QUANTITATIVE METHODS IN ECONOMICS
Volume XXV, No. 1, 2024, pp. 10 – 18

ON JOINT DISTRIBUTION OF BIDS IN SOME *k*-TH PRICE AUCTION MODEL

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Abstract: In the era of increasingly widespread use of electronic communication tools, constant growth of e-commerce turnover is hardly surprising. One of its mechanisms are online auctions whose turnover, on a global scale, is enormous. This paper proposes a stochastic model of the value of subsequent actual bids appearing in the course of a dynamic internet k-th price auction. On this basis, it was possible to determine unconditional joint probability distributions, as well as univariate marginal distributions of successive bids appearing in the course of the auction type under consideration.

Keywords: auction bids, k-th records, exceedance distribution

JEL classification: C46, C58, C19

INTRODUCTION

Auctions other than first-price took place as early as the end of the 19th century, but the first time they gained a complete theoretical description was in 1961, thanks to [Vickrey 1961]. The original Vickrey-type auctions are second-price auctions. Over time, this type of auction was generalized to the k-th price auction. k-th price auctions, similarly to classic auctions, can be held both in a static and dynamic form.

The latter form began to develop dynamically with the emergence and widespread use of electronic media. The current wide-ranging availability of the Internet means that e-auctions no longer of interest only to a limited group of parties but to billions of customers worldwide. Consequently, they involve enormous financial flows. An introduction to the theory of auctions and a broad review of the

https://doi.org/10.22630/MIBE.2024.25.1.2

literature can be found, among others, in [Klemperer 1999], [Ockenfels et al. 2006], [Hickman et al. 2012].

One may understand static k-th price auction as a game consisting of two stages: 1) auction participants get to know the starting price and submit bids (knowing only their own offer); 2) the participant with the highest bid wins, but pays the k-th highest bid price for the auctioned object. Numerous papers describe this type of auction (especially for $k = 2$). They draw from game theory and aim to study the efficiency, equilibrium points, or limiting properties of the auction with an infinite increase in the number of participants. Occasionally, in the conducted analyses, risk-neutral bidders and risk-averse bidders are discerned or the possibility of collusion between certain bidders, and other specific circumstances, is assumed. See, e.g., [Fibich, Gavious 2008], [Fibich, Gavious 2010], [Mezzetti, Tsetlin 2009], [Gorelkina 2014], [Hickman 2010], [Mihelich, Shu 2020], [Myerson 1981].

By contrast, in dynamic auctions, successive bids of various bidders (including repeated bids) are registered one by one within a predetermined time interval, and the current purchase price changes with successively incoming bids. The bidders know the price at all times. Such an auction is completed at the end of the set time interval, and the last current price becomes the purchase price. Here, one may apply the k-th price auctions.

Following the distinction indicated in [Athey, Haile 2007] and the definitions contained therein, we note that the literature proposes two types of stochastic models of bid flow in dynamic auctions: 1) models taking into account potential bids (bids from potential bidders (e.g. [Haile, Tamer 2003], [Canals-Cerdá, Pearcy 2013])); 2) models taking into account actual bids (bids of actual/active bidders (e.g. [Barbaro et al. 2006], [de Haan et al. 2008], [Namazi, Schadschneider 2006], [Roth and Ockenfels 2000])). In both cases (potential and actual bids), stochastic modeling of the sequence of bid values is reduced to the determination of an appropriately selected (finite) sequence of random variables. Potential bids make it possible to use a sequence of independent and identically distributed random variables.

Although such a sequence does not necessarily accurately describe reality, it makes it easier to use probabilistic tools, e.g., the theory of k-th record values (see [Dziubdziela, Kopociński 1976], [Dziubdziela 2018]). A general model of potential bids for any k was considered by Dziubdziela [2008], who examined limiting properties of relative increment of auction prices, using the results obtained in the papers of [Bieniek, Szynal 2000] and [Bieniek, Szynal 2003]. At the same time, the special case of $k = 2$ was analyzed more thoroughly in [de Haan et al. 2008], [de Haan et al. 2009].

In contrast, the values of actual bids – by their very nature – cannot be modeled with the use of a sequence of independent random variables with the same probability distribution. That is because in this case it is necessary to use sequences of random variables that assume ever greater values from a certain moment.

Consequently, these variables are dependent and have different probability distributions.

The bid value model proposed in the article refers to the principles of conducting eBay on-line auctions which are very similar to dynamic second-price auctions (see [Namazi, Schadschneider 2006]). This model does not consider the minimum bid increment and the possibility of using a proxy bidding system. Although our model describes one aspect of an auction (bid amounts) and uses certain simplifying assumptions, it is innovative on the account of a simple ploy of using exceedance distributions to describe actual bids that are stochastically dependent.

RESULTS

Model definition

Without loss of generality, we assume that a seller price $X_0 = 0$. Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables that represent actual bids – announced successively by active bidders – in a dynamic 'k-th-price' auction, for a priorly fixed $k \geq 2$, $k \in \mathbb{N}_+$. Then, $X_k < X_{k+1} < X_{k+2} < \cdots$ almost surely, which means that the auction is ascending from the k-th bid.

Let F be a common cumulative distribution function (cdf) of $X_1, X_2, \ldots, X_k$. Since $X_1, X_2, \ldots, X_\mathrm{k} \geq 0$, then $F(0) = 0$. We additionally assume that $X_1, X_2, \ldots, X_k$ are absolutely continuous, which means that F is differentiable. So let $f = F'$ be probability density function (pdf) of these variables.

With respect to the given cdf F and pdf f, we consider a family of 'exceedance distributions' indexed by all real $a \geq 0$, thus we work on two families $\{F_{\bar{a}}\}_{a\geq 0}$, and $\{f_{\bar{a}}\}_{a\geq 0}$ of cdf's, and pdf's, given by the following formulas:

$$F_{\bar{a}}(t) = \frac{F(t)-F(a)}{1-F(a)} \cdot \mathbb{I}_{(a,+\infty)}(t),$$

$$f_{\bar{a}}(t) = \frac{f(t)}{1-F(a)} \cdot \mathbb{I}_{(a,+\infty)}(t).$$

Note that $F_{\bar{0}} = F$, $f_{\bar{0}} = f$, and $F_{\bar{a}}(0) = 0$ for any $a \geq 0$

This family of distributions was used by [Haile, Tamer 2003], followed by [Canals-Cerdá, Pearcy 2013], which is a very practical approach (Haile and Tamer called these distributions as 'truncated distributions'). However, we must clearly emphasize that these papers considered models of the distributions of private valuations coming from potential bidders, and not, as in our case, the distributions of actual bids.

By our definition, the course of bidding assumes $X_1, X_2, \ldots, X_k$ to be independent and identically distributed with a common cdf F, while on the basis of the given family of exceedance distributions, conditional distributions of variables $X_{k+1}, X_{k+2}, \ldots$ are:

$$(X_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1,) \sim F_{\overline{x_{n-k:n-1}}}.$$

Model properties

Let $\mathcal{D}_n^{(k)}$ designate the support of joint distribution of vector $(X_1, X_2, \dots, X_n)$. Therefore:

- $\mathcal{D}_n^{(k)} = \mathbb{R}_+^n$, for $n \in \{1, \dots, k\}$
- $\mathcal{D}_n^{(k)} = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \forall_{j \in \{k+1,\dots,n\}}\, x_j \; > \; x_{j-k:j-1}$, for $n > k$

where $x_{1:n} \le \dots \le \; x_{n:n}$ stand for the ordered configuration of numbers $x_1, \dots, x_n$

Let $x \in \mathcal{D}_n^{(k)}$ for a given k and n. Then we define:

$$\Sigma_x^{(k,n)} \; = \; \left\{ \sigma \in \mathcal{P}(n) : \bar{\sigma}(x) \in \mathcal{D}_n^{(k)} \right\},$$

where $\bar{\sigma}(x) = \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right)$ and $\mathcal{P}(n)$ designates the permutations' set of $\{1, \dots, n\}$.

Lemma 1.
If $n \ge k$, then for any $x \in \mathcal{D}_n^{(k)}$ we have $\#\Sigma_x^{(k,n)} \; = k^{n-k}k!$.

Proof: For a fixed value of k we use induction with respect to n.

1. In the basic case, since $n = k$, $\mathcal{D}_n^{(k)} \; = \mathbb{R}_+^n$, then $\#\Sigma_x^{(k,n)} \; = k!$.
2. In the inductive step (where $n > k$), assuming that for any $x \in \mathcal{D}_n^{(k)}$ we have $\#\Sigma_x^{(k,n)} \; = k^{n-k}k!$, we show that for any $x' \in \mathcal{D}_{n+1}^{(k)}$ we obtain $\#\Sigma_{x'}^{(k,n+1)} \; = k^{n-k+1}k!$.

   For a given $x' = (x'_1, \dots, x'_n, x'_{n+1}) \in \mathcal{D}_{n+1}^{(k)}$, it follows that $x \coloneqq (x'_1, \dots, x'_n) = \pi_{(1,\dots,n)}(x) \in \mathcal{D}_n^{(k)}$. Let $\sigma \in \Sigma_x^{(k,n)}$ and $(y_1, \dots, y_n) \coloneqq \left(x'_{\sigma(1)}, \dots, x'_{\sigma(n)}\right)$. By definition of $\mathcal{D}_n^{(k)}$, we get $\forall_{j \in \{k+1,\dots,n\}}\, y_j \; > \; y_{j-k:j-1}$. Next, by definition of $\mathcal{D}_{n+1}^{(k)}$, we have $x'_{n+1} \; > \; y_{j-k+1:n}$.
   In such circumstances, there are exactly k possibilities to expand the permutation $\sigma \in \mathcal{D}_n^{(k)}$ to such a permutation $\sigma' \in \mathcal{D}_{n+1}^{(k)}$ that $\sigma'|_{\{1,\dots,n\}} = \sigma$:

   $$\bar{\sigma}'_1(x') = (y_1, \dots, y_{n-k+1}, x'_{n+1}, y_{n-k}, \dots, y_n),$$
   $$\bar{\sigma}'_2(x') = (y_1, \dots, y_{n-k+2}, x'_{n+1}, y_{n-k+1}, \dots, y_n),$$
   $$\vdots$$
   $$\bar{\sigma}'_k(x') = (y_1, \dots, y_n, x'_{n+1}),$$

   which encompasses exactly all the possible permutations in $\mathcal{D}_{n+1}^{(k)}$.

Steps 1. and 2. justify the proof.

□

Since the number $\#\Sigma_x^{(k,n)} \; = k^{n-k}k!$ does not depend on the choice of $x \in \mathcal{D}_n^{(k)}$, we denote it hereinafter as $N(k, n)$.

Theorem 2.
The joint unconditional pdf of vector $(X_1, X_2, \dots, X_n)$ has a form:

$$f_n(x_1, x_2, \dots, x_n) = \frac{f(x_1)\cdot f(x_2)\cdot\ldots\cdot f(x_n)}{G(x_{1:k})\cdot G(x_{2:k+1})\cdot\ldots\cdot G(x_{n-k:n-1})} \cdot \mathbb{I}_{\mathcal{D}_n^{(k)}}(x_1, \dots, x_n),$$

where $G(t) = 1 - F(t)$.

Remark 3.
Definition of $\mathcal{D}_n^{(k)}$ yields that if $(x_1, x_2, \dots, x_n) \in \mathcal{D}_n^{(k)}$, then $x_{p:r} = x_{p:s}$ for any ordered entries $x_{p:r}$, $x_{p:s}$ such that $k \le p + k - 1 \le r \le s \le n$. For that reason the pdf in Theorem 2 may be rewritten as:

$$f_n(x_1, x_2, \dots, x_n) = \frac{f(x_1)\cdot f(x_2)\cdot\ldots\cdot f(x_n)}{G(x_{1:n-1})\cdot G(x_{2:n-1})\cdot\ldots\cdot G(x_{n-k:n-1})} \cdot \mathbb{I}_{\mathcal{D}_n^{(k)}}(x_1, \dots, x_n) =$$
$$= \frac{f(x_1)\cdot f(x_2)\cdot\ldots\cdot f(x_n)}{G(x_{1:n})\cdot G(x_{2:n})\cdot\ldots\cdot G(x_{n-k:n})} \cdot \mathbb{I}_{\mathcal{D}_n^{(k)}}(x_1, \dots, x_n).$$

Proof of Theorem 2: On the basis of the proposed definition of k-th-price-auction model it emerges that if number of bids n is smaller or equal to k, then the bids are independent, which leads to joint pdf given by a formula:

$$f_n(x_1, x_2, \dots, x_n) = f(x_1) \cdot f(x_2) \cdot \ldots \cdot f(x_n).$$

So as a special case we have:

$$f_k(x_1, x_2, \dots, x_k) = f(x_1) \cdot f(x_2) \cdot \ldots \cdot f(x_k).$$

Otherwise ($n > k$), $n - k$ last bids depend on the previous ones. In this case, we derive recursively joint pdf of $(X_1, X_2, \dots, X_n)$ with the use of conditional pdf's:

$$\begin{aligned} f_n(x_1, \dots, x_n) &= f_{n-1}(x_1, \dots, x_{n-1}) \cdot f_{X_n|X_1=x_1,\dots,X_{n-1}=x_{n-1}}(x_n|x_1, \dots, x_{n-1}) = \\ &= \cdots = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1) \\ &= f_{X_n|X_1=x_1,\dots,X_{n-1}=x_{n-1}}(x_n|x_1, \dots, x_{n-1}) \cdot \ldots \cdot \\ &\quad \cdot f_{X_{k+1}|X_1=x_1,\dots,X_k=x_k}(x_{k+1}|x_1, \dots, x_k) \cdot f(x_k) \cdot \ldots \cdot f(x_1). \end{aligned}$$

Now, since the conditional distributions are modeled by a given family of exceedance distributions, then the direct substitution in (1) completes the proof.

□

The obtained joint distribution formula leads to unconditional marginal distributions of all variables $X_{k+1}, X_{k+2}, X_{k+3}, \dots$, which is stated in:

Theorem 4.
For fixed k and n, $n > k$, the unconditional marginal pdf of X_n has a form:

$$f_{X_n}(t) = \left(\frac{k}{k-1}\right)^{n-k} f(t) \left(1 - G^{k-1}(t) \sum_{i=0}^{n-k-1} \frac{(1-k)^i}{i!} \ln^i G(t)\right).$$

Before we proceed with the proof of Theorem 4, we consider auxiliary integrals that are recursively defined for a fixed $t > 0$ as follows:

$$c_1(x) := \int_x^t \frac{f(s)}{G(s)} G^{k-1}(s)ds, \quad c_{m+1}(x) := \int_x^t \frac{f(s)}{G(s)} c_m(s)ds$$

where $0 < x < t, m \in \mathbb{N}_+$.

Lemma 5.
The auxiliary integrals are given by:

$$c_m(x) = \frac{1}{(k-1)^m} G^{k-1}(x) + \frac{1}{k-1} G^{k-1}(t) \sum_{j=0}^{m-1} \frac{1}{j!} a_{m-1-j}(t) \ln^j G(x), \tag{2}$$

where

$$a_j(t) = -\sum_{i=0}^{j} \frac{(-1)^i}{i!(k-1)^{j-i}} \ln^i G(t).$$

Proof: We use induction with respect to m.

1. In the basic case, the calculation yields that:
$$c_1(x) = \int_x^t f(s)G^{k-2}(s)ds = \frac{1}{k-1}\left(G^{k-1}(x) - G^{k-1}(t)\right).$$
In this case $a_0(t) = 1$.
2. In the inductive step (where $m > 1$), assuming that (2) holds, and knowing that:
$$\int_x^t \frac{f(s)}{G(s)} \ln^j G(s)\, ds = \frac{1}{j+1}\left(\ln^{j+1} G(x) - \ln^{j+1} G(t)\right),$$
for $j \in \mathbb{N}$, we calculate:
$$\begin{aligned}\int_x^t \frac{f(s)}{G(s)} c_m(s)ds &= \frac{1}{(k-1)^{m+1}}\left(G^{k-1}(x) - G^{k-1}(t)\right)\\ &\quad + \frac{1}{k-1} G^{k-1}(t) \sum_{j=0}^{m-1} \frac{1}{j!} a_{m-1-j}(t) \frac{1}{j+1}\left(\ln^{j+1} G(x) - \ln^{j+1} G(t)\right)\\ &= \frac{1}{(k-1)^{m+1}} G^{k-1}(x) + \frac{1}{(k-1)^{m+1}} a_0(t)G^{k-1}(t)\\ &\quad + \frac{1}{k-1} G^{k-1}(t) \sum_{j=1}^{m} \frac{1}{j!} a_{m-j}(t)\left(\ln^j G(x) - \ln^j G(t)\right)\\ &= c_{m+1}(x) - \frac{1}{k-1} G^{k-1}(t)a_m(t) - \frac{1}{k-1} G^{k-1}(t) \sum_{j=1}^{m} \frac{1}{j!} a_{m-j}(t) \ln^j G(t)\\ &\quad + \frac{1}{(k-1)^{m+1}} a_0(t)G^{k-1}(t)\\ &= c_{m+1}(x) - \frac{1}{k-1} G^{k-1}(t) \sum_{j=0}^{m} \frac{1}{j!} a_{m-j}(t) \ln^j G(t) + \frac{1}{(k-1)^{m+1}} G^{k-1}(t).\end{aligned}$$
Now let $S = \sum_{j=0}^{m} \frac{1}{j!} a_{m-j}(t) \ln^j G(t)$. Then $S = -(k-1)^{-m}$, because:
$$\begin{aligned}S &= -\sum_{j=0}^{m} \frac{1}{j!} \sum_{i=0}^{m-j} \frac{(-1)^i}{i!(k-1)^{m-j-i}} \ln^i G(t) \ln^j G(t)\\ &= \frac{-1}{(k-1)^m} \sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{j+i}{j} \frac{(k-1)^{j+i}}{(j+i)!} \ln^j G(t) \left(-\ln G(t)\right)^i\\ &= \frac{-1}{(k-1)^m} \sum_{l=0}^{m} \sum_{i=0}^{l} \binom{l}{j} \frac{(k-1)^l}{l!} \ln^j G(t) \left(-\ln G(t)\right)^{l-j}\end{aligned}$$

$$= \frac{-1}{(k-1)^m} \sum_{l=0}^{m} \frac{(k-1)^l}{l!} (\ln G(t) - \ln G(t))^l$$
$$= \frac{-1}{(k-1)^m},$$

where we substitute $l = j + i$, and assume that $0^0 = 1$.

□

Proof of Theorem 4: For a given k and n, $n > k$, we calculate:

$$f_{X_n}(t) = \int_{\{(x_1,x_2,\dots,x_{n-1})\in\mathbb{R}^{n-1}\}} f_n(x_1, x_2, \dots, x_{n-1}, t) d(x_1, x_2, \dots, x_{n-1})$$
$$= \frac{N(k,n)}{k!} \int_{\substack{\{(x_1,x_2,\dots,x_{n-1})\in\mathbb{R}^{n-1}:\\ x_1<x_2<\dots<x_{n-k},\\ x_{n-k+1},\dots,x_{n-1},t>x_{n-k}\}}} \frac{f(x_1)f(x_2)\dots f(x_n)}{G(x_{1:k})G(x_{2:k+1})\dots G(x_{n-k:n-1})} d(x_1, x_2, \dots, x_{n-1})$$
$$= k^{n-k} f(t) \int_0^t \frac{f(x_1)}{G(x_1)} \int_{x_1}^t \frac{f(x_2)}{G(x_2)} \dots \int_{x_{n-k-1}}^t \frac{f(x_{n-k})}{G(x_{n-k})} \int_{x_{n-k}}^{+\infty} f(x_{n-k+1}) \dots \int_{x_{n-k}}^{+\infty} f(x_{n-1}) dx_{n-1} dx_{n-2} \dots dx_2 dx_1$$

and notice that $f_{X_n}(t) = k^{n-k} f(t) c_{n-k}(0)$ which completes the proof.

□

FINAL REMARKS

This paper proposes a stochastic model of the course of actual bid values in a dynamic internet k-th price auction. The model takes into account the existence of stochastic dependence and different probability distributions describing subsequent bids. Despite conducting an extensive search of the literature on the subject, we were not able to find studies that would explicitly model actual bid values appearing in the course of a dynamic action in a way similar to our proposal. In our opinion, the determination of unconditional joint probability distributions of bid values, and especially marginal probability distributions thereof, is an interesting mathematical problem – even if considered separately from the auction framework.

We observe that in the case of $k = 1$, the probability distributions of subsequent bids in our model are identical to the distributions of maxima of independent and identically distributed random variables. This is so provided that the latter variables are given by the same cumulative distribution function F as the sequence of bids in the model. The mentioned maxima of independent and identically distributed random variables are the records in the sense of [Chandler 1952], and also they are the first record values ($k = 1$) in the sense of [Dziubdziela, Kopociński 1976]. However, we notice that for $k \geq 2$ such an equality of marginal bid distribution in k-th price auction and of k-th record values no longer holds.

Though the presented model describes bid amounts only, it can be combined with models describing two other aspects of dynamic auctions, i.e., time arrivals and the number of bids. For example, an interesting model of bid times has been proposed by [Shmueli et al. 2008]. Simultaneously, the number of bids can be naturally modeled using the Poisson process, as indicated in [de Haan et al. 2009], [Hickman et al. 2012], [Saeedi, Hopenhayn 2015], [Shmueli et al. 2008], [Wang et al. 2008].

REFERENCES

Athey S., Haile P. A. (2007) Nonparametric Approaches to Auctions. Handbook of Econometrics, eds. J. J. Heckman and E. Leamer, 6A, 3847-3965.

Barbaro S., Mainz O., Bracht B. (2006) Shilling, Squeezing, Sniping: Explaining Late Bidding in Online Second-Price Auctions.

Bieniek M., Szynal D. (2000) Limiting Distributions of Differences and Quotients of Successive k-th Upper and Lower Record Values. Probability and Mathematical Statistics, 20, 189-202.

Bieniek M., Szynal D. (2003) Limit Distributions of Differences and Quotients of Non-Adjacent k-th Record Values. Probability and Mathematical Statistics, Vol. 23, Fasc. 1, 19-38.

Canals-Cerdá J. J., Pearcy J. (2013) Arriving in Time: Estimation of English Auctions with a Stochastic Number of Bidders. Journal of Business & Economic Statistics, 31(2), 125-135.

Chandler K. N. (1952) The Distribution and Frequency of Record Values. Journal of the Royal Statistical Society: Series B (Methodological), 14(2), 220-228.

de Haan L., de Vries C., Zhou C. (2008) The Extent of Internet Auction Markets. SSRN Electronic Journal.

de Haan L., de Vries C. G., C. Zhou (2009) The Expected Payoff to Internet Auctions. Extremes, 12, 219-238.

Dziubdziela W. (2008) Probabilistyczny model aukcji internetowych ebay. Rola Informatyki w Naukach Ekonomicznych i Społecznych: Zeszyty Naukowe 8(2), 319-326.

Dziubdziela W. (2018) k-th Record Values and their Basic Properties. Mathematica Applicanda, 46, 293-324.

Dziubdziela W., Kopociński B. (1976) Limiting Properties of the k-th Record Values. Applicationes Mathematicae, 2(15), 187-190.

Fibich G., Gavious A. (2008) Asymptotic Analysis of Large Auctions with Risk-Averse Bidders. https://www.openu.ac.il/gametheoryconference2008/download/large_auctions_risk_aversion_MOR2.pdf [access: 2023.11.20].

Fibich G., Gavious A. (2010) Large Auctions with Risk-Averse Bidders. International Journal of Game Theory, 39(3), 359-390.

Gorelkina O. (2014) Bidder Collusion and the Auction with Target Bids. Discussion Paper Series of the Max Planck Institute for Research on Collective Goods 2014_10, Max Planck Institute for Research on Collective Goods.

Haile P., Tamer E. (2003) Inference with an Incomplete Model of English Auctions. Journal of Political Economy, 111(1), 1-51.

Hickman B. (2010) On the Pricing Rule in Electronic Auctions. International Journal of Industrial Organization, 28(5), 423-433.

Hickman B., Hubbard T., Saglam Y. (2012) Structural Econometric Methods in Auctions: A Guide to the Literature. Journal of Econometric Methods, 1, 67-106.
Klemperer P. (1999) Auction Theory: A Guide to the Literature. Journal of Economic Surveys, 13(3), 227-286.
Mezzetti C., Tsetlin I. (2009) Auctions in which Losers Set the Price. Games and Economic Behavior, 66(2), 855-864.
Mihelich M., Shu Y. (2020) Analytical Solution of kth Price Auction. International Journal of Game Theory, 49, 875-884.
Myerson R. B. (1981) Optimal Auction Design. Mathematics of Operations Research, 6(1), 58-73.
Namazi A., Schadschneider A. (2006) Statistical Properties Of Online Auctions. International Journal of Modern Physics C (IJMPC), 17(10), 1485-1493.
Ockenfels A., Reiley D., Sadrieh A. (2006) Online Auctions. NBER Working Papers 12785, National Bureau of Economic Research, Inc.
Roth A. E., Ockenfels A. (2000) Last Minute Bidding and the Rules for Ending Second-Price Auctions: Theory and Evidence from a Natural Experiment on the Internet. Working Paper 7729, National Bureau of Economic Research.
Saeedi M., Hopenhayn H. A. (2015) Dynamic Bidding in Second Price Auction. 2015 Meeting Papers 1346, Society for Economic Dynamics.
Shmueli G., Russo R., Jank W. (2008) The BARISTA: a Model for Bid Arrivals in Online Auctions. The Annals of Applied Statistics, 1, 412-441.
Vickrey W. (1961) Counterspeculation, Auctions, and Competitive Sealed Tenders. Journal of Finance, 16(1), 8-37.
Wang S., Jank W., Shmueli G. (2008) Explaining and Forecasting Online Auction Prices and Their Dynamics Using Functional Data Analysis. Journal of Business & Economic Statistics, 26(2), 144-160.