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RISK AVERSION AND A CALCULUS FOR FINITELY GENERATED PIECEWISE LINEAR FUNCTIONS: A CALCULUS THAT ECONOMISTS OUGHT TO DEVELOP?

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Abstract: In this note we propose a calculus for piece-wise linear functions, in order to obtain derivatives and second derivatives at points where the function is not differentiable. Such derivatives can be used to calculate coefficients of risk aversion at initial wealth for piece-wise linear utility functions for gains, which display loss aversion-and hence non differentiability at zero gains.

Keywords: piece-wise linear functions, non-differentiable, derivative, second derivative, utility function, risk aversion

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INTRODUCTION

This paper is a consequence of our concerns with trying to extend the concepts of absolute and relative risk aversion to utility functions, that are piecewise linear, with a kink at the origin, in order to account for "loss aversion" (at initial wealth) that many individuals seem to display. In mathematics and statistics, a **piecewise linear**, **PL** or **segmented** function is a real-valued function of a real variable, whose graph is composed of straight-line segments. (See [Stanley 2004] page 143). Needless to say, piecewise linear functions have the scope for wide applicability in economics, of which utility theory of gains and losses is just one example.

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The following statement can be found in page 5 (Chapter 1) of the well-known work by Eeckhoudt, Gollier and Schlesinger [Eeckhoudt, Gollier and Schlesinger 2005]: "… most human beings do not extract utility from wealth. Rather, they extract utility from consuming goods that can be purchased with this wealth."
Our understanding is that most human beings do not extract "satisfaction" from wealth. Rather, they extract "satisfaction" from consuming goods that can be purchased with this wealth. Although wealth may not yield "satisfaction", it has use-value since it can be used to purchase goods that yield satisfaction. We are at this point drawing a clear distinction between satisfaction derived from consumption and use-value of an instrument that is used to derive satisfaction from consumption. The use-value of wealth- purely as an instrument used to extract satisfaction from consumption- may be referred to as **utility** of wealth.

Since it may be difficult to measure or quantify the satisfaction extracted from consuming goods, if possible and quantifiable, the utility of wealth that is used to purchase goods, could be used as a "proxy" for the satisfaction that is derived or extracted from consuming these goods.

Given this understanding of utility functions for wealth-and hence utility functions for gains and losses with initial wealth being located at the origin- in the next section we proceed to develop "a calculus" (allowing for the possibility of other approaches) to tackle the problem of measuring risk aversion, both absolute and relative, when utility functions for gains (losses being negative "gains") are piecewise linear and possibly non-differentiable, particularly at the origin. In the same section, we extend the calculus-from our perspective-to multi-piecewise linear functions. An example of such a utility function is the one in page 224 (the fourth section of Chapter 13) of the work cited above by Eeckhoudt, Gollier and Schlesinger. We discuss issues related to risk aversion for this particular utility function in a final section of this paper. Prior to the discussion of risk aversion for the specific utility functions, we define the concepts of concave and convex piece-wise linear functions on closed and bounded intervals of the real line and obtain their equivalence with the expected properties in terms of second derivatives. Is Newtonian calculus, therefore, a "trivialization" of the mathematics required to understand the essentially "discrete" real world we live in?

FIRST AND SECOND DERIVATIVES FOR PIECEWISE LINEAR FUNCTIONS

A real-valued function f whose domain denoted $dom(f)$ is a non-degenerate interval in the real line is said to be a **real-valued finitely generated PL** if or some positive integer 'n' with n ≥ 2, there exists a non-empty finite set of real numbers $\{x_j | j = 1,..., n\}$ and satisfying $x_j < x_{j+1}$ for $j \in \{1,...,n-1\}$ if $n > 1$ such that:

(i) The restrictions of f to $(-\infty, x_1]\cap dom(f)$, $[x_n, +\infty)\cap dom(f)$ and $[x_j, x_{j+1}]$ for $j\in\{1,...,n-1\}$ if $n>1$ are all affine, i.e. the restriction of f on each segment mentioned above satisfies $f(px + (1-p)y) = pf(x) + (1-p)f(y)$ for all $p\in[0,1]$ and x,y belonging to the segment.
(ii) If $(-\infty, x_1)\cap dom(f) \neq \phi$, then $(-\infty, x_1)\cap dom(f)$ is a non-degenerate open interval in the real line.
(iii) If $(x_n, +\infty)\cap dom(f) \neq \phi$, then $(x_n, +\infty)\cap dom(f)$ is a non-degenerate open interval in the real line.
(iv) For all x_j with $j\in\{2,...,n-1\}$: $\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}} \neq \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}$.
(v) If $x_1-h\in dom(X)$ for some $h>0$, then $\frac{f(x_1)-f(x_1-h)}{h} \neq \frac{f(x_2)-f(x_1)}{x_2-x_1}$.
(vi) If $xn+h\in dom(X)$ for some $h>0$, then $\frac{f(x_n+h)-f(x_n)}{h} \neq \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$

In this case we say that f is a **real-valued finitely generated PL function generated by** $\{x_j| j = 1,..., n\}$.
Let f be "a real valued finitely generated" PL function generated by $\{x_j| j = 1,..., n\}$. Let $x\in dom(f)$. For all $j\in\{1,...,n-1\}$: (i) For all $x\in[x_1, x_n)\cap[x_j, x_{j+1})$, let $D^+f(x) = \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}$; and (ii) For all $x\in(x_1, x_n]\cap(x_j, x_{j+1}]$, let $D^-f(x) = \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}$. Thus $D^+f(x_j) = D^-f(x_{j+1})$ for all $j\in\{1,\ldots,n-1\}$.

If $(-\infty, x_1)\cap dom(f) \neq \phi$, then let $D^-f(x_1) = D^-f(x) = \frac{f(x_1)-f(x)}{x_1-x} = D^+f(x)$ for all $x\in(-\infty, x_1)\cap dom(f)$.
If $(x_n, +\infty)\cap dom(f) \neq \phi$, then let $D^+f(x_n) = D^+f(x) = \frac{f(x)-f(x_n)}{x-x_n} = D^+f(x)$ for all $x\in(x_n, +\infty)\cap dom(f) \neq \phi$.
Clearly, for all $x\in dom(f)\backslash\{x_1,..., x_n\}$, $D^+f(x) = D^-f(x)$ and for all x_i satisfying $(-\infty, x_i)\cap dom(f) \neq \phi$ and $dom(f)\cap(x_i, +\infty) \neq \phi$, it is the case that $D^+f(x_i) \neq D^-f(x_i)$.

Given $\alpha\in[0,1]$ and $x\in dom(f)$ such that $(-\infty, x)\cap dom(f) \neq \phi$ and $(x, +\infty)\cap dom(f) \neq \phi$, let the α-**first derivative** of f at x, denoted $D^{(\alpha)}f(x) = \alpha D^+f(x) + (1-\alpha)D^-f(x)$ and the **second derivative** of f at x denoted $D^2f(x) = D^+f(x)$- $D^-f(x)$.

Note the difference between (α) in the definition of $D^{(\alpha)}f(x)$ and the positive integer 2, written without any brackets in $D^2f(x)$. Also note that given $\alpha\in[0,1]$ and $x\in dom(f)$ such that $(-\infty, x)\cap dom(f) \neq \phi$ and $(x, +\infty)\cap dom(f) \neq \phi$, $D^{(\alpha)}f(x) = D^-f(x)+ \alpha D^2f(x)$.
Thus, $\{x\in dom(f)| D^2f(x) \neq 0\} = \{x_i| (-\infty, x_i)\cap dom(f) \neq \phi$ and $dom(f)\cap(x_i, +\infty) \neq \phi\}$
Given any real valued finitely generated PL f, any $x\in dom(f)$ such that $Df(x)$ is well-defined, and $m\in[min\{D^+f(x), D^-f(x)\}, max\{D^+f(x), D^-f(x)\}]$, the straight-line

$z = my + [f(x)-mx]$ is said to be **a tangent to** f **at** x, with **the slope of the tangent to** f **at** x being 'm'.

Note: Given a PWL function f, $x \in dom(f)$ is said to be a **local maximum** of f if there exists $\varepsilon > 0$ such that $f(x) \geq f(y)$ for all $y \in (x-\varepsilon, x+\varepsilon) \cap dom(f)$ and a **local minimum** of f if there exists $\varepsilon > 0$ such that $f(x) \leq f(y)$ for all $y \in (x-\varepsilon, x+\varepsilon) \cap dom(f)$. It is easily verified that $x \in dom(f)$ is a local maximum of f if and only if the following two conditions are satisfied: (i) If $(x, +\infty) \cap dom(f) \neq \phi$, then $D^+f(x) \leq 0$; and (ii) If $(-\infty, x) \cap dom(f) \neq \phi$, then $D^-f(x) \geq 0$.

The calculus discussed above can be easily extended to the multi-variable case.

For a positive integer $n \geq 2$, let $N = \{1,2,...,n\}$ and for any non-empty subset J of N and any non-empty set Y, let Y^J denote the set of functions from J to Y, where that for all $y \in Y^J$ and $j \in J$, $y(j)$ is written as y_j.
For each $j \in \{1,...,n\}$, $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}$, let (y_{-j}, ξ) denote $x \in \mathbb{R}^N$ such that $x_i = y_i$ for $i \neq j$ and $x_j = \xi$.

Let f be a real-valued function f whose domain denoted $dom(f)$ is a non-empty subset of $\mathbb{R}^N$.
For each j and $y \in \mathbb{R}^N$, let $dom^j(f|y_{-j}) = \{\xi \in \mathbb{R} | (y_{-j}, \xi) \in dom(f)\}$, and if $dom^j(f|y_{-j}) \neq \phi$, let $f^{(j)}(.|y_{-j})$ be the real valued function defined on $dom^j(f|y_{-j}) = dom(f^{(j)}(.|y_{-j})$ such that for every $\xi \in dom(f^{(j)}(.|y_{-j}), f^{(j)}(\xi|y_{-j}) = f(y_{-j}, \xi)$.

We are interested in the case where $dom(f^{(j)}(.|y_{-j})$ is the union of a finite set of mutually disjoint non-degenerate intervals of $\mathbb{R}$, $\{I_1,...,I_K\}$.
Suppose that the restriction of $f^{(j)}(.|y_{-j})$ to I_k for some $k \in \{1,...,K\}$ is a real valued finitely generated PL function and suppose $\xi \in I_k$. To keep notations simple we will use $f^{(j)}(.|y_{-j})$ to denote its restriction to I_k.

If $I_k \cap (\xi, +\phi) \neq \phi$, then let $D_j^+ f(y_{-j}, \xi) = D^+ f^{(j)}(\xi|y_{-j})$.
If $I_k \cap (-\infty, \xi) \neq \phi$, then let $D_j^- f(y_{-j}, \xi) = D^- f^{(j)}(\xi|y_{-j})$.
If both $I_k \cap (\xi, +\phi) \neq \phi$ and $I_k \cap (-\infty, \xi) \neq \phi$, then for $\alpha \in [0,1]$ let $D_j^{(\alpha)} f(y_{-j}, \xi) = \alpha D_j^+ f(y_{-j}, \xi) + (1-\alpha) D_j^- f(y_{-j}, \xi)$ and $D_j^2 f(y_{-j}, \xi) = D_j^+ f(y_{-j}, \xi) - D_j^- f(y_{-j}, \xi)$.
Such functions may be referred to as **multi-piecewise linear (M-PL) functions.**

$D_j^{(\alpha)} f(y_{-j}, \xi)$ may be called the **j[th] partial α-derivative of f at** (y_{-j}, ξ) and $D_j^2 f(y_{-j}, \xi)$ may be called the **j[th] partial second derivative of f at** (y_{-j}, ξ).

CONCAVE AND CONVEX FUNCTIONS

A PL function f generated by $\{x_1,...,x_n\}$ on $dom\ (f) = [x_1, x_n]$ is said to be **concave** if for all $x,\ y \in dom(X)$ satisfying $x < x_i < y$ for some $i \in \{2,...,n-1\}$ and $t \in (0,1)$, $f(x + t(y-x)) > f(x) + t[f(y)-f(x)]$.

A PL function f generated by $\{x_1,...,x_n\}$ on $dom\ (f) = [x_1, x_n]$ is said to be **convex** if for all $x,\ y \in dom(X)$ satisfying $x < x_i < y$ for some $i \in \{2,...,n-1\}$ and $t \in (0,1)$, $f(x + t(y-x)) < f(x) + t[f(y)-f(x)]$.

It is easy to see that f is concave if and only if $-f$ is convex.

Proposition 1:
Suppose f is a PL function generated by $\{x_1,…,x_n\}$ on dom (f) = $[x_1, x_n]$.
(i) $[D^2f(x_j) < 0$ for all $j \in \{2,…,n-1\}]$ if and only if f is concave.
(ii) $[D^2f(x_j) > 0$ for all $j \in \{2,…,n-1\}]$ if and only if f is convex.

Proof:
(i) Suppose $D^2f(x_j) < 0$ for all $j \in \{2,...,n-1\}$.
Let $x,y \in dom(f)$ with $x < x_j < y$ for some x_j. Let $x \in [\beta_k, \beta_{k+1}]$, $y \in [\beta_h, \beta_{h+1}]$. Since $x < x_j < y$, it must be the case that $\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} \neq \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$ which along with $D^2f(x_j) < 0$ for all $j \in \{2,...,n-1\}$ and $x < y$ implies $k < h$ and $\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} \neq \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$
For $t \in (0,1)$, consider the point $(x + t(y-x), 0)$ on the horizontal axis. The vertical line on $(x + t(y-x), 0)$ meets the chord joining $(x, u(x))$ and $(y, u(y))$ at the point $(x + t(y-x), u(x) + t(u(y)-u(x)))$. This follows immediately by applying the relevant property of similar triangles.
If $\frac{f(y)-f(x)}{y-x} \geq \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}$ then since $\frac{f(x_{g+1})-f(x_g)}{x_{g+1}-x_g}$ is a strictly decreasing function of 'g', then $f(y) = f(x) + \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}(x_{k+1} - x) + ... + \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}(y-x_h)$ along with $\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} > \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$ implies
$f(y) < f(x) + \frac{f(y)-f(x)}{y-x}[(x_{k+1} - x) + ... + (y-x_h)] = f(x) + \frac{f(y)-f(x)}{y-x}(y-x) = f(y)$, i.e., $f(y) < f(y)$, which is not possible. Thus, $\frac{f(y)-f(x)}{y-x} < \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}$.

On the other hand, if $\frac{f(y)-f(x)}{y-x} \leq \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$, then $-\frac{f(y)-f(x)}{y-x} \geq -\frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$.
Thus, $f(x) = f(y) - \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}(y-x_h) - ... - \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}(x_{k+1} - x)$ along with $-\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} < -\frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$ implies

$f(x) < f(y) - \frac{f(y)-f(x)}{y-x}[(y-x_h)+ \ldots +(x_{k+1} - x)] = f(y) - \frac{f(y)-f(x)}{y-x}(y-x) = f(x)$, i.e. $f(x) < f(x)$ which is not possible.

Thus, $\frac{f(y)-f(x)}{y-x} > \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$. Thus, $\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} > \frac{f(y)-f(x)}{y-x} > \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$.
Thus, for $x + t(y-x) \in (x_k, x_{k+1}]$, $f(x+t(y-x)) = f(x) + t\,\frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}(y-x) > f(x) + t\frac{f(y)-f(x)}{y-x}(y-x) = f(x) + t(f(y)-f(x))$.

Suppose $h = k + 1$. Then since $x < x_j < y$ for some x_j, it must be the case that $y > x_h$. Suppose $x_{h+1} \geq y > x_h$. For $x + t(y-x) \in (x_k, x_{k+1}]$ we already have $f(x+t(y-x)) > f(x) + t(f(y)-f(x))$. Hence suppose $x_{h+1} > x+t(y-x) > x_h$.
Thus, $x_h = x + s(y-x)$ for some $s \in (0,t)$.

Consider the function $\xi| \rightarrow f(x) + \xi(f(y)-f(x))$ and the function, $\xi| \rightarrow f(x + \xi(y-x)) = f(x_h) + \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}(\xi-s)(y-x)$ *for* $\xi \in [s,1]$.
The latter holds since $x + \xi(y-x) - x_h = [x + \xi(y-x)] - [x + s(y-x)] = (\xi-s)(y-x)$.
Both are affine functions of ξ, with $f(x + s(y-x)) = f(x_h) > f(x) + s(f(y)-f(x))$.
Thus at $\xi = 1$, the two affine functions attain the same values $f(y)$.

Now, consider the function $\xi| \rightarrow f(x + \xi(y-x)) - [f(x) + \xi(f(y)-f(x))]$ for $\xi \in [s,1]$.
Since it is the difference of two affine functions, it must be affine.
At $\xi = s$, the value of the function is positive and at $\xi = 1$, the value of the function is zero. Hence for $t \in [s,1)$, the value of the affine function must be positive, i.e., $f(x + t(y-x)) > f(x) + t(f(y)-f(x))$ for $t \in [s,1)$.
Thus, if $h = k+1$, then $f(x + t(y-x)) > f(x) + t(f(y)-f(x))$ for $t \in (0,1)$.

Suppose that $f(x + t(y-x)) > f(x) + t(f(y)-f(x))$ for $t \in (0,1)$ if $h \in \{k+1,\ldots, k+j\}$ and now suppose $h = k+j+1$. Thus, $y \in (x_{k+j+1}, x_{k+j+2}]$.
By the induction hypothesis, $f(x + t(x_{k+j+1}-x)) > f(x) + t(f(x_{k+j+1})-f(x))$ for $t \in (0,1)$, i.e., $f(x + t(x_h-x)) > f(x) + t(f(x_h)-f(x))$ for $t \in (0,1)$.
$y > x_h$ implies $x_h = x + s(y-x)$ for some $s \in (0,1)$.
Thus, $x + ts(y-x) = (1-t)x + t[x + s(y-x)] = (1-t)x + tx_h = x + t(x_h-x)$ for all $t \in [0,1]$.
Thus, $f(x + st(y-x)) > f(x) + t(f(x_h)-f(x))$ for $t \in (0,1)$.

Towards a contradiction suppose $f(x_h) \leq f(x) + s(f(y)-f(x))$.
Now $f(y) = f(x_h) + \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}(y-x_h)$ and at the same time
$f(y) = [f(x) + s(f(y)-f(x))] + \frac{f(y)-f(x)}{y-x}(y-xh) \geq f(xh) + \frac{f(y)-f(x)}{y-x}(y-xh)$.

Since $\frac{f(y)-f(x)}{y-x} > \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}$, we get $f(y) \geq f(x_h) + \frac{f(y)-f(x)}{y-x}(y-x_h) > f(x_h) + \frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}(y-x_h) = f(y)$, i.e., $f(y) > f(y)$, which is not possible.
Thus, it must be the case that $f(x_h) > f(x) + s(f(y)-f(x))$, i.e., $f(x_h)-f(x) > s(f(y)-f(x))$.
Thus, applying the induction hypothesis, we get
$f(x + st(y-x)) = f(x + t(x_h-x)) > f(x) + t(f(x_h)-f(x)) > f(x) + ts(f(y)-f(x))$ for all $t\in(0,1]$.
Thus, $f(x + t(y-x)) > f(x) + t(f(y)-f(x))$ for all $t\in(0,s]$.

Consider the functions $t|\rightarrow [f(x + t(y-x))] - [f(x) + t(f(y)-f(x))]$ for all $t\in[s, 1]$.
$f(x + t(y-x)) = f(x_h) + [\frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}](t-s)(y-x)$ for all $t\in[s, 1]$.
$[f(x + t(y-x))] - [f(x) + t(y-x)] = f(\beta_h) + [\frac{f(x_{h+1})-f(x_h)}{x_{h+1}-x_h}](t-s)(y-x) - [f(x) + t(f(y)-f(x))]$ for all $t\in[s, 1]$.
Thus, $t|\rightarrow f(x + t(y-x)) - [f(x) + t(f(y)-f(x))]$ for all $t\in[s, 1]$ is an affine function.

At $t = s$, the value of the function is $f(x_h) - [f(x) + s(f(y)-f(x))] > 0$ and at $t = 1$, the value of the function is 0.

Thus, $[f(x + t(y-x))] - [f(x) + t(f(y)-f(x))] > 0$ for all $t\in[s, 1)$.
Combined with what we obtained earlier we get $f(x + t(y-x)) > f(x) + t(f(y)-f(x))$ *for all* $t\in(0, 1)$.

Now, suppose f is concave. Towards a contradiction suppose $D^2f(x_j) > 0$ for some $j\in\{2,\dots,n-1\}$. Thus $D^+f(x_j) > D^-f(x_j)$. Thus, $\frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j} > \frac{f(x_j)-f(x_{j-1})}{x_j-x_{j-1}}$.
Hence $(x_j-x_{j-1})f(x_{j+1}) + (x_{j+1}-x_j)f(x_{j-1}) > (x_{j+1}-x_{j-1})f(x_j)$.
Thus $\frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}f(x_{j+1}) + \frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}f(x_{j-1}) > f(x_j) = f(\frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}x_{j+1} + \frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}x_{j-1}) = f(x_{j-1} + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}(x_{j+1}-x_{j-1}))$, since $x_j = \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}x_{j+1} + \frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}x_{j-1} = x_{j-1} + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}(x_{j+1}-x_{j-1})$.
Further, $0 < \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}} < 1$.
This contradicts f is concave and proves that $D^2f(x_j) < 0$ for all $j\in\{2,\dots,n-1\}$.
(ii) Follows from (i) and the fact that f is convex if and only if $-f$ is concave. Q.E.D.
Note: With the algebraic proof of Proposition 1 being as noted above, now the following alternative proof of Proposition 1 is equally valid:
Proof: Draw a diagram. Q.E.D.

Applications of the theory discussed in this and the previous section, may be the computation of risk-aversion at initial wealth for utility functions which display "loss aversion" which is discussed in the next section.

RISK AND LOSS AVERSIONS FOR MONETARY GAINS

The "conjectures" discussed in the previous section may be applied to problems associated with measuring risk aversion for piecewise linear functions of the type discussed in page 224 (the fourth section of Chapter 13) of the work by Eeckhoudt, Gollier and Schlesinger cited earlier in this paper. Absolute risk aversion- as defined in [de Finetti 1952], [Arrow 1962] and [Pratt 1963]- is the rate of "percentage change in marginal utility of gains", with respect to "change in wealth". Relative risk aversion is the rate of "percentage change in marginal utility of gains", with respect to "percentage change in wealth".

In this case the utility function we are concerned with is a piece-wise linear, real valued and concave function u with $0 \in int(dom\ (u))$, $u(x) = ax$ for all $x \in dom(u)$ with $x \geq 0$ and $u(x) = bx$ for all $x \in dom(u)$ with $x < 0$, where $b > a > 0$. Clearly u is concave and further it is non-differentiable at 0.

Thus, $D^+u(0) = a > 0$, $D^-u(0) = b > 0$, $D^2u(x) = a-b < 0$ and for $\alpha \in [0,1]$, the α-first derivative of f at 0, $Du^{(\alpha)}(0) = \alpha a + (1-\alpha)b > 0$.

For $\alpha \in [0,1]$, the α-**absolute risk aversion** at x = 0 is given by $\frac{b-a}{\alpha a+(1-\alpha)b}$ and the α-**relative risk aversion** for initial wealth 'w' is given by $w\frac{b-a}{\alpha a+(1-\alpha)b}$.

We may refer to $\frac{D^-u(0)}{D^+u(0)}$ (which in this case is $\frac{b}{a}$) as the "**co-efficient of loss-aversion at initial wealth**". Thus, for $\alpha \in [0,1]$, the α-absolute risk aversion at $x = 0$ is $\frac{\frac{b}{a}-1}{\alpha+(1-\alpha)\frac{b}{a}}$ and the α-relative risk aversion for initial wealth 'w' is $w\frac{\frac{b}{a}-1}{\alpha+(1-\alpha)\frac{b}{a}}$.

For $\alpha = 1$, the 1- absolute risk aversion at $x = 0$ is $\frac{b}{a} - 1$ which increases as $\frac{b}{a}$ increases.

For $\alpha = 0$, the 0- absolute risk aversion at x = 0 is $1- \frac{1}{\frac{b}{a}}$ which increases as $\frac{b}{a}$ increases.

For $\alpha \in (0,1)$, consider the function $x| \rightarrow \frac{x-1}{\alpha+(1-\alpha)x}$ whose domain is the set of strictly positive real numbers.

$\frac{\alpha+(1-\alpha)x}{x-1} = \frac{-\alpha(x-1)+x}{x-1} = -\alpha + \frac{x}{x-1} = -\alpha + \frac{x-1+1}{x-1} = -\alpha + 1 + \frac{1}{x-1}$ decreases as x increases. Thus, the function $x| \rightarrow \frac{x-1}{\alpha+(1-\alpha)x}$ increases as x increases. Thus, for $\alpha \in (0,1)$, the α- absolute risk aversion at $x = 0$ is $\frac{\frac{b}{a}-1}{\alpha+(1-\alpha)\frac{b}{a}}$ which increases as $\frac{b}{a}$ increases.

It might seem strange that for the same utility function a wealthy person has higher "relative risk aversion" than a less wealthy person. However, it is very unlikely for a wealthy person to have the same utility function that a less wealthy person has. For one, the wealthy individual is likely to have a lower value for the coefficient of loss-aversion at initial wealth, than a less wealthy individual, thereby reducing absolute risk aversion and possibly relative risk aversion for the former.

SUMMARY

In this paper we propose and discuss a kind of calculus for piecewise linear functions, that cannot be generalized to a wider class of functions- in particular functions which are differentiable. There is a very wide scope for applying such a calculus in economics and we choose just one example- measurement of risk aversions for monetary gains and losses- to show how it may be applied. It may be worth noting that Ross in [Ross 1981] has questioned the validity of absolute risk aversion as a "good enough" measure of attitude towards risk. Ross suggests that willingness to pay a higher risk premium for the same amount of loss, indicates greater risk aversion. In order to incorporate risk premiums in our analysis the utility function invoked in the previous section would have to be slightly modified by including at least one more point where the second derivative of the utility function is negative and this point would have to correspond to a loss of monetary wealth. Typically, an individual with a lower second derivative- hence greater absolute value of the second derivative, since the second derivative is negative- would be the one who would be willing to pay a higher risk premium if the "accidental loss" exceeded the loss where the second derivative is negative, and such an individual would be the one who is more risk averse in the sense of Ross.

The framework and the mathematical technology of our discussion is within the subject area known as finite mathematics in the tradition of Kemeny, Snell and Thomson [Kemeny, Snell and Thomson 1957]. Finite mathematics is the mathematics (e.g., real analysis, matrix algebra and analysis, probability theory etc.) that is based on just the "ordered field" property (axiom) of the real number system but not "the least upper bound" property (axiom) of the real number system. The least upper bound property says that "every non-empty subset of real numbers that is bounded above has a least upper bound". That does not prevent us from defining the least upper bound of a non-empty subset of real numbers that is bounded above and explicitly showing that one exists, if that were possible. For instance, the statement "1 is the least upper bound of the closed interval [0,1]" is permissible in finite mathematics, since 1 is an upper bound of [0,1] and given any rational number $\varepsilon > 0$ (i.e., a positive real number that can be expressed as the ratio of two positive integers), we can always find a real/rational number, say max $\{\frac{1}{2}, 1-\frac{\varepsilon}{2}\}$ that belongs to [0,1] and is greater than $1-\varepsilon$. However, since the proof of the Archimedean property of the real number system, requires the least upper bound property, the

Archimedean property or any of its consequences, is not admissible in finite mathematics.

Note: This paper is a revised and considerably expanded version of an earlier note by the author entitled: Two observations on "Economic and Financial Decision Under Risk" by Eeckhoudt, Gollier and Schlesinger (2005) that is available at: https://drive.google.com/file/d/1-UnmzHX16xorCQ9sqzTpyMtijOTdl5Qn/view).

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