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SOME REMARKS ON GENERALIZED REGRESSION METHODS

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Abstract: As a result of studying certain phenomena gained on the plane, the unit circle and on the earth sphere we present here some introductory notations and remarks, concerning the problems in question.

Keywords: circular regression, Hilbert space, regression function, regression structure

INTRODUCTION

Continuous development of science has brought certain possibility of use of the newest and more efficient methods and research tools. Among them, mathematical and statistical tools occupy an important position. They play a key role in the construction of models of quantitative description of economic phenomena and processes. These models can take different forms, but it is necessary to remember that properly constructed model of these phenomena or economic process should ensure the preservation of relationships and logical correctness of the structure between the original and the image generated by this model.[1] In practice, making research in various scientific fields, one usually uses mathematical models, essentially simplifying the reality, in question. From one side it leads to possibilities of applying tools of the well known theory of linear mathematics but, on the another side, there are necessity to simplify the obtained numerical data, substituting them by a regressed ones, in any sense.

In both the operations some necessary invariants shall be preserved. Hence, our research, gained on this way, is still sensible. If these simplifications are to far

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[1] Fałda B. (2010) Modelowanie dynamiczne procesów ekonomicznych, Wydawnictwo KUL, Lublin, p. 25.

reaching then the usual linear techniques becomes misleading, if not entirely meaningless.

Simplifying, for instance, one of the basic principles known in physics, which is giving rise to the wave equation, we substitute the general dependencies by a linear equation. As a result we are losing the most interesting physical phenomena like non-linear waves known, in the case of the water waves, as solitons or tsunami. Therefore, the linearization procedure, obtained by using the linear regression method, often leads to an information set, that we shall decide if it is satisfactory.

Generally, in the authors opinion, the mentioned linearization procedure has feature of the well working local method which is not giving rise to results of global character, if linearity are not assured. Another generalization approach is to describe the situation in a case when there is not possible to use probability methods. In such a case we propose to use the general regression idea, by the use of which one may introduce the probability structure. This is concerning the case when the observed phenomena occurs in the surface of the unit sphere, what will be under discussion hereafter.

INTRODUCTION TO GENERAL REGRESSION THEORY

The theory of regression gone a long way of its development, from a simple considerations about linear forms to technically advanced, multidimensional nonlinear models.

The analysis of the relationship between two or more variables, given in such a manner that one variable can be predicted or explained by using information on the others is called regression analysis.[2] In standard linear statistical model we consider the following situation:

Let $X := \{x_0, x_1, \dots, x_n\}$ and $Y := \{y_0, y_1, \dots, y_n\}$ be arbitrary given sequences of numbers. Then there exists a unique $f_0 \in \mathcal{F}$ (the space of linear functions) satisfying the following condition

$$\sum_{k=0}^{n}\left[f(x_k) - y_k\right]^2 \geq \sum_{k=0}^{n}\left[f_0(x_k) - y_k\right]^2 \quad \text{for any} \quad f \in \mathcal{F}. \tag{1}$$

The function f_0 is of the form

$$f_0(t) = a_0 t + b_0 \quad \text{as} \quad t \in \mathbb{R}, \tag{2}$$

where

$$a_0 := \frac{(n+1)\sum_{k=0}^{n} x_k y_k - \sum_{k=0}^{n} x_k \sum_{k=0}^{n} y_k}{(n+1)\sum_{k=0}^{n} x_n^2 - \left(\sum_{k=0}^{n} x_k\right)^2} \tag{3}$$

and

[2] Freund R. J., Wilson W. W., Mohr D. L. (2010) Statistical methods, 3rd edition, Elsevier, p. 323.

$$b_0 := \frac{\sum_{k=0}^{n} y_k - a_0 \sum_{k=0}^{n} x_k}{n+1}. \tag{4}$$

The function f_0 is usually named the linear regression for a given empiric sequences X and Y. The function f_0 plays the role of optimal function with the smallest quadratic deviation from the mentioned above observations. In literature, cf. [Sen, Srivastava 1990, Seber, Wild 2003], one may find a number of its modifications, obtained by properly used diffeomorphic modifications and localization.

The generalization to the case, when instant of the space of the linear functions space $\mathcal{F}$, formed by linear functions, one uses an arbitrary, finite or infinite dimensional, Hilbert function space $\mathcal{H}$. An introduction and basic results are presented in [Partyka, Zając 2015]. The background for further considerations is a regression structure of the form

$$\mathfrak{R} := (A, B, \delta; x, y)\,, \tag{5}$$

where:

1. A, B are given nonempty sets;
2. obtained by an experiment or observation the functions $x: \Omega_1 \to A$ and $y: \Omega_2 \to B$ for some nonempty sets Ω_1 and Ω_2;
3. $\delta: (\Omega_1 \to B) \times (\Omega_2 \to B) \to \overline{\mathbb{R}}$ is a matching measure of theoretical function to empirical data x and y.

Hence, one has given a theoretical functional model $\mathcal{F}$ of a considered structure $\mathfrak{R}$, such that $\mathcal{F} \subset (A \to B)$, where $A \to B$ denotes the class of all functions acting from A to B, where A and B are arbitrary sets[3]; usually subsets of $\overline{\mathbb{R}}$. Our purpose is to determine such functions $f_0 \in \mathcal{F}$, which satisfy the following condition of the best matching to the empirical data

$$F(f) := \delta(f \circ x, y) \geq F(f_0)\,, \tag{6}$$

Here, instead of extremality condition (1) one considers much more general condition (6).

Generalization of the classic square deviation, calculated with respect to any measure $\mu: \mathcal{B} \to [0, +\infty)$, is defined by

$$\delta(u, v) = \int_{\Omega_1 \times \Omega_2} |u(t_1) - v(t_2)|^2 \, \mathrm{d}\mu(t_1, t_2)\,, \tag{7}$$

assuming that the family of subsets of the Cartesian product $\Omega_1 \times \Omega_2$ form a σ-field $\mathcal{B}$ and functions

$$\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to u(t_1) \quad \text{and} \quad \Omega_1 \times \Omega_2 \ni (t_1, t_2) \to v(t_2) \tag{8}$$

are measurable. The set of all $f_0 \in \mathcal{F}$ satisfying inequality (6) is denoted by $\mathrm{Reg}(\mathcal{F}, \mathfrak{R})$, whereas each of $f_0 \in \mathrm{Reg}(\mathcal{F}, \mathfrak{R})$ is said to be the regression function in $\mathcal{F}$ with respect to $\mathfrak{R}$.

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[3] Partyka D., Zając J. (2015) Generalized approach to the problem of regression, Anal. Math. Phys., DOI 10.1007/s13324-014-0096-7, Anal. Math. Phys. (2015) 5, 251.

The usually used synchronous case one obtains by setting $t_1 = t_2 = t$ and $\mu(t_1, t_2) = \mu(t)$, which covers the case of the classical measure, used in linear regression theory.

In [Partyka, Zając 2015] one may find a precise mathematical description, leading to solution of the extremal problem (6) both in the case of finite as well as infinite dimensional Hilbert or pseudo Hilbert space. Moreover, it is showed there, that the solution, called generalized regression function, is constructed there as a linear combination of the basis vectors, same in the case of finite as infinite dimensional Hilbert space; see Theorem 4.3 and Theorem 5.1 in [Partyka, Zając 2015].

As a sort of special advantage of this theory we would like to point out that one has here freedom in choosing the basis vectors. By this we may properly adopt the space in question to the observed phenomena.

The regression problem for $\mathcal{F}$ with respect to $\mathfrak{R}$ is to determine all functions $f_0 \in \mathcal{F}$ minimizing the functional F and satisfying the following equality

$$F(f) = \int_{\Omega_1\times\Omega_2} |f \circ x(t_1) - y(t_2)|^2 \,\mathrm{d}\mu(t_1, t_2)\,, \quad f \in \mathcal{F}\,, \tag{9}$$

which is the discrete case is leading to

$$F(f) = \sum_{k=0}^{n} |f \circ x(k) - y(k)|^2 = \sum_{k=0}^{n} |f(x_k) - y_k|^2\,, \quad f \in \mathcal{F}\,. \tag{10}$$

These forms are suggesting to consider the family $\mathcal{L}_1(\mathfrak{R})$ of all functions $f\colon A \to B$, such that $f \circ x(k)$ are measurable of finite L^2-norm.

In symmetry to this one may consider the family $\mathcal{L}_2(\mathfrak{R})$ of all functions $g\colon B \to B$ such that $g \circ y(t_2)$ is measurable of finite L^2-norm. Hence, the structure $\mathcal{H}(\mathfrak{R}) := (\mathcal{L}_2(\mathfrak{R}), +, \cdot, \langle \cdot \mid \cdot \rangle)$ is a Hilbert space, where $(\mathcal{L}_1(\mathfrak{R}), +, \cdot)$ is a complete linear space, where

$$\langle u|v\rangle := \int_{\Omega_1\times\Omega_2} u \circ x(t_1)\overline{v \circ x(t_1)}\,\mathrm{d}\mu(t_1, t_2) \tag{11}$$

is well defined scalar product.

To each $g \in \mathcal{L}_2(\mathfrak{R})$ we associate the functional

$$g^*(u) = \int_{\Omega_1\times\Omega_2} u \circ x(t_1)\overline{g \circ y(t_2)}\,\mathrm{d}\mu(t_1, t_2) \tag{12}$$

well defined for all $u \in \mathcal{L}_1(\mathfrak{R})$.

Within this notations we may present[4] the solution of the regression problem which, taking into account the ortogonal decomposition procedure, reeds as

Theorem: Given $p \in \mathbb{N} \cup \{\infty\}$ let $h_k \in \mathcal{L}_1(\mathfrak{R})\backslash\{\Theta\}$ be an orthogonal sequence in the $\mathcal{H}(\mathfrak{R})$ and $g \in \mathcal{L}_2(\mathfrak{R})$. If $p \in \mathbb{N}$, then

$$\mathrm{Reg}\big(\mathcal{F}, \mathfrak{R}_g\big) = (\Theta \cap \mathcal{F}) + \sum_{k=1}^{p} \frac{\overline{g^*(h_k)}}{\|h_k\|^2} h_k, \tag{13}$$

[4] Partyka D., Zając J. (2015) Generalized approach to the problem of regression, Anal. Math. Phys., DOI 10.1007/s13324-014-0096-7, Anal.Math.Phys. (2015) 5, p. 258.

where $\mathcal{F} := \text{lin}(\{h_k : k \in \mathbb{Z}_{1,p}\})$ and Θ is the set of functions in $\mathcal{L}_1(\mathfrak{R})$ with zero norm. If $p = \infty$, then

$$\text{Reg}(\text{cl}(\mathcal{F}), \mathfrak{R}_g) = \Theta + \sum_{k=1}^{\infty} \frac{\overline{g^*(h_k)}}{\|h_k\|^2} h_k \,. \tag{14}$$

Here $\mathfrak{R}_g := (A, B; \delta, x, g \circ y)$ is a regression structure for each $g: B \to B$, which is properly described balancing function.[5]

By this each $f \in \text{Reg}(\mathcal{F}, \mathfrak{R})$ is of the form

$$f = \sum_{k=1}^{p} \lambda_k h_k \tag{15}$$

for a sequence $\{\lambda_k\}$, such that $\lambda_k \in \mathcal{B}$, as $k = 1,2, \ldots, p$.

The coefficient λ_k is of the form

$$\lambda_k = \frac{\overline{g^*(h_k)}}{\|h_k\|^2} \,, \quad k = 1,2, \ldots, p \,. \tag{16}$$

To present a connection between the classical regression function and the generalized one, let us see the example, given in [Partyka, Zając 2015]. To this let the real regression structure $\mathfrak{R}$ be given with $g \in \mathcal{L}_2(\mathfrak{R})$, where $\mathcal{F} := \text{lin}(\{h_1, h_2\})$ as $h_1, h_2 \in \mathcal{L}_1(\mathfrak{R})$ such that $\mathcal{F} \cap \Theta = \{\theta\}$.

Then

$$\text{Reg}\,(\mathcal{F}, \mathfrak{R}) = (\Theta \cap \mathcal{F}) + \sum_{k=1}^{2} \frac{\overline{g^*(h'_k)}}{\|h'_k\|^2} h'_k \,, \tag{17}$$

where $h'_1 = h_1$ and $h'_2 := h_2 - \frac{\langle h_2 | h_1 \rangle}{\|h_1\|^2} h_1$.

Denoting

$$a_2 := \frac{\overline{g^*(h'_2)}}{\|h'_2\|} \tag{18}$$

and

$$a_1 := \frac{\overline{g^*(h'_1)}}{\|h'_1\|^2} - \frac{\langle h_2 | h_1 \rangle}{\|h_1\|} h_1 \tag{19}$$

we obtain

$$\text{Reg}\,(\mathcal{F}, \mathfrak{R}) = (\Theta \cap \mathcal{F}) + a_2 h_1 + a_1 h_1 \,. \tag{20}$$

If, in particular, $h_2(t) = t$, $h_1(t) = 1$ and $g(t) = t$, $t \in \mathbb{R}$, we obtain the classical linear regression function.[6]

For some additional, theoretical and practical, examples of the general regression theory see [Fałda, Zając 2011, Fałda, Zając 2012, Partyka, Zając 2015, Partyka, Zając 2010 and Zając 2010].

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[5] Partyka D., Zając J. (2015) Generalized approach to the problem of regression, Anal. Math. Phys., DOI 10.1007/s13324-014-0096-7, Anal. Math. Phys. (2015) 5, 283.

[6] Partyka D., Zając J. (2015) Generalized approach to the problem of regression, Anal. Math. Phys., DOI 10.1007/s13324-014-0096-7, Anal. Math. Phys. (2015) 5, 294-299.

The generalized regression theory allows us to construct, in pretty easy way, nonlinear regression functions, with a very good matching with the phenomena described by them.

THE CASE OF THE UNIT CIRCLE

Many phenomena observed in biology, geography, medicine and economy barely submits to the description using the linear coordinate system, also called as the rectangular coordinate system. This great idea of geometrization of mathematics, introduced in the seventeenth century by Deskartes, approached difficult mathematical concepts and increased possibilities of their use. In this way he found a strict method of consideration of many scientific issues of a local nature, where straight line, plane or n-dimensional space with a system of rectangular coordinates are excellent centers of modelling. For the study of phenomena of a global nature, this idea applies with difficulty, leading to discrepancies with the data observed. However, it remains very useful in linear modelling, where the space is replaced with the tangent one. Global information can be obtained then by "gluing" local information.

While it is easy to find examples of the global character phenomena in biology, geography or medicine, the globalization of economic processes has intensified only in the last period of time; financial markets and capital, international logistics and demographic issues. This means that we have to take into account the geometric shape of the object on which these phenomena are observed. A lot of phenomena, occurring on the plane, we can much more easily describe using polar coordinates rather than the rectangular ones. These include demographic issues, urban development and transport, in which we consider transport real distance and its direction. This leads to a model with the plane of the polar coordinate system. In the case of financial and demographic phenomena their natural activity area is situated on the sphere, modelling the earth's surface, with a spherical coordinate system.

The first scientific publications on issues of statistics of random variables with values taken on the circle appeared in the 70s in connection with research in biology, geography and medicine. Economic development and related economic problems provides more reasons to apply the relevant descriptions and appropriate mathematical tools. This means that the problems of regression we need to replace eg. a straight line onto circle, and polynomial regression onto trigonometrical regression, etc.; [Fisher 1995].

The classical approach to the regression problem, when the probability space is constructed on a circle, can be found in [Jammalamadaka, Sengupta 2001 and Marida 1972]. But here, we try to adopt our approach, presented in the previous section to the case, when $\mathcal{B} = \mathbb{T}_r$. To this end let

$$\mathbb{T}_r := \{(x, y): x^2 + y^2 = r^2\} \tag{21}$$

or, equivalently, is defined by polar coordinates (r,φ), in the form $x = r\cos\varphi$, $y = r\sin\varphi$, where $r > 0$ and $0 \le \varphi < 2\pi$. For $r = 1$ we have $(1,\varphi) \leftrightarrow (\cos\varphi, \sin\varphi)$, which is the unit circle $\mathbb{T} := \mathbb{T}_1$.

For given $\alpha, \beta \in [0; 2\pi)$ we describe two measures:

$$d_0(\alpha,\beta) := \pi - (\pi - |\alpha - \beta|) \in [0,\pi] \tag{22}$$

and

$$d_1(\alpha,\beta) := 1 - \cos(\alpha - \beta) \in [0,2). \tag{23}$$

Let us consider the regression structure $\mathfrak{R} := (A, \mathbb{T};\ \delta, x, y)$, where $x: \Omega_1 \to A$, $y: \Omega_2 \to \mathbb{T}$. Here $\Omega_1 \neq \emptyset$ and $\Omega_2 \neq \emptyset$ are given. The function $\delta: (\Omega_1 \to \mathbb{T}) \times (\Omega_2 \to \mathbb{T}) \to \overline{\mathbb{R}}$ is given, for $u: (\Omega_1 \to \mathbb{T})$ and $v: (\Omega_2 \to \mathbb{T})$, by the formula

$$\delta(u,v) = \int_{\Omega_1\times\Omega_2} |u(t_1) - v(t_2)|^2 \mathrm{d}\mu(t_1,t_2) \tag{24}$$

if the function $u(t_1) - u(t_2)$ is $\mathcal{B}$-measurable and $\delta(u,v) := +\infty$ in other case.

To a given theoretic model $\mathcal{F} \subset (A \to \mathbb{T})$ of the structure $\mathfrak{R}$ we are going to find $\mathrm{Reg}(\mathcal{F},\mathfrak{R})$ consisting all functions $f_0 \in \mathcal{F}$ such that $F(f) \ge F(f_0)$ for each $f_0 \in \mathcal{F}$, where $\mathcal{F} \ni f \to F(f) := \delta(f \circ x, y) \in \overline{\mathbb{R}}$.

Then

$$F(f) = \int_{\Omega_1\times\Omega_2} \left|f\big(x(t_1)\big) - y(t_2)\right|^2 \mathrm{d}\mu(t_1,t_2) \ \text{ as } f \in \mathcal{F} \tag{25}$$

and

$$\int_{\Omega_1\times\Omega_2} |f \circ x(t_1) - y(t_2)|^2 \mathrm{d}\mu(t_1,t_2) \ge \int_{\Omega_1\times\Omega_2} |f_0 \circ x(t_1) - y(t_2)|^2 \mathrm{d}\mu(t_1,t_2) \tag{26}$$

Since $|f \circ x(t_1)|^2 = 1$ and $|y(t_2)|^2 = 1$ then

$$\int_{\Omega_1\times\Omega_2} \mathrm{Re}\big[f \circ x(t_1)\overline{y(t_2)}\big]\, \mathrm{d}\mu(t_1,t_2) \le \int_{\Omega_1\times\Omega_2} \mathrm{Re}\big[f_0 \circ x(t_1)\overline{y(t_2)}\big] \mathrm{d}\mu(t_1,t_2). \tag{27}$$

Then

$$\int_{\Omega_1\times\Omega_2} \mathrm{Re}\big[(f - f_0)\big(x(t_1)\overline{y(t_2)}\big)\big] \mathrm{d}\mu(t_1,t_2) \le 0. \tag{28}$$

By this we see that

$$f_0 \in \mathrm{Reg}(\mathcal{F},\mathfrak{R}) \Leftrightarrow y^*(f - f_0) \le 0, \quad f \in \mathcal{F}, \tag{29}$$

where

$$y^*(u) \coloneqq \int_{\Omega_1\times\Omega_2} \mathrm{Re}\big[u \circ x(t_1)\overline{y(t_2)}\big] \mathrm{d}\mu(t_1,t_2) \tag{30}$$

for arbitrary $u: A \to \mathbb{R}$ and $y: \Omega_2 \to \mathbb{R}$ such that the integral in question exists.

Using this notations we may show that $f_0 = \frac{y}{|y|}\mathbf{1}_{\mathbb{T}}$ or $f_0 = -\frac{y}{|y|}\mathbf{1}_{\mathbb{T}}$, where $y \coloneqq \int_{\Omega_1\times\Omega_2} y(t_2)\mathrm{d}\mu(t_1,t_2)$ and $\mathbf{1}_{\mathbb{T}}$ denotes a constant on $\mathbb{T}$. By this

$$\mathrm{Reg}(\mathcal{F},\mathfrak{R}) = \left\{\frac{y}{|y|}\mathbf{1}_{\mathbb{T}}\right\} = \frac{EY}{|EY|} \tag{31}$$

provided $EY \neq 0$ is an expected value on $\mathbb{T}$.[7]

THE UNIT SPHERE CASE

Our propose in this section is to present a sort of introduction to the case when, instead of $\mathbb{T}$ we consider the unit sphere

$$\mathbb{S}^2 := \{(x, y, z): x^2 + y^2 + z^2 = 1\}. \tag{32}$$

Using the polar coordinates $(1, \alpha, \beta)$, where $0 \leq \alpha < \pi$ and $0 \leq \beta < 2\pi$, we can see that

$$\begin{aligned} x &= \cos\alpha\cos\beta, \\ y &= \sin\alpha\cos\beta, \\ z &= \sin\beta. \end{aligned} \tag{33}$$

The distance between two points $P_1, P_2 \in \mathbb{S}$ is defined as the length of the shorter arc, distinguished on the unit circle on $\mathbb{S}^2$ centered at the origin and passing through the points P_1, P_2.

Similarly to the previous section we may apply here the generalized regression technique, which seems to be not so sensitive on the form of domain, where the functions x and y are described by (5). To this end we can distinguish the family $\mathrm{Reg}(\mathcal{F}, \mathfrak{R})$, where

$$\mathcal{F} := \left\{e^{i[\alpha,\beta]}\mathbf{1}_{\mathbb{S}}: 0 \leq \alpha < \pi,\ 0 \leq \beta < 2\pi\right\} \tag{34}$$

and

$$\mathfrak{R} = (A, \mathbb{S}, \delta; x, y),\ \ x: \Omega_1 \to \Omega_2,\ \ y: \Omega_2 \to \mathbb{S}. \tag{35}$$

Here Ω_1, Ω_2 are given.

The symbol $[\alpha, \beta]$ denotes fixed polar coefficients of a point on $\mathbb{S}$, described by (33), where α and β are fixed. The class $\mathcal{F}$ is a family of all constant functions on $\mathbb{S}$. Obviously,

$$A \ni t \to \mathbf{1}_A(t) \coloneqq 1 \tag{36}$$

for arbitrary $A \neq \emptyset$.

Within this notations one searches a function $f_0 = e^{i[\alpha_0,\beta_0]}\mathbf{1}_{\mathbb{S}}$, were $\alpha_0, \beta_0 \in \mathbb{R}$ and satisfy (6).

CONCLUSIONS

The particular motivation, leading to this kind of extremal mathematical problem, defined on the unit sphere, is strongly suggested by global transportation problems on the earth sphere. Important, from the mathematical point of view, is

[7] Jammalamadaka S. R., Sengupta A. (2001) Topics in Circular Statistics, World Scientific, River Edge, New York, p. 16.

the remark saying that the method, used here, can be applied to much general cases, including arbitrary Riemann surface, instead of the unit sphere $\mathbb{S}$.

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