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MODERNIZATION OF MEANS FOR ANALYSES AND SOLUTION OF NONLINEAR PROGRAMMING PROBLEMS

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Abstract: The problems of optimization for nonlinear programming (NLP) with constraints inequalities are considered. Definition of condition-indicator as quantitative criterion of the properties of Lagrange function is justified. Application of indicator to increasing degree of completeness for system in NLP for finance and business problems with constraints inequalities are obtained. The new Lagrange function with square of each component of vector Lagrange multipliers for nonlinear objective function simultaneously with criterion-indicator as a source of additional equations is investigated. The conditions, in which the dimensionality of the vector of strategies and the number of constraints doesn't effects on the uniqueness of the optimization problem solution is received and discussed.

Keywords: optimization strategies, NLP, quantitative criterion-indicator, new Lagrange function, source of additional equations.

INTRODUCTION

Applicability of modern methods of optimal design [Akoff and Sasieny 1971, Ventcel. 1980, Germaier 1971, Zaychenko 2000] to the analysis of socio-economic projects, including systems of management [Kondratenko et al. 2013, Kondratenko et al. 2014, Kondratenko 2015] is defined by capabilities of nonlinear modeling techniques [Trunov 2009]. Modernization of its for finding a solution of the maximization problem with constraints inequality for the objective function, structure of which is dynamically changed in according with proposed optimization strategies and decisions, are considered [Trunov 2011, Kondratenko 2015] as actual tasks. In particular, actuality acquires problems of optimal design, selection of solution, when a dynamic systems such as the production company with a stocks and logistic divisions and trading network, or stationary energy generation company, or system of distribution companies, or underwater vehicles, or mobile robots of production systems are designed and defined as systems of objects economic and technical parameters of which are described by nonlinear models [Trunov 2011]. Under these conditions, the task of formulating performance criteria and expressions for their quantitative evaluation and optimal design and decision-making are particularly important from a general methodological and practical point of view. In modern state of development of the NLP and optimal design and management theory, this scientific analysis carried out by expert's and decision making support systems [Zaychenko 2000, Kondratenko et al. 2013, Kondratenko 2015]. The most common approaches, realized in them, lead to the general problem of nonlinear simulation, which usually solves by the method of Lagrange multipliers. The application of these two approaches: profit maximization or losses minimization problems with constraints inequalities, in depending from the number of variables, leads to the problem of finding solutions of independent nonlinear algebraic equations or systems of nonlinear algebraic equations [Trunov 2011].

According to R. Bellman, Kuhn-Tucker, L. V. Kantorovich [Zaychenko 2000] the problem of constructing a standard search algorithm solution of such problems devoted a lot of work, but despite this the applicability of the known methods of successive approximations such as Newton - Kantorovich, Bellman's quasilinearization is limited by the cases, when the first derivative of the pattern, formed by the action of the operator is not zero [Trunov 1999]. The last condition for a long time limited the applicability of quasilinearization - one from the most effective methods to solution of nonlinear problems, in cases when the operator is strictly monotone and its effect on determination plural forms convex set, with one simple root. However, recurrent approximation method, proposed by the author in [Trunov 1999], doesn't impose constraints on the value of the first derivative of operator actions.

However, despite these significant advantages of the method at the present time its applicability to the problems of modeling and designing is constrained by the number of absent equations in the NLP system [Trunov 2011] and non-monotonicity of objective function or its partials derivatives [Trunov 2013]. Thus, despite the success in solving such equations the main problems are don't solved at the present time:

- Building effective generalized algorithm for solution of NLP problems with constraints inequality;
- Determining the conditions, which allow formulate additional independent equations, which complement of system for the Lagrange multipliers method with constraints inequality.

The main purpose of article is to:

- To explore the possibility of using the fundamental properties of the objective function to finding new conditions for the formation of additional equations to complement of NLP system and provide existence and uniqueness of solution of the problem of NLP.

TASK POINTING

Suppose, that $\overline{X}$ is a vector of system states, with *n* – components, that formed and it is defined in the space of real numbers - R^n. Also suppose, that an objective function - $F(\overline{X})$ and limitations provided in the form constraints of inequalities: $g_j(\overline{X}) \leq 0$; $j = \overline{1,m}$ are given. Then Lagrange function has the form [Zaychenko 2000]:

$$L(\overline{X},\overline{\Lambda}) = F(\overline{X}) + \sum_{j=1}^{m} \lambda_j g_j(\overline{X}) . \tag{1}$$

Search for minimum of Lagrange functions, according to Kuhn-Tucker theorem, leads to the problem of the saddle point. The last, under conditions of linearity for restrictions, in general is reduced to a system of equations in which are separated linear and nonlinear operators:

$$L_1(\overline{X}) + L_2(\overline{X},\overline{\Lambda}) = 0 , \tag{2}$$

where

$$L_1(\overline{X}) = \nabla_x\left[F(\overline{X})\right]; L_2(\overline{X},\overline{\Lambda}) = \sum_{j=1}^{m} \lambda_j \nabla_x\left[g_j(\overline{X})\right]$$

– patterns are created by the action of a linear operator (gradient) in the space of vectors $\overline{X}$ and $\overline{\Lambda}$ determine the formation of the system of equations by the Lagrange multipliers method. In the case of one-dimensionality of the vector $\overline{X}$ and the presence of only one of the constraints in the form of equation (2) are reduced to algebraic equation, which usually nonlinear. In other cases, (2) is reduced to a system of nonlinear equations. Thus, NLP is reduced to the problem of finding a root of nonlinear algebraic equation or roots of systems of nonlinear algebraic equations [Zaychenko 2000]. Generalization of this problem can be done by introducing of the nonlinear objective function $F(\overline{X})$, which is scalar-function of vector and is defined as N times differentiable in the space R^n of n measurable vector strategies $\bar{X}$ under constraints inequalities $g_j\left(\bar{X}\right) \geq 0$, $j = \overline{1,m}$, which will be written compactly due to implementation of vector-functions, i.e. $\bar{G}\left(\bar{X}\right) \leq 0$. We pose the problem of maximizing the total benefits from investments under the conditions of existing constraints inequalities:

$$\begin{cases} \max\limits_{\bar{X}} \quad F(\overline{X}), \\ \bar{G}\left(\bar{X}\right) \geq 0. \end{cases} \tag{3}$$

and in other cases problem of minimizing, for example, a total loses:

$$\begin{cases} \min\limits_{\bar{X}} \quad F_L(\overline{X}), \\ \bar{G}\left(\bar{X}\right) \le 0. \end{cases} \tag{4}$$

Let's notice, subscript L is indicated changes in general system for problems of minimizing. Assume that each component of the vector-function constraints also are defined as differentiated N times in half-space of real numbers - R^n and allowable region S is formed by intersection of half-space, each of which is the inequality formed by the component of the vector-function of constraints and is the convex polyhedron. Applying to the problem of the Kuhn-Tucker theorem, draw up a Lagrange function [Zaychenko 2000]:

$$L(\bar{X},\bar{\Lambda}) = F(\overline{X}) + \bar{\Lambda}^T \bar{G}\left(\bar{X}\right) \tag{5}$$

and served system of equations maximization problem with constraints inequalities:

$$\begin{cases} \nabla_x L(\bar{X},\bar{\Lambda}) + \bar{V} = 0, \bar{X}^T \bar{V} = 0, \\ \nabla_\lambda L(\bar{X},\bar{\Lambda}) - \bar{W} = 0, \bar{\Lambda}^T \bar{W} = 0, \end{cases} \tag{6}$$

or minimization problem with constraints inequalities

$$\begin{cases} \nabla_x L(\bar{X},\bar{\Lambda}) - \bar{V} = 0, \bar{X}^T \bar{V} = 0, \\ \nabla_\lambda L(\bar{X},\bar{\Lambda}) + \bar{W} = 0, \bar{\Lambda}^T \bar{W} = 0, \end{cases}$$

where are indicated:

$\bar{V} = \left[V_1,..V_i,...V_n\right]^T$: $\bar{W} = \left[W_1,..W_j,...W_m\right]^T$; $\bar{\Lambda} = \left[\lambda_1,..\lambda_j,...\lambda_m\right]^T$ - additional relevant n and m - dimensional vectors strategies and vector m - dimensional of Lagrange multipliers. System (6) contains $n+m+2$ equations with $2(n+m)$ unknowns. To increase degree of completeness system, as it is required in work [Zaychenko 2000], there is proposed approximation by MRA [Trunov 2011]. This approach increased number of equations till $2n+m+2$. Certainly that searching for independent and additional m equations, which will be to complete algebraic system (6), is an actual task [Trunov 2013].

JUSTIFICATION OF DEFINITION FORMULAS OF INDICATOR AND SOLUTION OF THE OPTIMIZATION PROBLEM WITH NONLINEAR CONSTRAINTS INEQUALITIES

Realization of this goal is challenging solution of problem, because to create the overalls principles of formation of additional equations it means to find out an conditions or criterions, based on the properties of the objective function and vector function of constraints as a fundamental properties. However, this problem currently not solved, what is stipulated of the difficulties of its nature [Mokhtar et al. 2006].

The search of fundamental properties of function and using them as indicators for solution of finance and business problems by means of NLP with constraints inequalities is discussed in work [Trunov 2011]. Application of comparing operator based on properties of function and derivatives of first and second orders is demonstrated as effective indicators of function states at the point of current approach of vector strategies [Trunov 2013]. Lets, consider another type of indicator for our purposes.

Pointing the goal: theoretically to justify expression of indicator. Define $f(\overline{X})$ in the space of real numbers R^n, as a function of *n* component of the state vector in a neighborhood of approach $\overline{X_n}$, assume that $f(\overline{X})$ is continuous and differentiated three times, while assuming that all approaches have one initial value, following to the method of recurrent approximation [Trunov 1999, Trunov 2011], decompose its in series:

$$f(\overline{X_{n+1}}) = f(\overline{X_n}) + \sum_{i=1}^{n} \frac{\partial f(\overline{X_n})}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i} \Delta x_j \Delta x_i + R(\overline{X_n}). \quad (7)$$

Finding partial derivative of (7), with respect to one of the variable x_i involved and the other variable the components of the vector $\overline{X}$ being treated, for example, as constants we can be put in the form:

$$\frac{\partial f(\overline{X_{n+1}})}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f(\overline{X_n})}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i} \Delta x_j \Delta x_i + \frac{\partial R(\overline{X_n})}{\partial x_i}. \quad (8)$$

Assuming, that in the direction of x_i - components of vector of the objective function becomes local extremum in according of it necessary condition is written:

$$\frac{1}{2} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i} \Delta x_j \Delta x_i + \frac{\partial R(\overline{X_n})}{\partial x_i} = 0 \quad (9)$$

or

$$\frac{\partial^2 f(\overline{X_n})}{\partial x_i^2} \Delta x_i + \frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i} \Delta x_j + \frac{\partial R(\overline{X_n})}{\partial x_i} = 0. \quad (10)$$

Differentiating (8) in the direction of the vector components x_j obtain:

$$\frac{\partial^2 f(\overline{X_n})}{\partial x_j^2} \Delta x_j + \frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i} \Delta x_i + \frac{\partial R(\overline{X_n})}{\partial x_j} = 0. \quad (11)$$

Solving the latter with respect to Δx_i and substituting it to (10) after algebraic transforms establish that the simultaneous execution of necessary conditions of local extremum for Δx_i and Δx_j directions and provided with:

$$\frac{\partial^2 f(\overline{X_n})}{\partial x_i^2}\left[\frac{\partial^2 f(\overline{X_n})}{\partial x_j^2}+\frac{\partial R(\overline{X_n})}{\Delta x_j \partial x_j}\right]-\left[\frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i}\right]^2-\frac{\partial R(\overline{X_n})}{\Delta x_j \partial x_i}\frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i}=0\,. \tag{12}$$

The latter form can be reduced only in two cases:

- $f(\overline{X})$ is the quadratic form;
- the remainder as negligibly small in comparing with another terms, value of which is determined by the maximum of module third order partial derivatives. For these conditions relation (12) can be rewritten:

$$\frac{\partial^2 f(\overline{X_n})}{\partial x_i^2}\frac{\partial^2 f(\overline{X_n})}{\partial x_j^2}-\left[\frac{\partial^2 f(\overline{X_n})}{\partial x_j \partial x_i}\right]^2=0\,, \tag{13}$$

what coinciding with the expression conditions of saddle point. It should be noted that this expression is obtained directly, based only on the properties of continuity and differentiability of objective function.

DISCUSSION OF RESULTS

Let's consider an example of application equation (12) as necessary conditions of simultaneous execution of local extremum in Δx_i and Δx_j directions to Lagrange function. For this case can be written:

$$\left[\frac{\partial^2 F(\overline{X_n})}{\partial x_i^2}+\sum_{i=1}^{m}\lambda_j\frac{\partial^2 g_j(X_n)}{\partial x_i^2}\right]\left[\frac{\partial^2 F(\overline{X_n})}{\partial x_j^2}+\sum_{i=1}^{m}\lambda_j\frac{\partial^2 g_j(X_n)}{\partial x_j^2}+\frac{\partial R(\overline{X_n})}{\Delta x_j \partial x_j}\right]-\left[\frac{\partial^2 F(\overline{X_n})}{\partial x_j \partial x_i}+\sum_{i=1}^{m}\lambda_j\frac{\partial^2 g_j(X_n)}{\partial x_j \partial x_i}\right]^2-$$

$$-\frac{\partial R(\overline{X_n})}{\Delta x_j \partial x_i}\left[\frac{\partial^2 F(\overline{X_n})}{\partial x_j \partial x_i}+\sum_{i=1}^{m}\lambda_j\frac{\partial^2 g_j(X_n)}{\partial x_j \partial x_i}\right]=0 \tag{14}$$

Let's consider other example of application equation (12) at saddle point in $\overline{X}$ and $\overline{\Lambda}$ directions. This case of problem can be applied to solution of finance or economic problems of optimization for NLP with linear constraints inequalities. For this type of problems according with Lagrange function as function of vector strategies and vector of Lagrange multipliers are reduced due to of it properties to

$$\left[\frac{\partial^2 F(\overline{X_n})}{\partial x_i^2}\right]\left[\frac{\partial R(\overline{X_n})}{\Delta x_j \partial \lambda_j}\right]-\left[\frac{\partial g_j(X_n)}{\partial x_i}\right]^2-\frac{\partial R(\overline{X_n})}{\Delta x_j \partial \lambda_j}\left[\frac{\partial\, g_j(X_n)}{\partial \lambda_j\, \partial x_i}\right]=0 \tag{15}$$

or

$$\begin{cases} \dfrac{\partial^2 L(\overline{X_n},\overline{\Lambda})}{\partial \lambda_j^2} = 0, \\ \dfrac{\partial g_j(\overline{X_n})}{\partial x_i} = 0, \qquad i=\overline{1,n} \qquad j=\overline{1,m} \end{cases} . \tag{16}$$

Latest results (16) can be interpreted as condition of searching of solution along of the axis, which is parallel for x_i axis, other result (15) can be interpreted as searching in along of two intersected directions, but for linear function $g_j(\overline{X_n})$ they demonstrate violation (16). That indicates: components of vector Lagrange multipliers had to be excluded from consideration in another words: equation (14), for linear constraints, can generate additional equation for system (6) only for objective function and number of them is limited.

For resolution of this contradiction will be introduced as analog of Lagrange function, with non-linear function $g_j(\overline{X_n})$:

$$L(\overline{X}) = F(\overline{X}) + \sum_{i=1}^{m} \lambda_j^2 g_j(\overline{X}).$$

For this case under condition $g_j(\overline{X_n}) \geq 0$, and $\dfrac{\partial^2 F(\overline{X})}{\partial x_i^2} \leq 0$, and $\dfrac{\partial^2 g_j(\overline{X})}{\partial x_i^2} \leq 0$ equation (13) transform to operator:

$$D_{ij}(\overline{X}) = \left(\frac{\partial^2 F(\overline{X})}{\partial x_i^2} + \sum_{i=1}^{m} \lambda_j^2 \frac{\partial^2 g_j(\overline{X})}{\partial x_i^2} \right) \left(2g_j(\overline{X}) \right) - \left(2\lambda_j \frac{\partial\, g_j(\overline{X})}{\partial x_i} \right)^2 \leq 0, \qquad i=\overline{1,n},\, j=\overline{1,m} \tag{17}$$

and in the case of existence its simultaneously solution system (6) and equation (17) at point (λ_j^*) will be saddle point for any of i -th component of vector strategies and other than j - th component of vector Lagrange multipliers. Thus, the system (6) can be added by $n \times m$ additional equations, thus now the system will be consists from $(n+m+n\times m)+2$ of equations with $2\ (n+m)$ unknowns, i.e. the dimension of the vector of strategies and vector of the restrictions doesn't effect onto existence and uniqueness of optimization problem solution and the number of constraints, that provides a unique solution doesn't limited by two. This result is demonstrate, that application of new form of Lagrange function simultaneously with indicator - operator (17) is a source of additional equations. It should be noted, that the equations (13) are satisfy the conditions of continuity and uniqueness and consequently to them are preliminary quadratic recurrent approximation for new form of Lagrange function, completed by objective function of the vector strategies and sum of products of square of each components of vector Lagrange multipliers on components of vector constraints, will be satisfy of condition existence and uniqueness, since created by linear operator. Also it should be noted, that for linear constraints and new form of Lagrange function in condition (14) generates additional equations for system (6) of NLP resolved in simple form for value λ_j :

$$\lambda_j \geq \left[\frac{\partial \ g_j(\overline{X})}{\partial x_i} \right]^{-1} \sqrt{\frac{\partial^2 F(\overline{X})}{2\partial x_i^2} g_j(\overline{X})}, \qquad i=\overline{1,n}, j=\overline{1,m} \ .$$

This approach, open new modern instruments for solution of optimization problem and control on the bases of this solutions for social-economy project.

CONCLUSION

1. The application of recurrent approximations of Lagrange function simultaneously with condition overlaid on the value of partial derivatives of first, second and third order can increase the degree of completeness of the NLP system, in cases of monotonicity, differentiability of Lagrange function, but for linear constraints can generate only limited number of additional equations.
2. In the case of linear or non-linear constraints due to obtaining new Lagrange functions with square of each component of vector Lagrange multipliers for non-linear objective function simultaneous with indicator-operator (17) provide the conditions in which the dimensionality of the vector of strategies doesn’t effects on the uniqueness of the solution for optimization problem and the number of constraints, that provides a unique of solution of NLP system, doesn’t limited by two.
3. Condition of saddle point (13), which is usually used is approximate expression, accuracy of which determined by maximum of third order derivative of objective function and only for quadratic form of objective function it take exact form and it’s a source of additional equations.

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