QUANTITATIVE METHODS IN ECONOMICS
Vol. XVI, No. 1, 2015, pp. 25 – 36

OVERLAPPING MULTIGRID METHODS AS AN EFFICIENT APPROACH FOR SOLVING THE BLACK-SCHOLES EQUATION

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Abstract: In this paper the modification of a two-level multigrid method by allowing an overlap between adjacent subdomains and its application to a one-dimensional Black-Scholes equation is described. The method is based on the finite-difference schema known as implicit Euler. Numerical experiments confirm the superiority of the proposed method in relation to the classic multigrid method in form of shortening computation time, memory savings and ease of parallelization. The comparison shows the advantages of overlapping grids vs method without them, mainly due to improved accuracy of the solution.

Keywords: option pricing, Black-Scholes model, multigrid method, finite-difference scheme

INTRODUCTION

One of the fast-evolving areas of financial mathematics is the modeling of dynamics of changes of the prices of financial instruments, in particular the problem of the option pricing. Many of proposed algorithms, which were developed over the years for the partial and stochastic differential equations [Sauer 2012] and are based on the finite-difference [Knabner and Angerman 2003] or the finite-element method [Zienkiewicz et al. 2005]. There exists also a class of methods that proved to be an effective alternative to those classic numerical algorithms that is class of multigrid methods [Shapira 2003], [Wesseling 2004]. Their effectiveness has been also noticed in option pricing tasks. In this paper the modification of a two-level multigrid method by allowing an overlap between subdomains is described. This approach improve the convergence of the iteration process and the approximation of the solution at the expense of less susceptibility to parallelization. Application of this idea to a one-dimensional Black-Scholes equation is

presented. To be more exact the European call option is considered (for details on option pricing theory see [Hull 2006] or [Haug 2007]):

$$\begin{cases} \dfrac{\partial V(s,t)}{\partial t} + \dfrac{\partial V(s,t)}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V(s,t)}{\partial s^2} - rV(s,t) = 0 \\ V(s,T) = \max\{s-K,\,0\} \\ \lim\limits_{s\to 0} V(s,t) = 0 \\ \lim\limits_{s\to\infty} \dfrac{V(s,t)}{s} = 1 \end{cases} \tag{1}$$

where *s* is a stock price of the underlying asset, *t* is a time, *r* is a risk-free interest rate, σ is a volatility of the stock and K is a strike price. The solution $V:(0,\infty)\times[0,T]\to R$ is the price of the option as a function of *s* and *t* and the analytic solution is known as a Black-Scholes formula.

Over the years many methods of solving the Black-Scholes model were developed, see [Fries 2007], in particular algorithms used to solve partial differential equations were successfully adapted to options pricing. The main dissimilarity between these two applications are the boundary conditions. For tasks from science subjects boundary conditions are exactly specified. In Black-Scholes model the domain is infinite and therefore only an approximation of boundary conditions is known. This issue influences the accuracy of the solution computed by the tested method and makes much harder to developed new, reliable algorithms.

In finite-difference schema the infinite domain $(0,\infty)\times[0,T]$ is change to the finite $(S_{\min}, S_{\max})\times[0,T]$ in such a way, that all probable values of *s* are within the domain. Let *M* be the number of parts, to which the space interval was divided and *N* the number of parts being the result of the partition of the time interval. Using the implicit Euler method, see [Hull 2006] or [Pascucci 2011], the continuous task (1) could be discretized as follows

$$\begin{cases} \dfrac{V_i^{\,j} - V_i^{\,j-1}}{\tau} + ih\dfrac{V_{i+1}^{\,j-1} - V_{i-1}^{\,j-1}}{2h} + \dfrac{1}{2}\sigma^2 (ih)^2 \dfrac{V_{i-1}^{\,j-1} - 2V_i^{\,j-1} + V_{i+1}^{\,j-1}}{h^2} - rV_i^{\,j-1} = 0 \\ V_0^{\,j} = 0 \\ V_M^{\,j} = S_{\max} - Ke^{-r\tau(N-j)} \end{cases} \tag{2}$$

for $i = 0,1,2,\ldots,M$ and $j = 0,1,2,\ldots,N$, where by $V_i^{\,j} = V(S_{\min} + ih,\, j\tau)$, *h* is a step of a space interval and τ a step of a time interval.

The approximate solution is calculated only in the nodes of the grid. The finer grid the better approximation you may expect. This however involves greater computational complexity. The implicit Euler schema on each time layer requires solving system of linear equations, which is usually done by some kind of iterative meth-

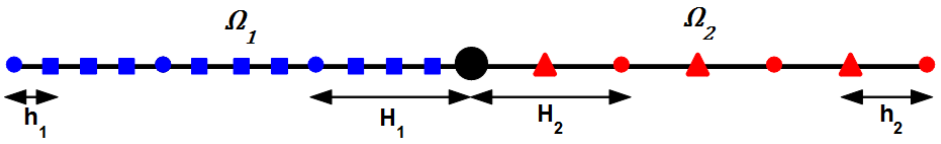
ods for sparse linear systems [Saad 2003] or dedicated methods using more sophisticated preconditioners [Bernardelli 2008], [Toselli & Widlund 2005], [Bhowmik & Stolk 2011]. In case of really big systems of linear equations, even when they are sparse, the memory and computational complexity of algorithms usually increases drastically. As an alternative, efficient approach, the multigrid methods may be considered, see [Shapira 2003], [Wesseling 2004].

THE MULTIGRID ALGORITHM FOR NON-OVERLAPPING GRIDS

The idea behind multigrid methods is to use a hierarchy of discretizations instead of one grid. The acceleration of the convergence of the iteration process is achieved by solving the task only on a coarse grid, which is faster due to the size of the problem, than solving the whole task on the fine grid. The definition of an interchange of information between coarser and finer layers is, besides the number and shape of the grids, at the same time a definition of the multigrid method. Multigrid methods are consistent with any discretization technique, starting from finite-difference [Strang 2007] and finite element [Zhu & Cangellaris 2006] methods. Also they are flexible if it comes to the regularity of the domain or boundary [Berridge & Schumacher 2002] and are known by their ease of parallelization, compare [Chiorean 2005]. In this paper the two-level multigrid method is described. For the comparison purpose two different splittings into grids are considered – the first with disjoint partitions and the second with overlaps.

Consider the decomposition of the space domain on each time layer into two disjoint subsets Ω_1 and Ω_2. Let Ω_H be the set of nodes on the coarse grid and by Ω_{H1} and Ω_{H2} denote Ω_H restricted to Ω_1 and Ω_2 respectively. We assume that these grids are uniform with the steps equal H_1 and H_2. Analogously let the fine grid Ω_h be decomposed into two subsets Ω_{h1} and Ω_{h2} with the steps denoted by h_1 and h_2 respectively. There is a simplifying assumption taken that the nodes of the coarse grid, are also nodes of the corresponding fine grid. The graphical representation of the two-level grid is presented on Figure 1.

Figure 1. Coarse grid and decomposition of the fine grid without overlap



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Let A_{H1}, A_{H1}, A_{h1} and A_{h2} be the matrices of the systems of linear equations related to the grids Ω_{H1}, Ω_{H2}, Ω_{h1} and Ω_{h2} respectively. Let V_{H1}^j, V_{H2}^j, V_{h1}^j and V_{h2}^j be the solutions of these systems of linear equations on the time layer $j = 0, 1, 2, ..., N-1$.

For the simplicity of notation let $V_H^j = V_{H1}^j \cup V_{H2}^j$ and $V_h^j = V_{h1}^j \cup V_{h2}^j$. To describe the multigrid method let introduce two operators:

- $R : \Omega_h \to \Omega_H$ – a restriction operator from the fine to the coarse grid,
- $I : \Omega_H \to \Omega_h$ – a prolongation operator from the coarse to the fine grid.

The algorithm for the non-overlapping decomposition of the grids could be describe in the following three steps on each time layer[1] $j = N, N-1, N-2, \ldots, 2, 1$:

STEP 1: Use the restriction operator R to calculate right side vectors of the matrix equations for the coarse grid

$$V_H^j = RV_h^j . \tag{3}$$

STEP 2: Use the newest calculated value at the common point of the two subdomains as the boundary condition for two systems of linear equations ($i = 1, 2$):

$$A_{Hi} V_{Hi}^{j+1} = V_{Hi}^j . \tag{4}$$

Solve these matrix equations in parallel.

STEP 3: Use the prolongation operator I to calculate the values of the solution in the nodes of the fine grid

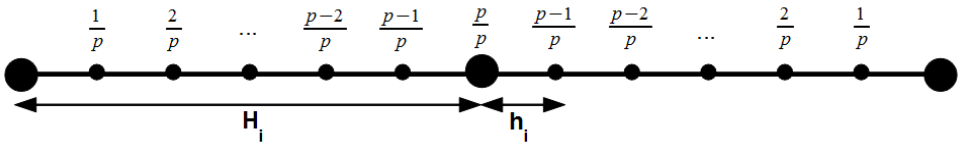
$$V_h^{j-1} = IV_H^j . \tag{5}$$

It can be shown [Strang 2007] that for the appropriately chosen operators R and I solutions of the equation defined on the fine and on the coarse grid are equivalent. In this paper we assume, that

$$R = \frac{1}{p} I^T \ , \tag{6}$$

if the step of the coarse grid is p times greater than the step of the fine grid. Interpolation operator I is defined on the stencil presented on Figure 2.

Figure 2. Graphical representation of the stencil for the interpolation operator I



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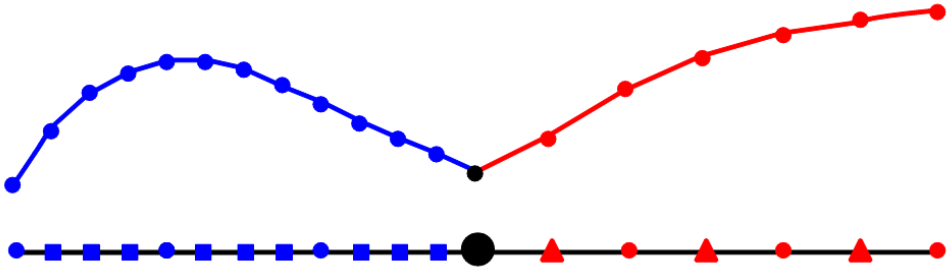
The solution limited to each subdomain should be smooth, but unfortunately, such an approach can result in the lack of smoothness in the whole domain. Demonstration of such a situation is given on Figure 3. The remedy could be an algorithm, which uses decomposition with overlapping grids.

[1] Black-Scholes equation is backward in time.

Figure 3. Example of non-smooth solution obtained by the multigrid algorithm with disjoint decomposition of the grids



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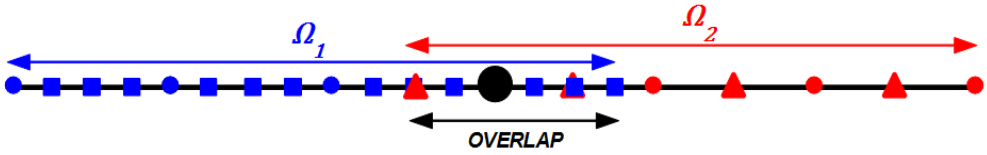
THE MULTIGRID ALGORITHM FOR OVERLAPPING GRIDS

A modification of the algorithm described in the previous subsection is made, by letting grids to overlap. Visualization of this idea is given on Figure 4. The concept of splitting the domain into smaller non-disjoint pieces is not new, [Toselli & Widlund 2005] or [Henshaw 2005], but explored rather only for partial differential equations, which have strict boundary conditions specified. Contrary, in the Black-Scholes equation only asymptotic character of the boundary conditions can be established. Therefore construction of algorithms for option pricing is much harder than it is for example for elliptic partial differential equations.

Figure 4. Coarse grid and decomposition of the fine grid with overlap



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To describe the algorithm presented already notation with minor changes can be used. For the clarity of the description we decompose the prolongation I and the restriction R operators into separate operators related to each subdomain, that is I_1, R_1 for Ω_1 and I_2, R_2 for Ω_2.

The algorithm for the overlapping decomposition of the grids could be summarized in the following steps on each time layer $j = N, N\text{-}1, N\text{-}2, \ldots, 2, 1$:

STEP 1: Use the restriction operator R_1 to calculate right side vector of the matrix equations for the coarse grid of the Ω_1:

$$V_{H1}^{j} = R_1 V_{h1}^{j}. \tag{7}$$

STEP 2: Based on values from the previous time layer use linear interpolation to calculate the value at the right end of the subdomain Ω_1 and use it as the boundary condition for the system of linear equations:

$$A_{H1}V_{H1}^{j+1} = V_{H1}^{j}. \tag{8}$$

STEP 3: Use the prolongation operator I_1 to calculate the values of the solution in the nodes of the fine grid of the Ω_1:

$$V_{h1}^{j-1} = I_1 V_{H1}^{j}. \tag{9}$$

STEP 4: Use the restriction operator R_2 to calculate right side vector of the matrix equations for the coarse grid of the Ω_2:

$$V_{H2}^{j} = R_2 V_{h2}^{j}. \tag{10}$$

STEP 5: Based on values from step 3 use linear interpolation to calculate the value at the left end of the subdomain Ω_2 and use it as the boundary condition for the system of linear equations:

$$A_{H2}V_{H2}^{j+1} = V_{H2}^{j}. \tag{11}$$

STEP 6: Use the prolongation operator I_2 to calculate the values of the solution in the nodes of the fine grid of the Ω_2:

$$V_{h2}^{j-1} = I_2 V_{H2}^{j}. \tag{12}$$

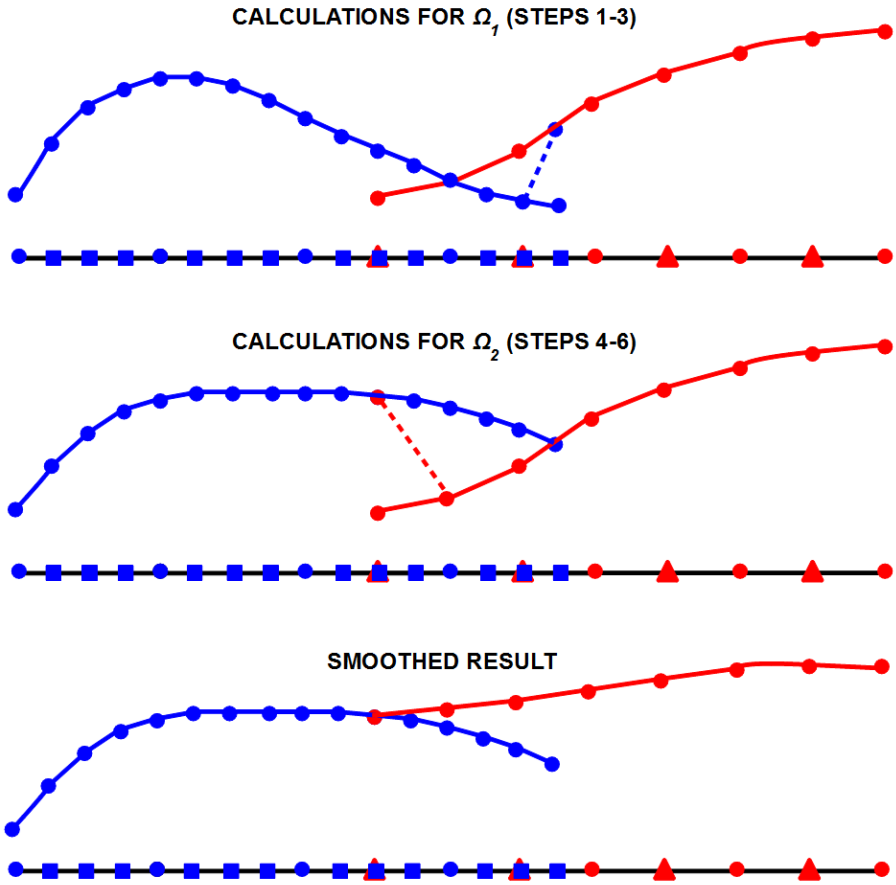
The exemplary difference between results of two algorithms on the given time layer is visualized on Figure 3 (non-overlapping grids) and Figure 5 (overlapping grids). Introducing overlaps firstly gives a fresh information about the boundary conditions for computations on each subdomain, and secondly makes the final solution smoother. Of course there are some disadvantages of this approach. Most of all computations for Ω_1 and Ω_2 can't be performed in parallel, steps 4-6 may be executed only after steps 1-3 are finished. The difference in performance however could be observed probably only on a computer cluster with many machines, not on the standard, even multi-processor computer.

NUMERICAL EXPERIMENTS

To verify the effectiveness of the proposed method with overlapping grids numerical experiments were conducted. As the test case the European call option problem (1) with the following parameters was chosen

$$T = 2,\ K = 10,\ r = 0.2,\ \sigma = 0.25,\ S_{min} = 0,\ S_{max} = 20. \tag{13}$$

Figure 5. The example of smoothed solution obtained by the multigrid algorithm with decomposition of the grids with overlaps



Source: own elaboration

Other parameters were changed during the experiments. The complete list of the parameters is as follows:
M – number of parts of the space interval (number of grid nodes on each time layer is equal $M+1$),
N – number of time layers used in the calculation, in experiments assumed as $N = round\left(M\left(2\max\{p_1, p_2\}\right)^{-1}\right)$
$overlap_1$, $overlap_2$ – size of the overlap measured as a number of fine grid points outside the corresponding subdomain,
p_1, p_2 – ratio of the step sizes between the coarse and fine grids on the subdomains, that is $p_i = H_i h_i^{-1}$ for $i = 1, 2$.

For every result there were measured the time of the computation and two types of errors:

- maximum residual error ($NORM_\infty$): $\left\| V - V^* \right\|_\infty = \max_{i=0,1,\ldots M} \left| V_i^0 - \tilde{V}_i^0 \right|$,
- absolute error at K (ABS_K): $\left| V(K) - \tilde{V}(K) \right|$,

where V is the computed solution and $\tilde{V}$ is the analytic solution from the Black-Scholes formula. The series of numerical experiments were divided into parts designed to verify dependence of errors on the ratio of number of grid nodes between subdomains, size of overlaps and ratio of the step sizes between the coarse and fine grids on the subdomains. Results of each part of the experiments and their brief descriptions are given in the remaining part of this section.

Selected results of the numerical experiments that verified the dependence between solution's accuracy and the number of grid nodes of the two subdomains are presented in Table 1.

Table 1. Dependence between errors and the ratio of number of grid nodes of two subdomains

Ratio	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
10% : 90%	12 000	0.030552	0.102905	0.009131	0.000673
30% : 70%		0.018038	0.011974	0.012346	0.000560
50% : 50%		0.015608	**0.011030**	0.011778	**0.000081**
70% : 30%		**0.014711**	0.017558	0.010776	0.002198
90% : 10%		0.033411	0.048296	**0.004385**	0.038145
10% : 90%	60 000	0.014191	0.021621	0.010784	0.000143
30% : 70%		0.014462	**0.002535**	0.011386	0.000120
50% : 50%		0.014556	0.003103	0.011268	**0.000024**
70% : 30%		0.014474	0.004428	0.011065	0.000431
90% : 10%		**0.013784**	0.010971	**0.009760**	0.007581

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $p_1 = p_2 = 6$, $overlap_1 = overlap_2 = 5$. In bold are the smallest values of errors for each M.

Source: own calculations

Maximum residual error ($NORM_\infty$) for the algorithm with overlaps seems to be worse or at most comparable to corresponding errors of the algorithm with disjoint decomposition of grids. In contrast, absolute error at K (ABS_K) is (except the extreme case of ratio 90% : 10%) much smaller for the algorithm with the overlap modification. In practice we are rather interested in the best possible value at the given point (or interval containing that point) not in the solution in the whole domain. This is why minimization of ABS_K error is desirable. Due to the computation complexity and given in Table 1 results in other two parts of the numerical experiment the ratio 50% : 50% was used.

The second part of the series of numerical experiments is related to the clue of the idea behind the described algorithm. It is intended to check how the size of an overlap influence the value of the errors. Selected results of these part of experiments are presented in Table 2.

Table 2. Dependence between errors and the size of overlaps, assuming $overlap_1 = overlap_2$

Overlap	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
1	12 000	**0.015608**	**0.008397**	**0.011778**	0.002106
2			0.009056		0.000317
3			0.009714		0.002023
5			0.011030		0.000081
6			0.011686		**0.000073**
7			0.012342		0.001103
1	60 000	**0.014556**	**0.002573**	**0.011268**	0.0004316
2			0.002705		0.0000719
3			0.002838		0.0004144
5			0.003103		0.0000244
6			0.003236		**0.0000065**
7			0.003368		0.0002299

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $p_1 = p_2 = 6$, ratio between number of fine grid nodes of two subdomains 50% : 50%. In bold are the smallest values of errors for each *M*.

Source: own calculations

In any considered case adding an overlap improved the accuracy of the solution. An observable fact resulting from comparison between different sizes of an overlap, is that increasing an overlap increases the $NORM_\infty$ error. Because of the approximation of the derivatives that was used in the discretization, the size of an overlap should be equal one step of the fine grid. Nevertheless concerning the second measured error (ABS_K) we can noticed, that it is rather effective to increase an overlap (taking into account the size *M* of the grid on the space interval). It seems to improve the accuracy (measured as a point error) of the solution for any reasonable set of parameters. In almost all numerical experiments parts of the experiments the size of an overlap was set to 5.

The last, third part of numerical experiments is directly related to the key concept of the multigrid method. There were measured errors while changing the parameters p_1 and p_2. Results are gathered in Table 3. As was expected, decreasing the size of the coarse grid makes errors greater. However ABS_K error of the solutions of the algorithm with overlapping grids may be and, depending on the size of the problem, usually is smaller for $p_1 = p_2 \in \{3,6\}$. Of course for greater values of considered in this part of numerical experiments parameters computations are noticeably faster.

Table 3. Dependence between errors and the ratio of the step sizes between the coarse and fine grids on the subdomains

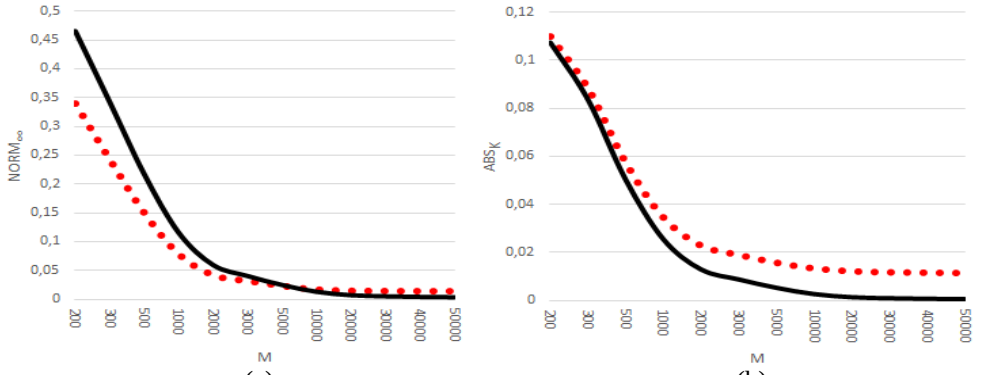
$p_1 = p_2$	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
2	10 000	**0.014518**	**0.003498**	**0.011335**	0.000352
3		0.014729	0.005880	0.011468	**0.000026**
4		0.015056	0.008259	0.011611	0.001133
6		0.016121	0.013006	0.011901	0.000091
10		0.019881	0.022429	0.012500	0.003678
2	60 000	**0.014462**	**0.001510**	**0.011173**	3.91531e-06
3		0.014478	0.001908	0.011195	**3.85165e-06**
4		0.014501	0.002307	0.011219	9.00507e-06
6		0.014556	0.003103	0.011268	2.44427e-05
10		0.014694	0.004694	0.011368	6.16543e-05

Source: own calculations

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $overlap_1 = overlap_2 = 5$, ratio between number of fine grid nodes of two subdomains 50% : 50%. In bold are the smallest values of errors for each M.

In case of both measured errors algorithm with overlaps is performing considerably better than the algorithm with disjoint subdomains. The dependence of the $NORM_\infty$ and the ABS_K error on the value of M is given on Figure 6.

Figure 6. The value of the (a) $NORM_\infty$ error and (b) ABS_K error with increasing value of M. Used parameters: $overlap_1 = overlap_2 = 5$, $p_1 = p_2 = 6$, ratio between number of fine grid nodes of two subdomains 50% : 50%. Dotted line – no overlaps, continuous line – with overlaps



(a) (b)

Source: own elaboration

SUMMARY

In this paper the proposition of the multigrid method with overlapping grids is presented. The description includes the specification of the algorithm steps, its pros and cons as well as results of the numerical experiments. The method is designed to solve one-dimensional Black-Scholes problem for the European call option and was compared to the analogous method without overlaps. Based on the theory and numerical experiments one can formulate the following conclusions. Comparing to the method without overlapping grids presented method has

- improve convergence rate,
- better accuracy of the solution,
- comparable memory and computer power requirements,
- comparable time of computations,
- less parallelization possibilities.

Comparing to the multigrid method without decomposition of the domain proposed method has

- slightly worse accuracy of the solution,
- much faster time of computations,
- more parallelization possibilities,
- much less memory and computer power requirements.

Nevertheless the one-dimensional task is considered as reasonably easy to solve. The true advantage of the described method should become visible in more than one dimension. Such a generalization is one of the main goals of the further research. Also worth considering would be implementation for more than two subdomains and most of all checking the effectiveness of the method for other option pricing problems, especially those, for which the analytic solution is unknown. Generalization of the classic Black-Scholes model could include for example American options, model with variable coefficients or adding a dividend term to the model.

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