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PRECISE ESTIMATES OF RUIN PROBABILITIES

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Abstract: In this paper we investigate a sequence of accurate approximations of ruin probabilities in discrete time models. We prove its convergence to the exact ruin probability without any restrictive assumptions on the claim distribution. Numerical studies show that the sequence, from the first term on, accurately approximates ruin probabilities. A formula for ruin probabilities in the finite horizon is also proposed.

Keywords: discrete time risk models, ruin probabilities, approximations, Solvency II

INTRODUCTION

Changes taking place on the worldwide financial market cause reforms of the supervision systems. The forthcoming EU directive Solvency II gives each insurance company a possibility to submit a motion (to the relevant supervision authority) for approving the internal model for determining the solvency capital requirement (SCR).

In this paper we investigate estimates of ruin probabilities in such a natural risk model. The discrete time setup, where the financial situation is reported on to the supervisor at the end of each fixed time period, has several advantages [Cheng et al. 2000], [Jasiulewicz 2013], [Gajek and Rudź 2013], [Rudź 2015] and others.

There are numerous papers investigating approximations of ruin probabilities in various risk models [De Vylder 1978], [Rolski et al. 1999], [Asmussen 2000], [Grandell 2000], [Čížek et al. 2005], [Dickson 2005], [Asmussen and Albrecher 2010] and others. Some of the approximations (for instance the De Vylder method) are based upon purely empirical grounds[1].

[1] See Asmussen S. [2000] Ruin Probabilities, World Scientific, Singapore, p. 80.

In this paper we investigate, along theoretical as well as numerical lines, some precise estimates of ruin probabilities in the following risk model.

A discrete time risk model

In this section we summarise, after [Bowers et al. 1997], [Klugman et al. 1998], [Rolski et al. 1999], [Gajek 2005], [Gajek and Rudź (2013)] and [Rudź 2015], the relevant material on a discrete time risk model.

All stochastic objects considered in the paper are assumed to be defined on a probability space $(\Omega,\ \Im,\ P)$. Let N denote the set of all positive integers and $\boldsymbol{R}$, the real line. Set $N^0 = N \cup \{0\}$, $R_+ = (0,\ \infty)$, $R_+^0 = [0,\ \infty)$, and $\overline{R_+} = (0,\ \infty]$.

Let a non-negative random variable X_i denote the aggregated sum of the claims in the i th time period; a positive real γ, the amount of aggregated premiums received each period; and a non-negative real u, the insurer's surplus at 0. We assume that $X_1,\ X_2, \ldots$ are i.i.d. random variables with a common distribution function F. Let $\{S(n)\}_{n\in N^0}$ denote the insurer's surplus process defined by

$$S(0) = u \tag{1}$$

and

$$S(n) = u + \gamma n - \sum_{i=1}^{n} X_i,\ \ n \in N. \tag{2}$$

The moment of ruin is defined by

$$\tau = \tau(u) = \inf\{n \in N:\ \ S(n) < 0\} \tag{3}$$

under the convention that $\inf \varnothing$ means ∞. The probability of ruin in a finite horizon n is defined by

$$\Psi_n(u) = P(\tau(u) \le n) \tag{4}$$

and the infinite-horizon probability of ruin by

$$\Psi(u) = P(\tau(u) < \infty). \tag{5}$$

Fix $\gamma \in \boldsymbol{R}_+$. Define $M:\ R \to \overline{R_+}$ by

$$M(r) = Ee^{-r(\gamma - X_1)},\ \ r \in R. \tag{6}$$

The positive real solution r_0 of the following equation

$$M(r_0) = 1, \tag{7}$$

if it exists, is called adjustment coefficient. Denote $R_0(u) = e^{-r_0 u}$, $u \ge 0$ and $M_0 = \{r \in R:\ \ M(r) < \infty\}$.

The following result provides a sufficient condition for the existence of r_0.

Lemma 1. [Gajek 2005, p. 15]. Assume that $EX_1 < \gamma$, $F(\gamma) < 1$ and the set M_0 is open. Then there exists a unique adjustment coefficient $r_0 > 0$.

Under the assumptions of Lemma 1 the following equality holds [Bowers et al. 1997]

$$\Psi(u) = \frac{R_0(u)}{E[R_0(S(\tau))|\tau < \infty]}. \tag{8}$$

Research objectives

There are some difficulties when working with the probability of ruin in the finite horizon. Let us quote at least one opinion of eminent researchers in this field [Rolski et al. 1999, p. 148]:

"Unfortunately, in many cases it is difficult to express the finite-horizon ruin probabilities in a closed form. The infinite-horizon ruin probabilities are mathematically simpler".

Thus, the research objectives are:

1. Finding a formula for the probability of ruin Ψ_n in the finite horizon.
2. Construction of a sequence of Ψ's approximations by means of Ψ_n and some other functionals.
3. Investigating the convergence of the sequence of Ψ's approximations.
4. Finding a close connection between the approximations and Formula (8).
5. Numerical studies which illustrate precision of the approximations.

MAIN RESULTS

From now on, we will use the convention that $\sum_{m=1}^{0} x_m = 0$. For simplicity of notation, we write $\int_a^\infty$ instead of $\int_{(a,\ \infty)}$ and $\int_0^b$ instead of $\int_{[0,\ b]}$, where $a,\ b \in R_+^0$.

In the present section we prove that

$$\Psi_n(u) = \sum_{k=1}^{n} \overbrace{\int \ldots \int}^{k}_{A_k(u)} dF(x_k)\ldots dF(x_1),\ n \in N,\ \ u \geq 0, \tag{9}$$

where the set $A_k(u),\ \ k \in \{1,\ ...,\ n\}$ consists of ordered tuples $(x_1,\ ...,\ x_k) \in R^k$ satisfying the following conditions:

- $x_k > u + k\gamma - \sum_{m=1}^{k-1} x_m,$
- $x_i \leq u + i\gamma - \sum_{m=1}^{i-1} x_m$ for every $i \in \{1,\ ...,\ k-1\}.$

Denote $\Gamma = R_+ \setminus (0,\ \gamma)$ and $d_0 = \sup\{d \in \Gamma :\ F(d) < 1\},$ if such a $d_0 \in \overline{R_+}$ exists. We will assume that $d_0 = \infty,$ $\gamma > \boldsymbol{E}\boldsymbol{X}_1$ and the set M_0 is open.

Formula (9) will be used to define a sequence $\{\tilde{\Psi}_n\}_{n \in N}$ given by

$$\tilde{\Psi}_n(u) = \frac{\Psi_n(u)}{D_n(u)},\ \ n \in N,\ u \geq 0, \tag{10}$$

where

$$D_n(u) = \sum_{k=1}^{n} \overbrace{\int \ldots \int}^{k}_{A_k(u)} e^{-r_0(k\gamma - \sum_{m=1}^{k} x_m)} dF(x_k)\ldots dF(x_1). \tag{11}$$

Throughout the paper we will investigate the approximations of Ψ given by (10).

Theorem 1.

i) for all $n \in N$ and $u \geq 0,$ it holds

$$\Psi_n(u) = \sum_{k=1}^{n} \overbrace{\int \ldots \int}^{k}_{A_k(u)} dF(x_k)\ldots dF(x_1). \tag{12}$$

Assume that $d_0 = \infty,$ $\gamma > \boldsymbol{E}\boldsymbol{X}_1$ and the set M_0 is open. Then:

ii) for all $n \in N$ and $u \geq 0$

$$\tilde{\Psi}_n(u) = \frac{R_0(u)}{E[R_0(S(\tau)) | \tau \leq n]}, \tag{13}$$

iii) the sequence $\{\tilde{\Psi}_n\}_{n \in N}$ converges pointwise, as $n \to \infty,$ to the exact probability of ruin $\Psi.$

Proof: Observe that $\{\tau=1\}, ..., \{\tau=n\}$ are disjoint events whose union is $\{\tau \le n\}$. Moreover, under the condition $\tau=1$: $S(\tau)=S(1)=u+\gamma-X_1$ and

$$\tau=1 \Leftrightarrow S(1)<0 \Leftrightarrow X_1>u+\gamma. \tag{14}$$

In much the same way as above, under the condition $\tau=2$: $S(\tau)=S(2)=u+2\gamma-(X_1+X_2)$ and

$$\tau=2 \Leftrightarrow (S(2)<0,\ S(1)\ge 0) \Leftrightarrow (X_2>u+2\gamma-X_1,\ X_1\le u+\gamma). \tag{15}$$

In general, under the condition $\tau=n,\ n\in N: S(\tau)=S(n)=u+\gamma n-\sum_{m=1}^{n} X_m$ and

$$\begin{aligned}\tau=n &\Leftrightarrow (S(n)<0, \forall_{1\le i<n} S(i)\ge 0)\\ &\Leftrightarrow (X_n>u+n\gamma-\sum_{m=1}^{n-1}X_m, \forall_{1\le i<n} X_i\le u+i\gamma-\sum_{m=1}^{i-1}X_m).\end{aligned} \tag{16}$$

i) Thus,

$$\begin{aligned}\Psi_n(u) &= P(\tau\le n)=P(\tau=1)+P(\tau=2)+...+P(\tau=n)\\ &= \int_{u+\gamma}^{\infty} dF(x_1)+\int_0^{u+\gamma}\int_{u+2\gamma-x_1}^{\infty} dF(x_2)dF(x_1)+...\\ &+\int_0^{u+\gamma}\int_0^{u+2\gamma-x_1}...\int_0^{u+(n-1)\gamma-\sum_{i=1}^{n-2}x_i}\int_{u+n\gamma-\sum_{i=1}^{n-1}x_i}^{\infty} dF(x_n)...dF(x_1)\\ &= \sum_{k=1}^{n}\overbrace{\int ... \int_{A_k(u)}}^{k} dF(x_k)...dF(x_1).\end{aligned} \tag{17}$$

ii) Furthermore, under the assumptions, $\int_{u+\gamma}^{\infty} dF(x)>0$ for every $u\ge 0.$ Thus, by (17), $\Psi_n(u)>0,\ \ u\ge 0.$ Following Lemma 1, there exists a unique adjustment coefficient $r_0>0$. Therefore, for all $n\in N$ and $u\ge 0$

$$E[R_0(S(\tau))|\tau\le n]=\frac{R_0(u)D_n(u)}{P(\tau\le n)}=\frac{R_0(u)D_n(u)}{\Psi_n(u)}, \tag{18}$$

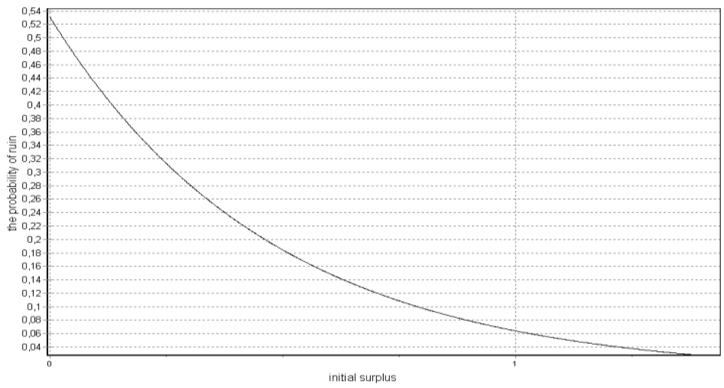
where the above-mentioned observations and some properties of the conditional expectation [Jakubowski and Sztencel 2001, p. 124] were used. Thus, ii) and iii) hold as well. ■

NUMERICAL EXAMPLES

In this section we present the results of numerical studies carried out by the author. We compare the approximations discussed in the previous section with the exact ruin probabilities obtained by means of the method described in [Rudź 2015]. The bisection method was also used.

We consider: exponentially distributed claims with the expected value $EX_1 \approx 0.22$ (Figure 1), a gamma claim distribution with the shape parameter 2 and the scale one 5.5 (Figure 2), a mixture of two gamma distributions with the shape parameters $(2,\ 2)$, the scale ones $(3,\ 7.5)$ and weights $(0.85,\ 0.15)$ respectively (Figure 3), a mixture of five exponential distributions with the scale parameters $(1,\ 2,\ 3,\ 7,\ 13)$ and weights $(0.3,\ 0.2,\ 0.3,\ 0.1,\ 0.1)$ respectively (Figure 4) and a mixture of five exponential distributions with the scale parameters $(1,\ 3,\ 5,\ 8,\ 10)$ and weights $(0.1,\ 0.3,\ 0.2,\ 0.2,\ 0.2)$ respectively (Figure 5).

Figure 1. Exponential distribution with the scale parameter 4.5 (i.e. the expected value $EX_1 \approx 0.22$). The amount of aggregated premiums $\gamma = 0.3$. The $\tilde{\Psi}_1$ approximation coincides with the exact probability of ruin Ψ (the solid line)



Source: own computations

The approximation $\tilde{\Psi}_1$ might be faultless (Figure 1) or almost faultless (Figures 2 and 3). We have confirmed the above observation by means of a number of simulation experiments.

Figure 2. Gamma distribution with the shape parameter 2 and the scale one 5.5 (i.e. the expected value $EX_1 \approx 0.36$). The amount of aggregated premiums $\gamma = 0.45$. The $\tilde{\Psi}_1$'s graph (the dotted line) almost coincides with the exact ruin probability's one (the solid line)

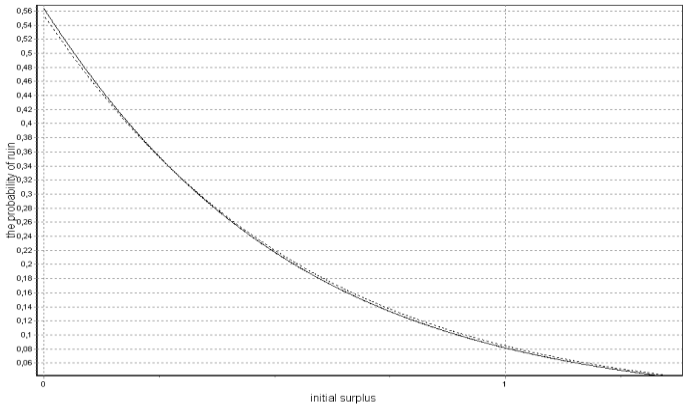


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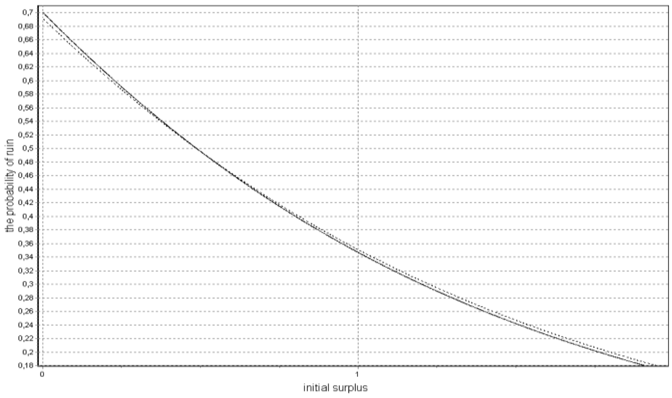
Figure 3. A mixture of two gamma distributions with the shape parameters $(2,\ 2)$, the scale ones $(3,\ 7.5)$, weights $(0.85,\ 0.15)$ respectively and the amount of aggregated premiums $\gamma = 0.7$. The $\tilde{\Psi}_1$'s graph (the dotted line) almost coincides with the exact ruin probability's one (the solid line)



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One distinguishes the graph of $\tilde{\Psi}_1$ from the Ψ's one in Figures 4 and 5. The latter one illustrates the approximations $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ and $\tilde{\Psi}_3$.

Figure 4. A mixture of five exponential distributions with the scale parameters $(1,\ 2,\ 3,\ 7,\ 13)$, weights $(0.3,\ 0.2,\ 0.3,\ 0.1,\ 0.1)$ respectively and the amount of aggregated premiums $\gamma = 0.6$. The $\tilde{\Psi}_1$ approximation (the dotted line) is presented in relation to Ψ (the solid line)

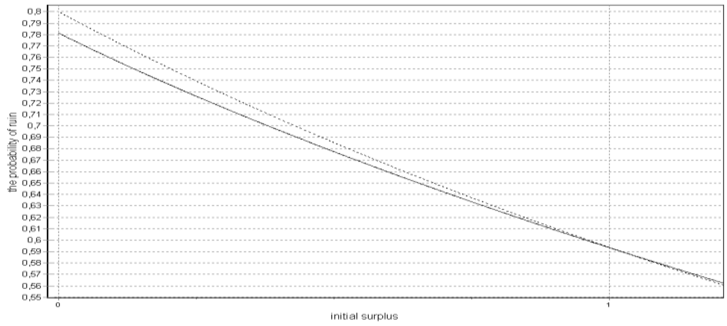

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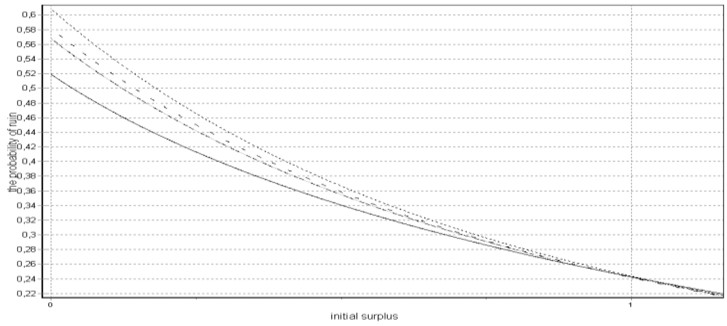
Figure 5. A mixture of five exponential distributions with the scale parameters $(1,\ 3,\ 5,\ 8,\ 10)$, weights $(0.1,\ 0.3,\ 0.2,\ 0.2,\ 0.2)$ respectively and the amount of aggregated premiums $\gamma = 0.41$. The approximations: $\tilde{\Psi}_1$ (the dotted line), $\tilde{\Psi}_2$ (the dashed line) and $\tilde{\Psi}_3$ (the dashed-dotted line) are presented in relation to Ψ (the solid line)



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CONCLUSION

In this paper we investigated Sequence (10) of Ψ's approximations in discrete time models. All research objectives have been achieved.

Theorem 1 i) gives a formula for the probability of ruin in the finite horizon. Theorem 1 ii) and iii) shows a close connection between Sequence (10) and Formula (8). In particular, it is shown that Sequence (10) converges pointwise to the exact probability of ruin without any restrictive assumptions on the claim distribution. The results of a numerical study lead to the conclusion that Sequence (10), from the first term on, accurately approximates ruin probabilities in the considered cases of claim distributions.

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