinary vector dominance for non-descendingly ordered vectors [8].

Mark vector y^1 dominates symmetrically prefers a vector y^2 if the following condition is met:

$$y^1 \ge_a y^2 \Leftrightarrow T(y^1) \ge T(y^2) \tag{5}$$

Solving a decision problem involves the determination of symmetrically efficient decision corresponding to preferences of a decision maker.

SCALARIZATION OF A PROBLEM

Special multiobjective task is solved for indicating a solution that is symmetrically efficient for a multiobjective task (1). It is a task with coordinates of a mark vector that are ordered non-descendingly that is the following task:

$$\max_y \{(T_1(y), T_2(y),..., T_m(y)) : \quad y \in Y_0\} \tag{6}$$

where: $y = (y_1, y_2,..., y_m)$ – mark vector,

$T(y) = (T_1(y), T_2(y),..., T_m(y))$, where $T_1(y) \le T_2(y) \le .... \le T_m(y)$ - mark vector that is ordered non-descendingly,

Y_0 - set of achievable mark vectors.

An efficient solving of a task of multiobjective optimization (6) is a symmetrically efficient solution of a multiobjective task (1).

In order to provide a solution of a multiobjective task (6) scalarization of this task is solved that has a scalarizing function $s : Y \times \Omega \rightarrow R^1$:

$$\max_x \{s(y, \bar{y}) : x \in X_o\} \tag{7}$$

where: $y=(y_1,y_2,...,y_m)$ – mark vector,

$\bar{y}=(\bar{y}_1,\bar{y}_2,...,\bar{y}_m)$ - controlling parameter.

It is a task of one-criterion optimization of specially created scalarizing function of two variables - mark vector $y \in Y$ and a controlling parameter $\bar{y} \in \Omega \subset R^m$ with the actual value that is function $s: Y \times \Omega \to R^1$. The parameter $\bar{y}=(\bar{y}_1,\bar{y}_2,...,\bar{y}_m)$ is at disposal of a decision maker, which enables him or her reviewing the set of symmetrically efficient solutions.

Optimum solution of a task (7) should be a solution of a multiobjective task. Scalarizing function should meet some properties - completeness property and sufficiency property. Sufficiency property means that for each controlling parameter $\bar{y}$, solving a scalarizing task means a solution that is symmetrically efficient that is $\hat{y} \in \hat{Y}_{0S}$. Completeness property means that thanks to adequate changes of a parameter $\bar{y}$ any result can be obtained $\hat{y} \in \hat{Y}_{0S}$. Such function fully characterizes symmetrically efficient solutions. Each maximum of this function is a symmetrically efficient solution. Each symmetrically efficient solution can be obtained by assuming appropriate values of controlling parameters $\bar{y}$.

Completeness and insufficient parameterization of the set of symmetrically nondominated vectors $\hat{Y}_{0S}$ can be obtained by applying the reference point method to a task (6). This method is used as controlling parameters of aspiration levels. Aspiration levels are such values of mark functions, which are satisfactory for a decision maker.

Scalarizing function in the method of the point of reference has the following form:

$$s(y,\bar{y})=\min_{1\le i\le m}(T_i(y)-T_i(\bar{y})_i)+\varepsilon\cdot\sum_{i=1}^{m}(T_i(y)-T_i(\bar{y})_i) \tag{8}$$

where: $y=(y_1,y_2,...,y_m)$ – mark vector,

$T(y)=(T_1(y),T_2(y),...,T_m(y))$,

where $T_1(y)\le T_2(y)\le ....\le T_m(y)$

- mark vector is ordered non-descendingly,

$\bar{y}=(\bar{y}_1,\bar{y}_2,...,\bar{y}_m)$ - vector of aspiration levels,

$T(\bar{y})=(T_1(\bar{y}),T_2(\bar{y}),...,T_m(\bar{y}))$,

where $T_1(\bar{y}) \le T_2(\bar{y}) \le ... \le T_m(\bar{y})$
- mark vector is ordered non-descendingly,
ε - arbitrarily small, positive regularizing parameter.

Such scalarizing function is called the function of achievement. This function measures closeness of a given solution to the aspiration level. The aim is to find a solution that is as close as possible to achieve definite requirements - aspiration levels.

Optimum values of this function can be used not only to calculate a symmetrically nondominated vector, but also to evaluate achievability of a given aspiration point $\bar{y}$. If a maximum of the achievement function $s(y, \bar{y})$ is negative then the aspiration point $\bar{y}$ is not achievable, however, the maximum point $\hat{y}$ of this function is symmetrically nondominated vector in some sense equally closest to the point $\bar{y}$. If a maximum of the achievement function $s(y, \bar{y})$ is equal to zero then the aspiration point $\bar{y}$ is achievable and is a symmetrically nondominated vector. If a maximum of the achievement function $s(y, \bar{y})$ is positive then the aspiration point $\bar{y}$ is achievable, however, the maximum point $\hat{y}$ of this function is symmetrically nondominated vector in some sense equally improved to the point $\bar{y}$ [12].

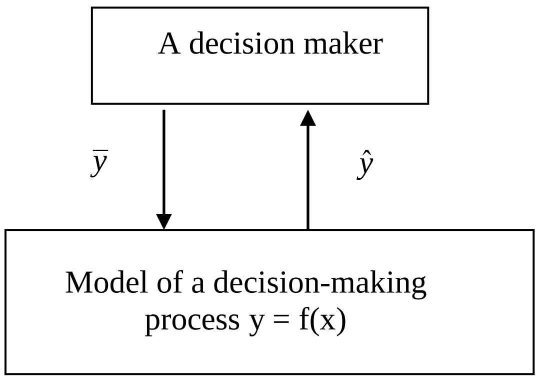
Maximization of such function due to y means a symmetrically efficient solution $\hat{y}$ and a decision that generates a symmetrically efficient decision $\hat{x}$. An indicated symmetrically efficient solution $\hat{x}$ depends on values of aspiration levels $\bar{y}$.

METHOD OF SELECTING SYMMETRICALLY EFFICIENT DECISIONS

The solution of a task of multiobjective optimization is the whole set of solutions, so a decision maker should select a decision with the help of the interactive computer system. Such system enables a controlled preview of the set of solutions. On the basis of the parameter values provided by a decision maker the task is solved and the system presents a solution for analysis that corresponds to current values of these parameters.

The tool for previewing the set of solutions is the function (8). Maximum of this function depends on the parameter $\bar{y}, i = 1,...,m$, which is used by a decision maker to select a solution. A decision maker, when solving a problem, defines aspiration levels $\bar{y}, i = 1,,...,m$ as desirable values of individual marks. A decision maker expresses his or her preferences in the reference point method by defining such a value for each mark function, which will be fully satisfactory for him or her. These values constitute the aspiration level for a given mark function. The controlling parameter in the form of aspiration levels represents actual values that are easily understood by a decision maker and characterize his or her preferences. Aspiration levels are expressed in the terms of values of individual mark functions. The method of supporting selection of a decision is presented on the Figure 1.

Figure 1. The method of supporting a decision-making process



Source: own work

Such manner of making decisions does not impose on a decision maker any rigid way of analyzing a decision problem and enables the possibility of modifying his or her preferences while analyzing a problem. A user has a master role in this method of making decisions.

EXAMPLE

The problem of selecting the best investment out of 10 investments is presented in order to illustrate the method of supporting a decision under risk. Probabilities of payments for individual investments are the following:

Investment 1:

payment [thousand PLN]	7	8	10	14	15
probability	0,3	0,2	0,2	0,2	0,1

Investment 2:

payment [thousand PLN]	7	8	9	10	13	14	15
probability	0,1	0,1	0,1	0,1	0,3	0,3	0,2

Investment 3:

payment [thousand PLN]	8	9	10	13	14	15
probability	0,1	0,2	0,1	0,2	0,2	0,2

Investment 4:

payment [thousand PLN]	8	9	10	11	13	14	15
probability	0,1	0,2	0,1	0,1	0,1	0,3	0,1

Investment 5:

payment [thousand PLN]	6	7	9	10	11	13	14	15
probability	0,1	0,1	0,1	0,1	0,1	0,1	0,3	0,10

Investment 6:

payment [thousand PLN]	8	9	13	14
probability	0,3	0,2	0,2	0,3

Investment 7:

payment [thousand PLN]	6	7	8	11	13	14
probability	0,1	0,2	0,2	0,3	0,1	0,1

Investment 8:

payment [thousand PLN]	6	7	9	11	13	15
probability	0,3	0,1	0,1	0,2	0,2	0,1

Investment 9:

payment [thousand PLN]	8	9	11	13	14
probability	0,1	0,3	0,1	0,3	0,2

Investment 10:

payment [thousand PLN]	7	8	9	10	11	14	15
probability	0,1	0,1	0,1	0,1	0,3	0,2	0,1

Each investment requires to invest 10000 PLN.

In order to characterize possible investments we set their values on the set that contains all possible values, which they can assume with non-zero probability:

payment in [thousand PLN]	6	7	8	9	10	11	13	14	15
Investment 1	0	0,3	0,2	0	0,2	0	0	0,2	0,1
Investment 2	0	0,1	0,1	0,1	0,1	0	0,3	0,3	0,2
Investment 3	0	0	0,1	0,2	0,1	0	0,2	0,2	0,2
Investment 4	0	0	0,1	0,2	0,1	0,1	0,1	0,3	0,1
Investment 5	0,1	0,1	0	0,1	0,1	0,1	0,1	0,1	0,1
Investment 6	0	0	0,3	0,2	0	0	0,2	0,3	0
Investment 7	0,1	0,2	0,2	0	0	0,3	0,1	0,1	0
Investment 8	0	0,1	0	0,1	0	0,2	0,2	0	0,1
Investment 9	0	0	0,1	0,3	0	0,1	0,3	0,2	0
Investment 10	0	0,1	0,1	0,1	0,1	0,3	0	0,2	0,1

Source: own calculations

While repeating appropriate scenarios we lead to the situation, where the probability of each scenario is the same and equals $p = \frac{1}{10}$. Then we get the situation, in which the results of each investment decision $i = 1,2,...,10$ are the following mark vectors with equally probable coordinates:

i	1	2	3	4	5	6	7	8	9	10
y^1	7	7	7	8	8	10	10	14	14	15
y^2	7	8	9	10	13	14	14	14	15	15
y^3	8	9	9	10	13	13	14	14	15	15
y^4	8	9	9	10	11	13	14	14	14	15
y^5	6	7	9	10	11	13	14	14	14	15
y^6	8	8	8	9	9	13	13	14	14	14
y^7	6	7	7	8	8	11	11	11	13	14
y^8	6	6	6	7	9	11	11	13	13	15
y^9	8	9	9	9	11	13	13	13	14	14
y^{10}	7	8	9	10	11	11	11	14	14	15

Source: own calculations

The set of symmetrically nondominated vectors is the following $\hat{Y}_{os} = \{y^2, y^3\}$. Two decisions: investment 2 and investment 3 are the symmetrically efficient decisions. When making a decision it is necessary to select between them and reject other investments, irrespective of individual preferences. Investment 2 and investment 3 are incomparable in relation to anonymous preference relations. Selection between them depends on individual preferences of a decision maker.

These two variants are selected by the reference point method with non-descendingly ordered coordinates with adequately determined aspiration level.

Assuming the biggest possible aspiration marks as aspiration levels, e.g. for the vector of aspiration $T(\bar{y}) = (8, \; 9, \; 9, \; 10, \; 13, \; 14, \; 14, \; 14, \; 15, \; 15)$ we get as a solution – investment 2.

Decreasing beginning aspirations, e.g. for the vector of aspiration $T(\bar{y}) = (7{,}5, \; 8{,}5 \; 9, \; 10, \; 13, \; 14, \; 14, \; 14, \; 15, \; 15)$ we get as a solution - investment 3.

This method enables to select any symmetrically efficient solution that corresponds to preferences of a decision maker.

CONCLUSION

The method of supporting decisions under risk was presented in this paper. Decision making process takes place by solving a task of multiobjective optimization. This method is characterized by using aspiration points and optimality of the achievement function in order to organize interaction with a user.

The method provides the whole set of solutions that are anonymously efficient and enables a decision maker to choose freely. However, such manner of conduct does not substitute a decision maker in his or her decision making process. The whole process of making decisions is controlled by a user.

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