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Faculty of Applied Informatics and Mathematics
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THE FOSTER-HART FORMULA FOR CASES OF SUPPLY CHAIN PROJECTS WITH VARIOUS CONSTRAINTS

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Abstract: In the publication it was mentioned that the tool supporting the process of reducing of the risk of bankruptcy within the framework of supply chain projects is the measure of the Foster-Hart, whose most important features make it unique. It takes into account the possibility of bankruptcy. The main purpose of the publication is to conduct a spatial graphic interpretation of the Foster-Hart formula for such projects, however, with entering into consideration different constraints, that should be taken into account by the companies (principal contractors).

Keywords: risk, projects, simulations, bankruptcy, production

JEL classification: G11, G17

INTRODUCTION

All projects are both unique and complex, so managing them, can be considered a complicated and problematic process, as it always involves planning, organizing and coordinating resources so that project goals can be successfully completed [Vanhoucke 2012]. Practice clearly indicates that the value of implemented projects is getting higher, and therefore they are exposed to various types of risks. They also include financial risk, which may significantly expose enterprises to reduced financial liquidity and, consequently, also to bankruptcy. All this makes project risk management more and more often supported by many tools

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and techniques, which include, among others, various risk models and active management of it [Boyce 2003]. This approach makes it easier to complete the project on time, taking into account its specificity and budget. This specificity is the result of the fact that in practice different types of projects are implemented, which includes also comprehensive supply chain projects, that can be considered interesting due to their features, because they are particularly exposed to various types of risks (including to the already mentioned financial risk), which should be managed to avoid bankruptcy.

In order to support the financial risk management process of these projects, an instrument seems to be necessary to measure risk, taking into account domestic and foreign currencies, which may distort its level, while taking into account the possibility of bankruptcy. Based on the literature research, it was found that such an instrument could be the Foster-Hart measure, taking the form of the formula presented in further parts of the work. Although its basic features have been studied in modern literature, no spatial graphic interpretation of this formula has been made for cases of complex supply chain projects, taking into account various constraints. Due to the possibility of bankruptcy for companies (principal contractors) implementing such projects, it may seem desirable, which is why it is the purpose of the work. For its needs, literature devoted to the Foster-Hart measure was used, as well as hypothetical data, whose analysis was carried out in the MATLAB software environment.

SUPPLY CHAIN PROJECTS – GENERAL CHARACTERISTICS

In the subject literature, projects are usually defined in a similar way. However, the most well-known is the definition of the Project Management Institute [PMI 2013], according to which „A project is a temporary endeavor undertaken to create a unique product, service, or result. The temporary nature of projects indicates that a project has a definite beginning and end. The end is reached when the project’s objectives have been achieved or when the project is terminated because its objectives will not or cannot be met, or when the need for the project no longer exists. A project may also be terminated if the client (customer, sponsor, or champion) wishes to terminate the project“. Practice shows that there are also projects in supply chain management, here the term “supply chain projects” is used to denote them. According to Chopra and Meindl (2010), a supply chain „consists of all parties involved, directly or indirectly, in fulfilling a customer’s request” and the Supply Chain Process Cycle consists of the order, replenishment, manufacturing and procurement cycle.

On the whole, there are no precise definitions of supply chain projects. To understand a meaning of such projects, it is necessary to characterize a project production, for which this chain is temporarily build. However, such a production is connected with the production of product (service) made by temporary organization which is created for a project. After finishing this project the

organization is liquidate [Modig 2007]. The most important element of this organization is a principal contractor which manages the project [Parrod, Thierry, Fargier, Cavaille 2007] and deliver it. It means that the whole temporary organization delivers the project product from raw materials to customer. All in all, supply chain project involves principal contractor delivering project to which belong complicated and non-routine tasks [Modig 2007]. A delivery of it may expose to the risk of bankruptcy, because the cost control is problematic. That is why assuming payments generated by such project we are able to use a Foster-Hart measure or riskiness in order to avoid a risk of bankruptcy. Such approach is not widely described in the literature. All in all, each supply chain project is typically unique whereas the customers are changing, and the wealth of a company can be too less for implementing it.

FOSTER-HART MEASURE AND SUPPLY CHAIN PROJECTS

The Foster-Hart measure [Foster, Hart 2009], due to its properties, seems to be appropriate for supporting the financial risk management process of the supply chain project. This is mainly due to the fact that it allows us to indicate one of them, the implementation of which may lead to bankruptcy of the enterprise. It's enough to mention that this is a feature not found among other measures. In addition, it is universal and objective, while being monotonic, which allows us to analyze investments and projects from various currency areas. Its form is expressed by the following formula:

$$E \left[\log \left(1 + \frac{1}{R(g)} g \right) \right] = 0, \quad (1)$$

whereas $E[X]$ is the expected value of the random variable X , g is the income that is generated by the investment (or project) with a certain probability (p_i) at the end of the given period and $R(g)$ means the critical value of the investor's (or business) wealth and, at the same time, the measure of investment risk.

The presented measure is also based on the following assumptions:

$$\sum_{i=1}^n p_i = 1, \quad \sum_{i=1}^n p_i g_i > 0, \quad \text{and} \quad (2a)$$

$$g_i < 0 \text{ then } g_j > 0, \quad \text{where} \quad (2b)$$

$$i \neq j \text{ and also,} \quad (2c)$$

$$i, j = 2, 3, \dots \quad (2d)$$

This critical value is calculated in order to compare with the current level of wealth of the investor (or company). It allows the division of investments (or projects) into two types because, if $R(g)$ is higher than the investor's current level of wealth, then it is considered too risky and therefore leading the investor to bankruptcy. If, on the contrary, $R(g)$ equals or is lower than this level, then undertakings can be considered as acceptable and increase the assets of the

company. Based on the research carried out in Polish and foreign literature, it was found that there is a need to support the financial risk management process of supply chain projects, using the Foster-Hart measure, moreover, it is a little-considered subject. Literature is focused on this measure in the context of the use for investment in shares [Leiss, Nax 2018], and not for other projects that may or even are always associated with income's generation. The problem that is associated with the use of this measure concerns the adaptation of its form to the mentioned process. However, assuming that it can be constructed using the classical theory of the decision tree [Aczel 2008], then an equivalent form of formula (1) can be used, which for four final payoffs is as follows:

$$\left(1 + \frac{g_1}{R(g)}\right)^{p_1} \times \left(1 + \frac{g_2}{R(g)}\right)^{p_2} \times \left(1 + \frac{g_3}{R(g)}\right)^{p_3} \times \left(1 + \frac{g_4}{R(g)}\right)^{p_4} = 1. \quad (3)$$

The presented dependence is possible assuming that during the implementation of the project, and thus after its start, at some point t_1 a specific scenario may occur, represented by two decision nodes, so that at the end of the project implementation period (t_2) four final can be assumed payoffs, two of which are the result of a given node. In this way one can get a payoff table, for which there is a dependence (3), where for example g_1 and g_2 are the result of occurrence at the moment t_1 of the positive scenario, and g_3 and g_4 - the negative scenario. Therefore, spatial graphic interpretation of the solution of Foster-Hart formula should help some main contractors evaluate supply chain project. Thanks to the spatial graphic a company can determine, whether to implement this project or not. For this purpose is needed hypothetical income of a project and hypothetical wealth of a pricipal contractor ($R(g)$). As it was shown, the too low value of wealth of a company. which does this implementation, can cause bankruptcy.

GRAPHIC INTERPRETATION OF THE FOSTER-HART FORMULA FOR SUPPLY CHAIN PROJECTS

The graphical interpretation of the Foster-Hart formula in the process of evaluating of supply chain projects riskiness, is based on the analysis of selected project cases. These cases relate to four projected payments with different probabilities and different scenarios. This analysis is based on a hypothetical supply chain project, which may take the form of two different variants (Table 1). Simulation tests were carried out in the MATLAB programming environment, while payoff was given without specifying the currency, because this is not important for the considerations. Three-dimensional drawings are prepared in such a way that they contain a hyper-plane that is the right side of the Foster-Hart formula and a plane with a value of "0", where their common part creates a trace that is the solution to this formula. On the other hand, in this two-dimensional drawings, this trace has been presented, taking certain values depending on the size of the withdrawals. The solution of the Foster-Hart formula is also presented on the three-dimensional charts. It is needed, because considering various constraints

there are more than one combination of payments, which can solve the Foster-Hart formula for one $R(g)$.

Table 1. Parameters of the sample supply chain project and its variants analyzed

Project parameters in the base version	
Parameter	Value
$R(g)$	250,000
Value of p_i ($p_1 = p_2 = p_3 = p_4$)	0.25
Range of g_1	(2,400;3,000) – change with step 20
Range of g_2	(-3,000;-2,500) – change with step 20
Value of g_3	-600
Value of g_4	800
Variant Number 1*	
Value of p_1	0.30
Value of p_2	0.25
Value of p_3	0.25
Value of p_4	0.20
Variant Number 2**	
$R(g)$	500,000
Value of g_3	600
Value of g_4	-800
Value of p_1	0.30
Value of p_2	0.25
Value of p_3	0.25
Value of p_4	0.20

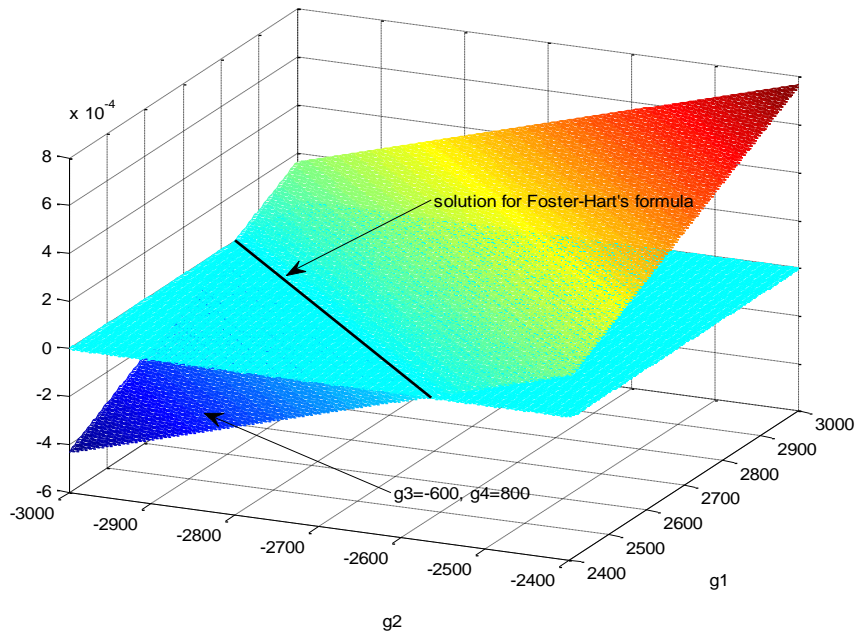
* For variant 1, the values of $R(g)$ and g are not given because they are the same as for the base version.

**For variant 2, the values g_1 and g_2 are not given because they are the same as for the base version. The most important features differentiating the presented investment cases are expressed using two-dimensional and three-dimensional charts.

Source: own study

The first numerical experiment of the project in the base version, of which $R(g)$ is immutable, but with replacement payout values, is shown in Figure 1.

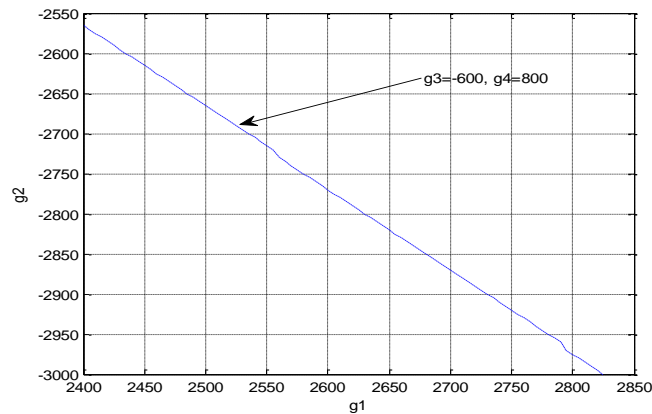
Figure 1. Three-dimensional graph of dependencies between the parameters of the analyzed project in the base version



Source: own study based on the results of the MATLAB program

Figure 1 presents the fact that not every combination of payments allows for the solution of the Foster-Hart formula. Therefore, not every project can be analyzed by enterprises. The rightness of the Hart-Foster rule occurs only for certain parameter ranges. Figure 1 shows a continuous and intermittent solution, which cover different ranges of variability g_1 and g_2 for accepted input assumptions. What's more, the changed payout values for the negative scenario significantly change the solution of the formula being described, making this case a completely separate project. The details of solution from Figure 1 is more convenient to analyze in a two-dimensional graph, as shown in Figure 2 including the variant of the base version. Thus, by changing the input parameters of the project, different solutions are obtained using the Foster-Hart measure.

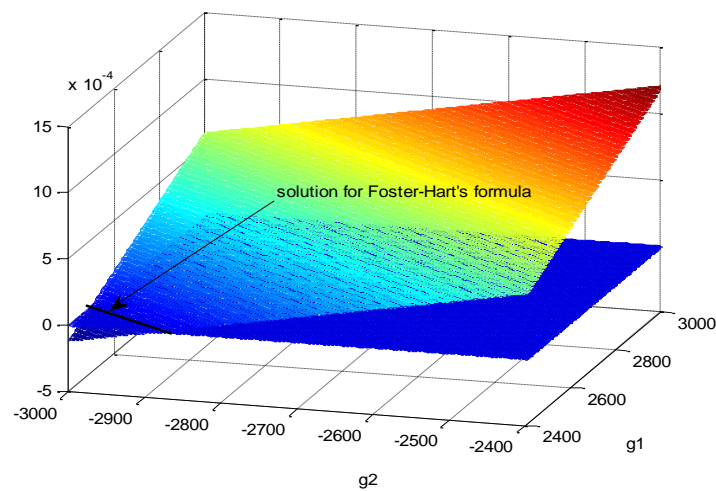
Figure 2. Two-dimensional graph presenting the solution of the Foster-Hart formula depending on the size g_1 and g_2 for the design in the base version



Source: own study based on the results of the MATLAB program

The first variant of the project is distinguished by the fact that the probabilities associated with the negative scenario are relatively low, and therefore the change in the size of payments generated by projects under such a scenario does not play a significant role for the enterprise. It is worth noting, however, that with the increase of p_1 and with the same p_2 , a higher loss g_2 can be accepted for a given payment value g_1 . This means that such a specific project should be treated as less risky and therefore more attractive for the company (Figure 3).

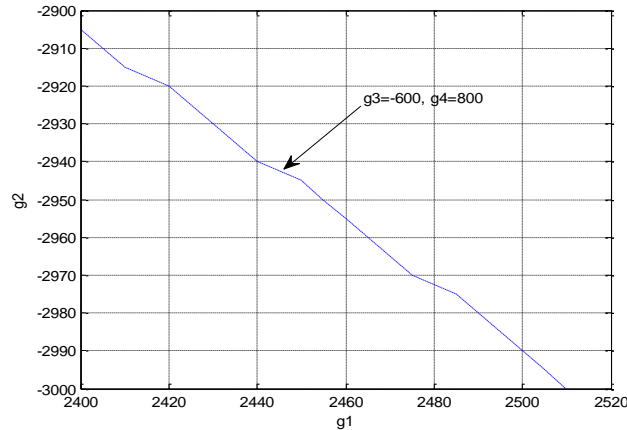
Figure 3. Three-dimensional graph of dependencies between the parameters of the analyzed project in the first variant



Source: own study based on the results of the MATLAB program

The reduction of the project risk is also illustrated in the two-dimensional graph (Figure 4).

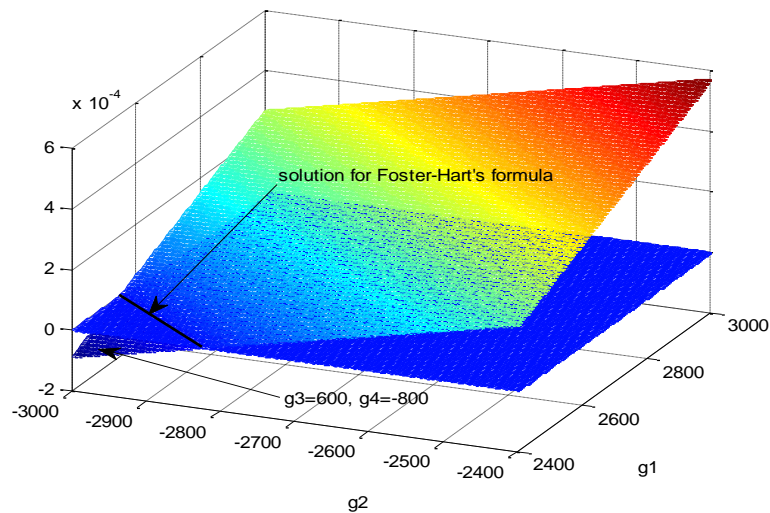
Figure 4. Two-dimensional graph presenting the solution of the Foster-Hart formula depending on the size g_1 and g_2 for the first variant



Source: own study based on the results of the MATLAB program

The second variant of the analyzed project assumes that $R(g)$ increases, but at the moment t_2 a company can get very low value of a project payment (Figure 5).

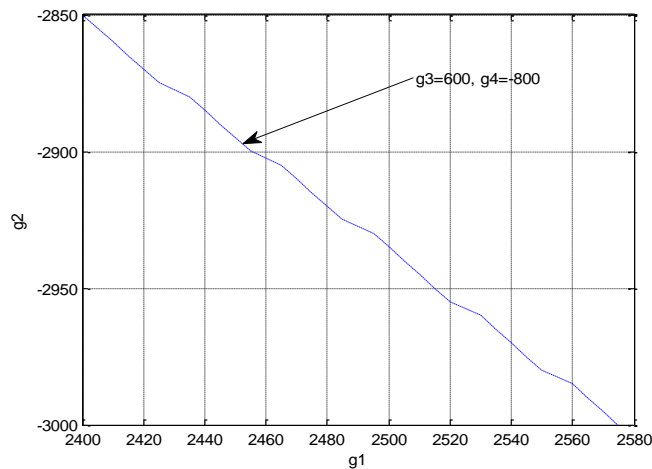
Figure 5. Three-dimensional graph of dependencies between the parameters of the analyzed project in the second variant with changed values of $R(g)$



Source: own study based on the results of the MATLAB program

The simulations carried out, reflect the fact that as the value of $R(g)$ increases, the project becomes less and less risky, and therefore the company may become less sensitive to changes in payments in hypothetical cases – even with low g_4 , because the hyper-plane and the "0" plane create with it an increasingly smaller angle.

Figure 6. Two-dimensional graph presenting the solution of the Foster-Hart formula depending on the size g_1 and g_2 for the second variant of the project



Source: own study based on the results of the MATLAB program

It should be added that the location of these hyper-planes also changes (Figure 6), which indicates that changes $R(g)$ may have to carry out other projects that the company originally set up. This is because, with the increase of $R(g)$, increasingly large ranges of values g_2 are acceptable to the company (in compare to the basic version).

CONCLUSIONS

The main problem with supply chain projects is that they are characterized by so many risks that the company realizing them, may lose liquidity. For this reason, the process of financial risk management of these projects should be supported by an instrument that measures risk, taking into account the possibility of bankruptcy. The work showed that it could be the Foster-Hart measure. Summarizing, in this paper it has been shown that with the appropriate assumption, this measure can be used in the process of evaluating of the supply chain project from the perspective of bankruptcy avoiding, however it is not always possible to resolve the Foster-Hart formula. Its graphic analysis confirmed this fact. In addition, it allowed to state that changes in the value of payments, generated by the project, change the solution of the formula. In addition, even a small decrease in

the probability level of the payout values, in the case of a positive scenario means that the project should be treated as more risky. It is logical, however, that with the increase in the value of $R(g)$ of a company, the supply chain project should be treated as less risky. Let it be pointed out, therefore, that the graphical interpretation of the Foster-Hart formula can be treated as a tool supporting the analysis of projects, because it is possible to compare a $R(g)$ of a project with an asset of company and to reject such project for which too high level of $R(g)$ is a need.

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ABOUT A CERTAIN “ANOMALY” IN THE PRICING OF DEBT SECURITIES

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Abstract: The intention of the paper is the presentation of some considerations concerned with the problem of debt securities pricing without simplifying assumptions that are commonly used in practice. In the paper the deterministic and discrete time approach is used. On financial markets coupon rates are strictly connected with interest rates described by yield curve. This relation is linear but different structures of bonds can be described by taking into account particular assumptions which refer to the coefficients in this dependence. It is shown that taking into account the dependence of coupons on forward rates leads to not standard dependence of intrinsic value on spot rates. It turns out that the intrinsic value of a bond is not a decreasing function of interest rates, what in a very fundamental way changes the investment risk that accompanies that kind of bonds.

Keywords: debt securities, intrinsic value, interest rate risk, coupon bonds, yield curve

JEL classification: C02, C65, G12,

INTRODUCTION

Bonds are immensely important instruments of the capital market. Hence the question of their pricing remains vital – for instance because the intrinsic value of those instruments depends on many factors. In particular, what matters is the method of coupon payment and its structure, or bond options (the right of early redemption by the issuer, the holders’ right to present bonds for redemption earlier); however, the primary factor is the spot rate structure (yield curve). It should be noted that as for the latter, what we observe is a sort of feedback: the term structure of interest rates is determined based on the prices of debt securities

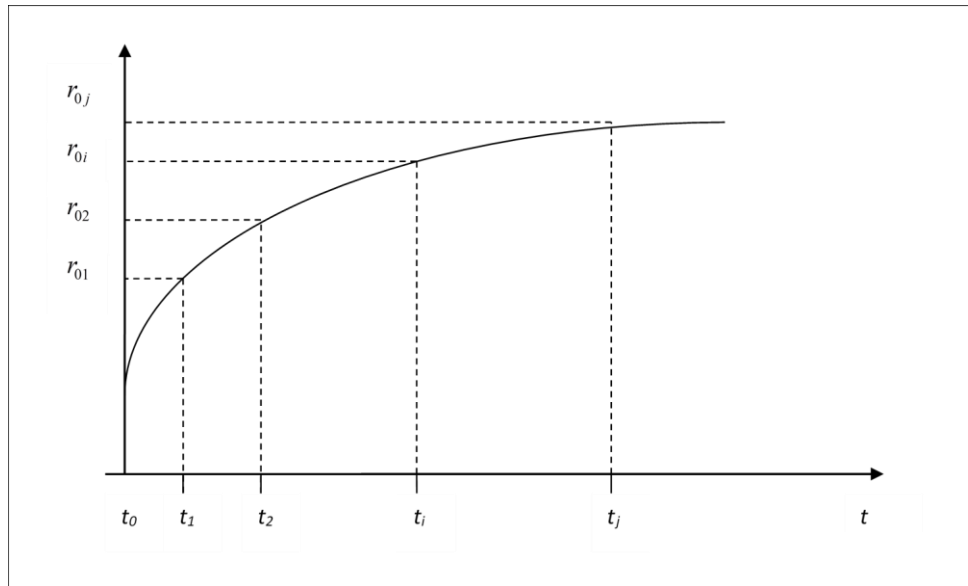
<https://doi.org/10.22630/MIBE.2019.20.1.2>

listed on the free market; on the other hand, it is the basis for pricing of such securities, which is particularly true for their new issuance. For this reason, bond pricing relies on highly advanced mathematical tools, such as non-linear approach methods [Deeba et al. 2002], partial differential equations [Zui-Cha et al. 2010], integration into trajectories known from the field of physics [Zanga et al. 2017], and Green's function [Pooe et al. 2004]. In the light of the aforementioned research, this paper relies on both elementary and deterministic methods. The underlying foundation is, of course, the notion of pricing based on the assumption that a bond intrinsic value is the sum of future financial flows discounted at present and strongly depend on market spot rates [Ritchken et al. 1966]. However, what motivates these considerations is an observation that in many cases the coupon rates depend on forward rates resulting from the yield curve. Consequently, it turns out that the intrinsic value (so the price, too) of some bonds is an increasing function of spot rates, rather than a decreasing one, as it is commonly known. This paper specifies conditions that need to be met for the "anomaly" to occur. Moreover, it touches on investment consequences and offers examples rooted in the Polish market of bonds that meet these requirements. The most introductory considerations are based on paper [Karpio 2010].

INTRINSIC BOND VALUE IN THE GENERAL CASE

A bond is a type of security, the pricing of which is based on expected interest rates. For this reason, let us start with a few comments on the term structure of interest rates (yield curve). We do not examine the curve structure methodology; instead we assume that this has been completed and the structure is given. Let us select the series of time moments (specific dates): t_0, t_1, \dots, t_N with t_0 being the investment start moment, and with the $t_N = T$ moment being its end. In one case, the series may be infinite (perpetual bonds). Let us also assume that the intervals between the selected dates are not homogeneous, and we will mark them with the $\Delta t_{i-1} = t_i - t_{i-1}$ symbols. We introduce a following convention: time intervals are expressed in years, and all the rates are nominal – referenced against years. The spot rates applying to the investments starting from the current moment t_0 and ending in the future moment t_i are marked with r_{0i} symbols. The forward rates related to the investments between moments t_i and t_j ($j > i$) are marked with f_{ij} . Figure no. 1 below presents a sample and somewhat idealistic spot rate curve.

Figure 1. Example of yield curve



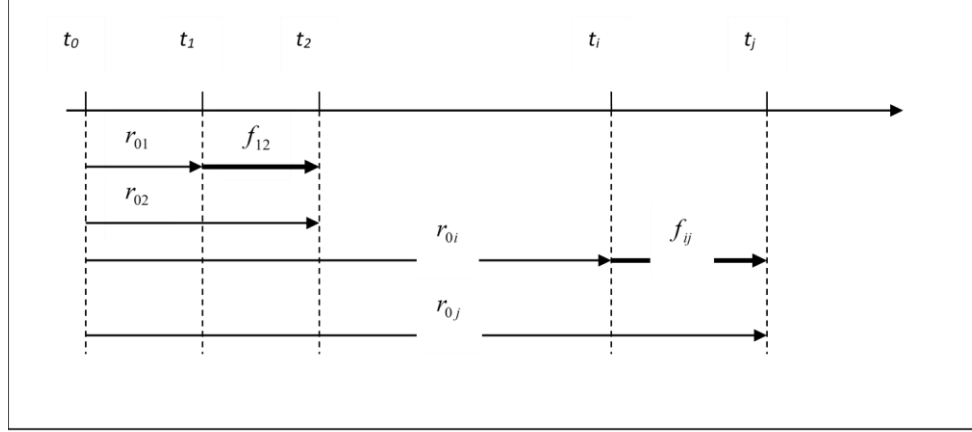
Source: own drawing

Please notice that at the starting moment t_0 , the spot rate does not equal to zero. This is because the curve describes nominal rates. They are referenced against a time unit; hence their border value is different from zero. This is observable when one examines interbank rates: daily rates do not usually differ much from the weekly rates or monthly ones; in fact, occasionally they tend to be higher. Obviously, with the investment time reverting to zero, the rate of return also tends to zero, but this is an investment-based rate, and not a nominal one. It is also noteworthy that some institutions offering access to the Forex currency market provide second-based capitalisation of security deposits at nominal rates substantially different from zero.

In practical terms, the curve above is called “normal” (spot rates grow in the time function); however, other shapes of the yield curve occur: flat, reverse, humped, etc. In the first case, the nominal investment rate is a constant time function. With the reverse yield, the longer the investment period is, the lower the nominal rates are. This yield curve type can be seen, for example, when inflation is high, and the central bank takes measures to reduce it. However, investors must believe that the measures will be effective; had it not been for this belief, the yield curve would be growing.

Let us analyse investments with different implementation periods. Bank deposits with different maturities may serve as a model. In schematic terms, a set of such investments, along with corresponding spot and forward rates, is shown in Figure 2.

Figure 2. Schematic representation of spot and forward rates



Source: own drawing

The spot rates are known, as they are observed in the form of the market prices of debt securities. The problem is the knowledge of forward rates, since they are the main factor affecting the decision whether an investment should be one-time in nature or have many stages. One of the most effective ideas followed on well-balanced financial markets – the principle of non-arbitration – justifies the following formula linking spot and forward rates:

$$\prod_{k=1}^i (1 + r_{0i} \Delta t_{k-1}) \prod_{k=i+1}^j (1 + f_{ij} \Delta t_{k-1}) = \prod_{k=1}^j (1 + r_{0j} \Delta t_{k-1}). \quad (1)$$

Hence, we arrive at a formula that allows one to determine forward rates (usually as an approximation):

$$\prod_{k=i+1}^j (1 + f_{ij} \Delta t_{k-1}) = \frac{\prod_{k=1}^j (1 + r_{0j} \Delta t_{k-1})}{\prod_{k=1}^i (1 + r_{0i} \Delta t_{k-1})}. \quad (2)$$

Based on this dependency, it is possible to prove a number of properties of forward rates [Karpio 2008]. For instance, with a normal (increasing) yield curve, the following relation takes place:

$$r_{0i} < r_{0j} < f_{ij}. \quad (3)$$

It has serious investment implications, since today's forward rates will be the spot rates in the future, which brings about very specific investors' actions.

In the case of the flat yield curve, forward rates are equal to spot rates. When determining the price of bonds, one-period forward rates are tremendously import. Consequently, the equation (2) makes it possible to precisely determine their value, which equals:

$$f_{i-1i} = \frac{1}{\Delta t_{i-1}} \left(\frac{\prod_{k=1}^i (1 + r_{0i} \Delta t_{k-1})}{\prod_{k=1}^{i-1} (1 + r_{0i-1} \Delta t_{k-1})} - 1 \right). \quad (4)$$

Let us assume that we work with a debt security that in the future, in the moments: t_1, \dots, t_N , will pay out coupons described with streams (payouts in the annual scale) h_{i-1i} applying to periods Δt_{i-1} . The payment will be made in the moments t_i , namely in arrears. The money coupons will then be equal to: $h_{i-1i} \Delta t_{i-1}$. The pricing takes place in the current moment t_0 . If P_M is the face value of the debt security, then the intrinsic value, defined as the current value of future cash flows, is equal to:

$$P_{t_0T} = \sum_{i=1}^N \frac{h_{i-1i} \Delta t_{i-1}}{\prod_{k=1}^i (1 + r_{0i} \Delta t_{k-1})} + \frac{P_M}{\prod_{k=1}^N (1 + r_{0N} \Delta t_{k-1})}. \quad (5)$$

Marking of the intrinsic value P_{t_0T} takes into account the pricing moment and the bond's maturity moment. Therefore, it is a $T - t_0$ forward bond. This formulation is free from any simplifying assumptions that are often used in literature. The interpretation is based on the reinvestment of the paid-out interest, which can be easily observed when the formulation is multiplied by $\prod_{i=1}^N (1 + r_{0N} \Delta t_{i-1})$, and when the definition of forward rates f_{ij} is used (equation (2)). In such case, the relation (5) takes the following form:

$$\begin{aligned} P_{t_0T} \prod_{i=1}^N (1 + r_{0N} \Delta t_{i-1}) \\ = \sum_{i=1}^N h_{i-1i} \Delta t_{i-1} \prod_{k=i+1}^N (1 + f_{i+1N} \Delta t_{k-1}) + P_M. \end{aligned} \quad (6)$$

The intrinsic value P_{t_0T} is the amount, the future value of which is equal to the sum of the face value P_M and the future value of the coupon payments $h_{i-1i} \Delta t_{i-1}$, for the period from the payment moment t_i to the maturity moment T , with forward rates equal to f_{i+1N} .

In practical terms, the notion of interest rate (coupon rate) is preferred over the stream of interest. The relation between these two values is very simple, having the following form:

$$s_{i-1i} = \frac{h_{i-1i}}{P_M}. \quad (7)$$

Hence, the formulation (5) may be expressed through coupon rates and demonstrated in the form that better lends itself to discussion:

$$\frac{P_{t_0T}}{P_M} - 1 = \sum_{i=1}^N \frac{(s_{i-1i} - f_{i-1i})\Delta t_{i-1}}{\prod_{k=1}^i (1 + r_{0i}\Delta t_{k-1})} \quad (8)$$

The left side refers the intrinsic value to the nominal value; hence it can be called a normalised intrinsic value. The right side clearly depends on the relation between coupon rates and forward rates. And this fact is of fundamental significance for further considerations.

ANOMALOUS INTEREST RATE RISK

Subsequent stages of the discussion on the bond intrinsic value call for some assumptions regarding coupon rates. In practice, two types of coupon bonds are most common: with fixed and variable coupon rates. As for the latter, the coupon depends on forward rates; an analysis of the structures of various bonds used on the Polish capital market (but also in other countries) shows that it is feasible to adopt the following dependency between coupon rates and forward rates.

$$s_{i-1i} = s + pf_{i-1i}. \quad (9)$$

The different values of the coefficients s and p ($s \geq 0$, $p \geq 0$) lead to various types of obligations encountered on financial markets. In particular, when $p = 0$ and $s > 0$, we get bonds with fixed coupons; $p > 0$ and $s = 0$ result in bonds with the coupons proportional to forward rates; in the case of $p = 0$ and $s = 0$, we observe zero-coupon bonds. If one attempts to interpret the aforementioned dependency, it is justifiable to conclude that the premium paid to a bondholder is composed of the part linked to the market interest rates pf_{i-1i} and the constant s that is independent of the changeable market situation reflected by the interest rate levels. Please notice that as far as the flat structure of forward rates is concerned, the coupon rate remains constant and equal to $s_{i-1i} = s + pr = \text{const}$. And so, in this case each bond (with the exception of the zero-coupon bond) is a fixed coupon bond as long as the rates remain constant.

Having applied the relation (9) to the formula (8), as well as having performed some transformations, we arrive at the following form of the normalised intrinsic value of a bond:

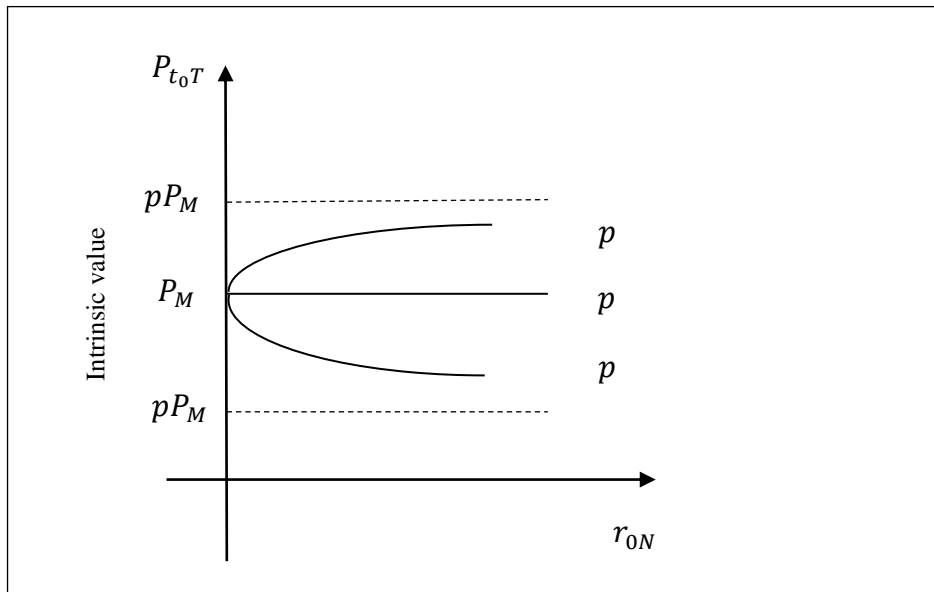
$$\begin{aligned} \frac{P_{t_0T}}{P_M} - 1 = s \sum_{i=1}^N \frac{\Delta t_{i-1}}{\prod_{k=1}^i (1 + r_{0i}\Delta t_{k-1})} \\ + (p - 1) \left(1 - \frac{1}{\prod_{k=1}^N (1 + r_{0N}\Delta t_{k-1})} \right). \end{aligned} \quad (10)$$

This relation clearly makes the intrinsic value dependent on the bond structure parameters. The commonly recognised claim that the intrinsic value (price) is the decreasing function of interest rates is not quite true. Let us consider a case when $s = 0$, then in the formula (10), but other component remains unchanged. If we make an additional assumption that $p > 1$, which is often the case in practice, one can easily notice that the intrinsic value is an increasing function of spot rates. This is a direct consequence of the fact that coupon rates are functions of forward rates. The claim that the intrinsic value is a decreasing function of interest rates is based on the output formula (5) for the intrinsic value; however, what is ignored in such case is the fact that the stream of interest is, in fact, a function of forward rates.

One should also note that when $s = 0$ the intrinsic value depends only on the spot rate related to the bond maturity date r_{0N} which, among others, makes it easier to calculate the basic measure of debt securities investment feasibility – namely the yield to maturity YTM. One can also easily provide asymptotic properties of the intrinsic value when $N \rightarrow \infty$ (i.e. $T \rightarrow \infty$) or $r_{0N} \rightarrow \infty$, if only $p \neq 1$. In an exceptional case $p = 1$, we are dealing with a constant function. In terms of quality, the dependency of P_{t_0T} on spot rate r_{0N} with various scopes of the parameter p can be illustrated with figure 3. The upper curve, when $p > 1$ and the lower curve, when $p < 1$, corresponds to the increase of maturity date $T - t_0$ with the predetermined discount rate r_{0N} .

The consequence of the obtained relations is, among others, specific investment decisions, depending on the expectations towards future interest rates. For instance: if $p > 1$, then investors, awaiting an increase of interest rates, will generate demand for these bonds, expecting a price growth; this means that they will be selling bonds with the parameter $p < 1$. Moreover, longer-term bonds (the upper curve) will attract bigger interest. The investors' behaviour will be different if a drop in interest rates is expected: they will be selling their bonds with the parameter $p > 1$ and buying bonds with the parameter $p < 1$, while the shorter-term bonds will be more popular.

Figure 3. The qualitative dependence of the bond intrinsic value and interest rate when $s = 0$



Source: own drawing

Finally, it should be noted that the described “anomalous” bonds have occurred and are present on the Polish capital market. In our reality, Treasury bonds are divided into wholesale bonds – on the primary market they are traded in tenders to the Treasury Securities Dealers, and retail bonds (savings bonds) offered to individuals. If we focus on the latter, currently, among the bonds on offer, three-year floating interest rate (TOZ) bonds can be found with a nominal value of PLN 100. Coupon rates are based on WIBOR 6M – this is the arithmetic average of the five days preceding the six-month coupon period. This rate is multiplied by the factor which in recent years is equal to 1, which in the presented considerations corresponds to the case $p = 1$. TOZ bonds have been issued since mid-1990s. For many years, the coupon has very much depended on the average yield of Treasury bills; interest was paid on a quarterly basis and the coefficient p was more than 1, when the National Bank of Poland tried to reduce inflation. However, it became less than 1, when inflation was under control. For these reasons, in 2012 issuance of Treasury bills was abandoned and the bonds were based on WIBOR 6M, while interest was paid out every six months. Since then, bonds have not been publicly traded and the coefficient p equals 1.

It was mentioned above that the coupon in subsequent interest periods is calculated on the basis of the arithmetic average of the WIBOR 6M rate from five days, the last of which is at least five days before the next interest period (all business days). The presence of the average does not undermine the previous

considerations regarding the term structure of spot rates. It is determined based on prices of listed bonds, and due to the different maturities and the lack of these instruments with all possible redemption moments, the curve is generated by averaging and interpolating the actual prices observed. Thus, by definition, spot rates and the resulting forward rates are average values.

SUMMARY

The considerations presented above have been illustrated with the use of TOZ bonds, but this is not the only example. It is possible to find among Treasury bonds examples of interest structures corresponding to any possible values of the parameters s and p . For instance, on the Polish market two-year zero-coupon bonds are found, i.e. OK ($s = 0$ and $p = 0$), ten-year and five-year bonds with constant coupon payments, respectively DS and PS ($s > 0$ and $p = 0$), or until recently – ten-year bonds were issued with variable coupon rate DZ ($s > 0$ and $p = 0$). Clearly, the examples above are not all the structures that can be found on the capital market. Consequently, the formula (10) presented herein covers a very large range of debt securities. Moreover, it is founded on the relation between forward and coupon rates, and the presented examples are limited to Treasury instruments. However, the considerations are directly transferred to the bonds of other issuers, business entities or local government units (municipal bonds). The only difference is the structure of the yield curve – it is determined separately for each group of issuers. The remainder of the considerations remains unchanged; it is also easy to provide examples of real bonds that are not Treasury bonds, which can be described using the formula (10) and different values of the parameter s and p .

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METHODOLOGICAL BARRIERS OF MONITORING AND RESEARCH OF FOOD LOSSES AND WASTE (FLW)

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Abstract: The subject of the paper is to perform a review and analysis of the methods and approaches to the problem of quantitative determination of the level of generating the food losses and waste in the food chain in the context of the application of quantitative methods for measurement of food waste generation. In the article, there have been reported the key needs, conditions and problems connected with the quantitative measurement of the food waste and the results of the key studies in this respect, including the definition problems and the results of big research projects, undertaken on a global level by FAO and on the level of the EU. The methods of quantitative determination and assessment of the food waste, including the life cycle analysis (LCA) have been presented. The application of the mentioned method allows satisfying the aims connected with the monitoring of a flow of the resources throughout the whole agri-food chain for the needs of creating a circular economy.

Keywords: food losses and waste, food waste, quantitative methods, food chain, life cycle analysis (LCA)

JEL classification: A11, A13, D78

INTRODUCTION

Food losses and waste (FLW), being also defined as food waste constitute a current, serious global problem and apart from a threat to the food safety of the world, they mean also wasting of the resources such as water [Duchin 2005], energy [Cuellar, Webber 2010] and other resources of natural environment and potential raw materials to be utilised in food economy in the future [Krajewski et al. 2016]. Generation of considerable quantities of the food waste is a symptom of a society being based upon the non-sustainable consumption which gradually with the increase in consumption and production generates higher and higher level of waste.

The assessment of the size of food losses and waste on the national level as well as in the particular branches and in the whole food sector has not been – until now – the subject of the complex studies in Poland, similarly as it refers to evaluation of the consequences of the phenomena relating to food waste. The mentioned problems cover a wide research area, being not undertaken until now due to a lack of available and reliable information, lack of appropriate documentation and source data. It restricts a scope of the studies and requires firstly, development of the method for quantification of the arising food waste on the macro level. Then, in the successive stage of the studies, the developed method may be verified and applied in relation to the national waste data in order to create the first analysis of quantitative definition of the food waste level in Poland.

The real scale and the level of the currently produced food waste are not known to the institutions of the state administration and to the societies, including also Poland. Apart from a lack of the verified methodology, the barrier comes from the fact that the entities which produce and utilize the discussed waste have not perceived until now the need of collecting the information where the waste were generated, where they were coming from and how they were processed [Beretta et al. 2013]. Only since the moment of appearance and application of life cycle analysis (LCA) method and material flow analysis (MFA), the mentioned data have become available what allowed mapping the streams of waste and monitoring of the mentioned phenomena in perspective of food products and chains [Corrado et al. 2017]. In such situation, the assessment of losses caused by food waste and of the potential resources when recovering the products became possible; the background for management of the discussed processes has been created. The process of food losses and waste management in the food chains should be treated as development of the process of food management, raising the level of effectiveness of managing the resources of food, water and energy [Krajewski et al. 2018] in food sector and in relation to the principles of a circular economy.

According to the scale and necessary accuracy in assessment of the level of food losses and waste, there are two main schools of thinking in respect of the methods for estimating and forecasting data on food wastes [Beigl, Lebersorger,

Salhofer 2008]. The first, bottom-up micro-approach assumes determination of indicator of waste generation (per capita, per plant, for particular products, for household, geographical area, etc.) which is then extended to the economy as a whole [Karadimas, Loumos 2000]. The exact estimation of the quantity of waste in micro-approach has many weak points due to difficulties in estimating of waste flows in the food chains and due to a huge amount of the data required to development of the estimate for each industrial branch. The second approach is a macro approach [Joosten et al. 1998] where is it assumed that the generated waste are proportional to manufacture in each sector and are analysed as a part of significant material flows in the economy.

THE AIM AND RANGE OF THE WORK

The studies on the food losses and waste, not only under the conditions of Polish food sector, meets important barriers of the lack of coherent definitions of the analysed phenomena, lack of adequate research methodologies and lack of data availability in economic documentation of the enterprises and institutions and in national statistics. It makes the processes of the studies and comparative analyses in the world scale difficult, limits the possibilities of monitoring the discussed phenomena for economic and administrative needs, running the national statistics and management of the processes.

Therefore, the aim of the considerations, as discussed in the present paper, will include the definition of the mentioned restrictions and evaluation of the research methodologies for estimating the food losses and waste, with a special reference to the possibilities of gaining and availability of the indispensable information. In the first stage of the work, it will be necessary to perform, therefore, the comparative analysis of the definition of key terms connected with the problems of food losses and waste in the context of the process of estimating the level of the discussed phenomena. Also, the comparative evaluation of the intentionally chosen research methodologies in relation to gaining information, modelling and quantification of the phenomena will be carried out. The obtained information will become a basis for conducting further studies within the frames of the research project, with acronym PROM, as implemented by the team within the frames of the competition NCBR "Gospostrateg"¹.

¹ The project entitled „Development of the System of Monitoring the Wasted Food and of the Effective Programme of Rationalization of Food Losses and Limitation of Food Waste”. Acronym PROM. Gospostrateg No 1 1/385753/1/NCMR/2018 dated 10.10.2018.

PROBLEMS OF DETERMINATION OF FOOD LOSSES AND WASTE IN FOOD CHAIN

The review of definitions of food losses and waste and of food waste for the needs of developing the methodology of the studies

The significant methodological problems, connected with the measurement of the level of food losses and waste in the food chain include a lack of clear and coherent definitions of basic categories of the subjects relating to the discussed problems, and, especially the subjects of food wastage and food waste [Bilska, Kołożyn-Krajewska 2016] and lack of common and harmonized methodology of quantitative measurement of generated losses and food waste. The report of the European Commission (EC) includes here also the delineation of the borders of the particular elements of the supply chain (food chain) and determination of the sources of data gaining [Caldeira, Corrado, Sala 2017]. The adequate methodology of the measurement requires, according to the European Commission, the following formulations:

- Appropriate definitions and terminologies, in particular of such categories as: edible/inedible parts of food; avoidable/unavoidable food waste ²;
- Borders of the particular stages and links of the supply chain of raw materials and agri-food products;
- Measurement units, so as the existing data between the regions, countries and types of the products could be comparable.

Until now, any legal definition of food losses and food waste has been adopted, neither in the EU food legislation, nor in the law concerning the waste³. The first work being conducted with the aim to make a complex examination of the mentioned problem in the EU included the preparatory study concerning food waste in EU 27 countries⁴. As a result of it, food waste were defined as a part of bio-waste, consisting of raw or processed food substances, including food products, being discarded at any time across the chain between agricultural farm and consumption, especially produced in the households before, during or after preparation of meals. The food waste may be classified into edible and inedible ones.

We should mention here a wide study on the European level, i.e. the Report of the European Project FUSIONS where a similar approach was adopted, with the definition of food waste as substances removed from the food chain. According to

² The studies of the food losses and waste with the application of categories „avoidable” and “unavoidable”, especially during the final stages of the supply chain, are conducted especially in Great Britain within the frames of the Initiative WRAP (2013). Household Food and Drink Waste in the United Kingdom, WRAP.

³ Ibid., p.12.

⁴ Preparatory study on food waste across EU 27 (2013) European Commission, BIOIS.

the mentioned approach, food waste is referred to as “any food, and inedible parts of food, removed from the food supply chain to be recovered or disposed (including composted, crops ploughed in/not harvested, anaerobic digestion, bioenergy production, co-generation, incineration, disposal to sewer, landfill or discarded to sea)”⁵. The reference definition on a global level has been generated as a result of the studies ordered by FAO; the mentioned studies consider all flows of substances which may be food, irrespectively of their destination but they treat food waste only as the edible parts of food, destined for human consumption which became losses or waste⁶.

In the Report of FUSIONS Project, dedicated to definitional framework for food waste, a comprehensive review of literature and definition of the terms connected with the problems of food losses and food waste for each stage of the food chain was carried out⁷. From the above mentioned review it is followed that the key terms may be differently defined in the particular stages of the chain and by its different participants and stakeholders (producers, processors, distributors, gross and retail trade, gastronomic units, households) and according to a given sector and branch. It is a significant factor of incoherence and methodological risk in the studies on losses, wastage and food wastes which should be appropriately estimated and managed. The results of the FUSIONS Project confirmed also a problem with the availability and quality of the data⁸.

In the English-language literature, there is conventionally employed term “Food loss and waste” – FLW, covering food losses and waste in total. In Polish language convention, there is adopted application of formula “food losses and wastage”. The univocal settlement of mutual overlapping of the semantic group of terms “losses” and “waste” and “wastage”, including their context scope, exceeds the frames of the present paper. It is, however, postulated that – following the definitions, adopted on the EU and global level – the wastage should be relating to generation of the waste and should be monitored via measurement of their amounts, i.e. the substances excluded from the food chain.

In the EU level, the food waste means all food, as defined in art. 2 of the Regulation (CE) 178/2002, that became the waste in accordance with art. 3 par.1 of the Waste Framework Directive⁹. It means that the food waste do not include either substances excluded from food definition (e.g. plants before harvesting, live

⁵ Food waste quantification manual to monitor food waste amounts and progression (2016), FUSIONS.

⁶ Global Food Losses and Food Waste: Extent, Causes and Prevention (2011), FAO.

⁷ FUSIONS Definitional Framework for Food waste Full Report, 2014, Annex C. Pp. 78-100.

⁸ Due to this reason, the data for the particular Member States were not published but only an estimate for the whole EU27 for each stage of food supply chain was given.

⁹ Directive 2008/98/EC of the European Parliament and of the Council of 19 November 2008 on waste.

animals if they are not prepared to introducing to turnover for human consumption), or substances excluded from waste definition (e.g. agricultural material used in agricultural farm, or by-products, employed in further industrial processes).

The data collected on the grounds of the Regulation (CE) 2150/2002 on waste statistics do not allow precise identification of food wastage within the scope of the generally collected data concerning production of waste. Similarly, the list of the waste as mentioned in art.7 of the Directive, and established in Annex to the Commission Decision 2000/532/EC9, does not allow, in many cases, the univocal identification of food losses. The measurement of food waste should however concern the material recognised as waste in accordance with the Directive. In connection with this fact, neither material excluded from the scope of the Directive, i.e. agricultural material as covered with art. 2.1 f) of the Directive, nor food destined for animal feed (as covered with art.2 par.2. e of the Directive), nor by-products of animal origin (as specified in art.2 par.2 b) should be monitored. Similarly, by-products coming from food production (as mentioned in art.5 of the Directive) should be not considered as food waste. The mentioned limitations should be, however, also estimated.

In the European Union countries, the temporary limits of the waste flows for the need of their measurement are designated by one-year period (since 1st January until 31st December) and the spatial limits – by territory of a given country. The problems of the possibility of classifying the arising food waste as possible to avoid or impossible to avoid are left by the European Commission to be settled on the level of country, region or sector (branch). Food wastes, in the current approach, constitute a separate category of waste, requiring the appropriate classification. It is not possible to obtain information of food waste exclusively on the grounds of the data, collected in within the frames of the European waste statistics WStatR, although the statistics of waste remain the main source of the available information on food wastage¹⁰.

In the Resolution of 16 May 2017 *On initiative on resource efficiency: reducing food waste, improving food safety* the European Parliament recommends the application of the following definition by the Commission and the Member States: “food waste means food intended for human consumption, either in edible or inedible status, removed from the production or supply chain to be discarded at primary production, processing, manufacturing, transportation, storage, retail and consumer levels, with the exception of primary production losses”¹¹.

¹⁰ According to WStatR, the data on food waste should be supplied in item 09 Animal and vegetal waste which are then divided into following sub-items: 09.1. Animal and mixed waste and 09.2. Vegetal waste. The second category of food waste is category 10. Mixed waste, in particular 10.1. Waste coming from households and similar. <https://rod.eionet.europa.eu/instruments/528>.

¹¹ Official Journal of Laws of the EU. C.2018.307.25.

In global (FAO, FLW Standard) and the EU (FUSIONS) definitions there is marked a tendency to consider the food losses and waste in total (FLW) and treating the wastage as generation of unnecessary food waste. There is a visible tendency to situate the losses on the production-supply side of the agri-food chain and the wastage on the demand-consumer side, being eventually considered totally as losses and waste.

In PROM Project, the definition obligatory on the European level (EU) was adopted in relation to food and food products. By-products are not defined as food because they are not intended for consumption but they are utilized for other purposes. If they are not utilized, they become waste but they are not food waste. Food waste is understood in conformity with the Directive on waste and means all food that is consistent with the definition specified in art. 2 of the Regulation (EC) 178/2002 of the European Parliament and of the Council and that became waste. According to the definition, adopted by the Polish Association of Food Technologists (PTTŻ) within the frames of MOST Project [Bilska, Kołożyn-Krajewska 2016], the term food losses and waste, as understood together, should be treated as food raw materials and products, manufactured for consumption purposes that have not been consumed by the people, so they have not been utilized in accordance with the primary destination of food in every stage of the food chain, from primary production, via processing and distribution to the final consumption in households. Such definition was also adopted in PROM Project. In the Project, the losses and waste are therefore defined and due to this fact, they are monitored and analyzed in total.

Identification of losses and waste in food chain

The processes of food management should be identified in a specific economic chain – in food chain, being also defined as the chain “*from farm to fork*”. The idea and definition of food chain is now the basis for ISO Standard 22000¹², settling the processes of health safety and quality assurance in food management. A justified cooperation in this respect, as occurring within the frames of the food chain, contains also a principle of sharing information; it ensures significant advantages for the companies - participants of the food chain, including limitation of waste and losses. Functioning of the chains of the particular food products is determined by many factors what will have an influence on processes of identification of the sites of generation of food losses and waste, monitoring of the level of the discussed phenomena and management of the processes.

On order to gain information of qualitative and quantitative nature in respect of the food losses and waste in the food chain, it is necessary to carry out the identification of the appropriate sources and ways of their obtaining. The data

¹² PN-EN ISO 22000:2006 Food safety management systems. Requirements for any organization in the food chain 2006, Polish Committee for Standardization (PKN), Warsaw.

sources in the qualitative methods may have a primary (direct) or secondary (indirect) nature. The direct measurement methods include: analysis of the composition of the waste, weighing or counting of the waste, evaluation of volume, surveys, diaries, records or observations. The indirect methods cover modelling, mass balance, food balance, use of the data from proxy servers and literature data. The direct obtaining of the data requires considerable expenditure. They are therefore usually employed in the particular stage of the supply chain, with the participation of a limited number of entities taking part in data collection.

The indirect measurements which use secondary data include a wider scope of the analysis and may ensure representativity in a higher scale, e.g. on the level of region or country. Most of the research described in literature is based upon the quantitative approach and data obtained from direct measurements, being based mainly or exclusively on the literature data. The so-far existing studies show that there are no universal methodologies for gaining the data and their choice depends on the specificity of a particular stage or link of the agri-food chain¹³. To estimate food losses and waste, the data obtained and collected primarily for other purposes may be also utilized, for example confirmation of the receipt of food waste, physical inventories or storehouse books. The presentation of direct and indirect methods is given in Table 1.

Table 1. Characteristics of direct and indirect methods in studies on food losses and waste

Direct methods	
Waste Composition Analysis (WCA)	It consists in physical separation of the waste constituents, their weighing and categorization. The method may be used to separate the fractions of food waste from the waste, containing other types of materials and substances as well as to get to know different substances which constitute food waste, including their types and sorts, as well as quantities of edible and inedible parts
Weighting	Use of weighing scales enables obtaining information on food waste mass (it may also include waste composition analysis)
Counting	Determination of the number of components of food waste with the application of counting methods, scanning, or use of visual scales in order to estimate the weight of the waste
Assessing volume	It consists in determination of the space, taken up by the food waste. The method is recommended for measurement of liquid waste but also, for solid and semisolid substances, including the determination of the waste amount, suspended in a liquid.
Garbage collection	Separation of fractions of food waste from other types of waste in the waste container in order to determine the weight

¹³ Kwasek M. (Ed.): Analysis of Food Losses and Waste in the World and in Poland. The Studies on Socially Sustainable Agriculture (37), Institute of Food Agriculture and Economy, State Research Institute, Warsaw (2016), p. 16.

	and proportion of food waste. It may or may not include waste composition analysis
Surveys	Gaining information from individual persons or entities on attitudes, beliefs and behaviours in relation to food waste via surveys (questionnaires)
Diaries	Gaining information on the waste on the grounds of daily records concerning the quantity and types of the generated food waste in the specified time intervals
Records	Determination of the quantity of the waste from the data coming from different documents, collected primarily for purposes different then recording of food waste (e.g. storehouse books)
Observation	Estimation of the waste volume, using comparative scales in order to determine visually the content of particular food residues
Indirect methods	
Modelling	It consists in estimation of the quantity of food waste, with the application of mathematical modelling methods based on the factors that determine generation of the waste
Mass balance	Estimation of the quantity of the waste by measurement of the outlays (e.g. raw materials and components in manufacturing plant) and the results (e.g. manufactured products) in the single stages of processing and related change in weight (e.g. water evaporation during boiling)
Use of Proxy data	Estimation of the quantity of waste on the grounds of the data obtained from enterprises or public entities. The method is often utilized for scaling of the data or gaining the aggregated data
Use of literature data	Gaining the data directly from literature or estimation of the quantity of food waste based upon the data from different publications

Source: C. Caldeira, S. Corrado, S. Sala (2017) Food Waste Accounting. Methodologies, Challenges and Opportunities. JRC Technical Reports, European Commission, 16-17

In the case of the studies reporting FW on the European level, the following approaches have been observed:

- Statistics concerning the waste, as based upon the Eurostat data in which the waste data contain classification into categories of waste according to three-digit European classification of the waste for statistical purposes and in conformity with the statistical classification of activity in the European Union (*Nomenclature Statistique des Activités économiques dans la Communauté Européenne, NACE*) where they are generated. EWC-Stat (*European Waste Catalogue*) is a substances-oriented classification and is connected with the administrative classification of waste list;
- Data from the national studies being extended to the European level;

- Combination of different data sources such as FAO, Eurostat, the European Food Safety Agency (EFSA) and scientific literature (Corrado, Sala 2013).

METHODOLOGIES FOR STUDIES ON FOOD LOSSES AND WASTE – GAINING INFORMATION, MODELLING, QUANTIFICATION

Methodologies of the studies on food losses and waste acc. to FAO – the case study: the EU countries

The studies conducted by Bio Intelligence Service (BIOIS) for the European Commission and the studies of Swedish Institute for Food and Biotechnology (SIK) for FAO belong to the major European and world data sources on food wastage. BIOIS conducts the studies on generation of the food waste on all stages of the food chain in the whole EU-27 but excluding agricultural production and without consideration of different product groups. SIK study concerns generation of food waste in all stages of the food chain, including agricultural production and with the classification into groups of the products. Unlike BIOIS, the studies of SIK have a global scope¹⁴.

In the methodology employed by FAO, the estimated level of losses and wastage for each of the analysed groups of products is determined using a mass flow model for each stage of the agri-food chain, in which the so-called food balance sheets coming from FAOSTAT database are utilized. Food Balance Sheets – FBS show a mass flow of food production in the country across a specified time interval. The following stages of the chain are considered: agricultural production, service and storage after harvesting, processing and packing, distribution and consumption. It should be mentioned that definition “consumption” covers domestic consumption as well as consumption outside the house i.e. in restaurants, coffeehouses, canteens, takeaway consumption, that is individual and collective consumption what results from the specificity of FBS which does not allow further differentiating¹⁵.

Food Loss and Waste Accounting and Reporting Standard – FLW Standard has been developed by organization Food Loss & Waste Protocol Partners, in order to determine the requirements and guidelines for governments, enterprises and other stakeholders, interested in identification of the sources of food losses and waste generation, their quantification, monitoring and appropriate management

¹⁴ Bräutigam K.-R., Jörissen J., Priefer C. (2014) The Extent of Food Waste Generation across EU-27: Different Calculation Methods and the Reliability of their Results. *Waste Management & Research*, 32(8), p. 684.

¹⁵ The method of mass balance to estimate food losses and waste is discussed in the report by J. Gustavsson et al. *Global Food Losses and Food Waste – Extent, causes and prevention* (2011), FAO, Rome.

with the aim to decrease their generation and reduce their impact on the environment.

The aim of the mentioned system is to facilitate measurement and monitoring of particular substances in the food chain and tracing their destination – target place. FLW Standard may be utilized on the level of a single enterprise as well as of the whole country in determination of the sites and scale of generating food losses, waste and wastage. The Standard consists of ten stages, from the definition of the aim of estimating the losses to the establishment of the way of monitoring the effectiveness and tracing the progress in time. Within the frames of the Standard, there were formulated the guidelines concerning the methods for quantification of food losses and food waste, including their direct measurement, composition analysis, calculation of mass balance and survey studies¹⁶.

According to FLW Standard methodology, the following items are excluded from the definition of waste: not harvested crops, food destined for animal feed, generation of bio-materials. Food waste include such food waste which will be utilized in aerobic (composting) or anaerobic (fermentation) processes, incineration, fertilization and storage, and also, the substances being discarded or disposed to sewer. The losses before harvesting are not included to the stream and weight of the waste. Similarly, composting of biomass in agricultural farm or its anaerobic fermentation for the needs of biogas production (agricultural biogas-producing plants) is not included – according to FLW methodology – to the stream of food waste. The destination and the way of disposal of a given waste play therefore a key role in its qualifying as food waste or not. It should be constantly monitored in order to preserve the representativity of the collected statistical data concerning the stream of the waste, losses and wastage. The mass of food packaging is not included to waste.

On a global scale (FAO), there are also being developed indicators aimed at facilitation of monitoring of the level of food losses and wastage, including food waste. The indicator, which is recommended to be used on the national level, is the index of food losses and waste *per capita*, covering the whole population of a given country; it is expressed in kg/person/year. It consists of two sub-indicators. The first one covers the losses resulting on the stage from agricultural farm to distribution points of the agri-food chain (Food Loss Index). The other one concerns the losses, generated on the stage from trade to households (Food Waste Index)¹⁷.

¹⁶ Food Loss and waste Accounting and Reporting Standard, Version 1.1, (2016), World Resources Institute.

¹⁷ State-of-play on the Global Food Loss Index to monitor SDG 12.3. 2017 https://ec.europa.eu/food/sites/files/safety/docs/fw_eu-platform_20171107_sub-fd_pres-03.pdf.

Methodology of the studies on food losses and wastage in the EU acc. to FUSIONS

On the level of the European Union, we should mention the FUSIONS Project, as implemented under the 7. Framework Programme of the European Commission in the years 2012 – 2016. It was a comprehensive research project dedicated to the development of the methodology for measurement and targeted counteracting food losses and wastage in the EU countries. In its methodological assumption, the mentioned FUSIONS Programme did not separate edible and inedible fractions of food but it covered the whole flow of the resources removed from the supply chain¹⁸. Redistribution, operation of transferring the food surpluses for charity purposes is treated equally as other targeted sites. FUSIONS considered the redistribution as a part of the food supply chain until the moment of its consumption. FUSIONS encouraged all entities, collecting the data on food wastage, to perform this activity in accordance to the FUSIONS guidelines in order to ensure the comparability of data, gained in all stages of the food supply chain in all EU-28 countries¹⁹.

According to FUSIONS, wastage includes food products which are still suitable for consumption but do not meet the specified criteria for becoming suitable for sale. Such food products are represented by seasonal articles, warehouse surpluses, food which is improperly labelled or was damaged during transport. The results of the project have demonstrated that there is no one universal method allowing obtaining the representative data, and for gaining their representativity, it is recommended to utilize few research methods in parallel²⁰.

Methodology of the studies and modelling with the application of LCA method

Life Cycle Assessment (LCA) is one of the methods and techniques of environment management, being recommended in many EU documents, *inter alia*, in *Directive 2008/98/EC of the European Parliament and of the Council of 19 November 2008 on waste* – in aspect of the choice of the methods and hierarchy of proceeding. LCA enables identification and assessment of potential environmental risks in the whole life cycle of the products and wastes across the supply chain. Its aim is to deliver the exhaustive information of the impact of a given waste or of a group of waste on the environment throughout the whole supply chain, so from the stage of production until the moment when a product become a waste and is subjected to the disposal processes, including recycling and utilization.

¹⁸ FUSIONS Definitional Framework, op.cot. p. 24.

¹⁹ Ibid., p. 9.

²⁰ FUSIONS. Report on review of (food) waste reporting methodology and practice. FUSIONS reducing food waste through social innovation, 2014.

LCA facilitates identification, quantification and hierarchization of organizational, logistic, technical and technological solutions from the viewpoint of their influence on the environment and developing the methods for its minimization.

The developed LCA methodology is defined in ISO standards (ISO 14001, 14041, 14044) and is aimed at increase of transparency in the application of LCA method and increase of the comparability between the studies of this type. The guidelines, as contained in the mentioned above standards are general and do not include detailed guidance concerning LCA use in the specified areas and economic sectors. LCA method, as being employed in the area of food waste management covers technical as well as biological processes. The features of the food waste differ from many other waste fractions as they are subjected to biological processes during the whole process of flow through single stages (links) of the supply chain. It has an impact on environment as well as on the potential ways of utilizing such type of the waste to minimize their level and by this, to counteract the losses and wastage. LCA covers such sectors as agricultural production, food economy, waste and sewage management and it requires a close definition of the particular borders of the system. For example, the studies limited to a company's door may omit certain important aspects (e.g. choice of package) which may, in turn, affect the generation of the losses in the successive stage of the agri-food chain [Bernstad, la Cour, Jansen 2012]. The available studies indicate the purposefulness of employing LCA method in the context of qualification of given categories of food losses and waste as being possible to be avoided or impossible to be avoided. Instructive experiences in this respect are available in British studies [Langley, Yoxall 2010].

Food losses occur in every stage of the supply chain. In global approach, as postulated by FAO, in edible parts of food are not treated as losses. In the context of LCA application, the mentioned parts may be recognized as agricultural or processing residues and may be disposed by the defined methods of waste processing (e.g. aerobic or anaerobic fermentation). The possibility of modelling food losses within the frames of LCA approach has a fundamental meaning for complex and detailed assessment of the burden on the environment connected with the food economy. It has a key importance in the case when the results of the studies are to be utilized in the determination of policies and initiatives aiming at reduction of the impact of agri-food system and the sustainable supply chains on the environment and climate [Corrado et al. 2017].

SUMMING UP AND RECOMMENDATIONS TO BE USED IN DEVELOPMENT OF METHODOLOGY OF THE STUDIES ON FOOD LOSSES AND FOOD WASTE

The conducted analyses confirm the thesis that there is no one universal method for gaining the quantitative data concerning food losses and wastage and the resulting food waste. The mentioned problems are of a complex nature and have their theoretical dimensions connected with the choice of the appropriate method of

gaining the quantitative data in the particular stages and links of the food chain. The discussed methods may be direct or indirect. From the performed review, it is followed that each method has a defined scope of applications what is indicated by the cited results of comprehensive research projects, undertaken on global (FAO) and European (FUSIONS Project) level. Life cycle analysis (LCA) is a complex research method, enabling gaining the data being appropriate for the process of implementation the principle of sustainable development and the resulting conception of circular economy. In the discussed model, food losses and wastage are eliminated at the source via optimization of management system in the food chain, including product management.

The key element of correct research methodology, facilitating gaining the data appropriate for the research needs is, therefore, an understandable and transparent system of definitions, concepts and basic descriptive, analytical and operational categories. The present paper, as implemented within the frames of PROM Project, will be used in the process of developing the adequate methodology of monitoring the losses and wastage for branches of food sector in the particular stages of the food supply chain.

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**IS THERE STILL ROOM FOR INCREASING SPEED
IN ALGORITHMIC AND HIGH-FREQUENCY TRADING?
THE CASE OF EUROPEAN OPTIONS PRICED
IN THE HESTON MODEL**

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Abstract: The purpose of the article was to investigate the possibility of increasing speed in transactions made within algorithmic and high-frequency trading. The analysis carried out for this purpose concerned the European options priced in the Heston model. Among issues being discussed, a new scheme of calculating Fourier and inverse Fourier transforms was proposed. It guarantees an increase of computational speed in relation to existing methods of generating final results.

Keywords: European options, the Heston model, Fourier transform

JEL classification: C02, G13

INTRODUCTION

In the financial literature algorithmic trading is variously defined. According to P. Teleaver, M. Galas and V. Lalchand [Teleaven et al. 2013] this term should be understood as any form of exchange of capital assets using advanced algorithms (computer systems) in order to automate entire or part of the transaction process. A. Cartea and S. Jaimungal [Cartea, Jaimungal 2013] are of a similar opinion. According to them algorithmic trading refers to the use of computer algorithms that make trading decisions, submit orders and manage those orders. In a slightly different way algorithmic trading is defined by A. P. Chaboud, B. Chiquoine, E. Hjalmarsson and C. Vega [Chaboud et al. 2014]. In their opinion this term is inextricably linked to the monitoring of markets, management and exchange of financial assets that are traded highly frequently. Such approach is in line with

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views of M. A. Goldstein, P. Kumar and F. C. Graves [Goldstein et al. 2014], who directly associate algorithmic trading with submitting, executing and cancelling buy or sell orders, analyzing financial data to identify short-term price opportunities, and trading via computers. In distinguishing between algorithmic and high-frequency trading helps the U.S. Securities and Exchange Commission (SEC). In one of the researches carried out by the SEC it was pointed out that despite the lack of a precise definition of high-frequency trading, algorithmic trading is a superior category in relation to high-frequency trading¹. In other studies performed by the SEC it is stated that high-frequency trading is a significant but not the only part of algorithmic and computer-aided trading². In this approach algorithmic trading covers wide range of activities including execution of orders on behalf of institutional clients and market makers by algorithms. Due to the fact that placed orders are of a large value they are placed on the market after dividing them into smaller suborders (or child orders) using appropriately designed queuing systems. For the purpose of this article, however, it is assumed that algorithmic trading is a trading system based on precise, previously prepared computer instructions in the process of capital assets' exchange [Yadav 2015].

The aim of the article is to show that there is still room for increasing speed in algorithmic and high-frequency trading. The article is organized as follows. In the first section development of algorithmic and high-frequency trading is presented. In the second section, the Heston model is briefly described. In the third section, some new approaches to the valuation of options in the Heston framework are proposed and an analysis of the speed of pricing European options is performed. Finally, in fourth section, the article is summarized and major conclusions are drawn.

DEVELOPMENT OF ALGORITHMIC AND HIGH-FREQUENCY TRADING

Algorithmic and high-frequency trading changed the functioning of contemporary capital markets. First of all, the automation of trades on stock exchanges significantly transformed the investment process. The involvement of financial resources in the transactions on the stock market to a lesser extent began to be related to the search for underestimated or overvalued securities and the selection of the appropriate moment of their purchase or sale. Greater weight began to be attributed to the continuous opening and closing positions at very short time intervals with the intention of generating short-term arbitrage profits. Secondly, some of the responsibilities related to making investment decisions were passed to

¹ <http://www.sec.gov/rules/concept/2010/34-61358.pdf> [access: 10.02.2016].

² https://www.sec.gov/marketstructure/research/hft_lit_review_march_2014.pdf [access: 15.02.2016].

the algorithms, in particular in areas related to the analysis of data and the selection of information that is most important for future changes in the prices of financial instruments. What's more, algorithms responsible for the purchase and sale began to take into account the impact of orders placed by other market participants, as well as the consequences of the use of algorithms by the buyers and sellers for the valuation of individual assets. Thirdly, investing financial resources based on mathematical instructions reduced the role of the human factor in the investment process to the development of financial models and the programming of computers.

The automation of stock trading, the use of algorithms to open and close positions, and the reduction of importance of the human factor in the investment process transformed algorithmic and high-frequency trading from a niche investment strategy into a dominant form of trade on many capital markets. This seems to be confirmed by M. A. Goldstein, P. Kumar and F. C. Graves [Goldstein et al. 2014]. According to them in the period of 2000-2012 the share of high-frequency transactions on the U.S. stock market increased from less than 10% to over 50%. On other segments of U.S. financial market a similar phenomenon was observed. The volume of high-frequency trading on the U.S. option and currency markets at the end of 2012 fluctuated from 40% to 60% of the total volume. According to N. Popper [Popper 2012] a similar tendency was observed in the most developed countries of the European Union, Japan and the remaining part of Asia. In their case, in 2012, high-frequency trading was responsible for 45%, 40% and 12% of stock trading respectively.

Development of algorithmic and high-frequency trading increased the speed of trade and shortened the time of holding open positions in financial instruments. As noted by A. G. Haldane [Haldane 2016], within 15 years starting from 2000, the average length of time for stock ownership was reduced from a few seconds to a few milli- or even microseconds. Moreover, as was noted by M. Narang [Narang 2016], the competition between traders on the financial market decreased the profit margin on the U.S. equity market to the value of one-tenth of a cent per share.

According to N. Popper [Popper 2016], in recent years, the role of algorithmic and high-frequency trading on the U.S. capital market has been gradually decreasing. In the period of 2009-2012, the annual number of shares bought or sold in computer-assisted transactions decreased from around 6 billion to around 3 billion. This meant a decrease of the share of algorithmic and high-frequency trading in the total volume from 61% to 51%. In the analyzed period, this phenomenon was accompanied by the decrease in profits of companies using these forms of trading from 4.9 billion USD in 2009 to 1.25 billion USD in 2012. It is worth noting that one of the reasons for the drop in profitability of algorithmic and high-frequency trading was the growing costs of both the maintenance of the ICT infrastructure and the acquisition and processing of market data. On the other side, in the period of 2009-2012, in the U.S., relatively stable increase in stock indices was observed. It was also a period of low market interest rates. Such circumstances could not ensure adequate profitability of momentum and reversal

strategies. In addition, as noted by L. Cardella, J. Hao, I. Kalcheva and Y. Ma [Cardella et al. 2014], the development of algorithmic and high-frequency trading and their profitability should be considered in a broader sense, i.e. they should include other financial instruments, and market segments. It was partly confirmed by N. Popper [Popper 2016] who observed an increase of the high-frequency trading in the total volume on the currency market (from 12% in 2009 to 28% in 2012).

Higher frequency of stock exchange transactions, increase in the speed of trade and shortening the time of holding open positions as well as popularization of algorithmic and high-frequency trading influenced the level of both informational and operational efficiency of the financial market.

In the next part of the article construction of the Heston model is briefly discussed and then alternative Fourier transform schemes are presented.

THE HESTON MODEL

There are two processes underlying the Heston model, i.e.:

$$dS_t = \mu S_t dt + \sqrt{\sigma_t^2} S_t dW_{1,t}, \quad (1)$$

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + v\sqrt{\sigma_t^2}dW_{2,t}, \quad (2)$$

where: S_t is the price of the underlying asset in period t , μ is the (constant) drift, σ_t^2 is the variance of the rates of return on the underlying asset, κ is the mean-reverting speed, θ is the average long-term variance of the rate of return on underlying asset (long-run mean), v is the volatility of volatility, and $W_{1,t}$, $W_{2,t}$ are two correlated Wiener processes such that $E(W_{1,t}W_{2,t}) = \rho$.

From eq. 2 it can be easily concluded that the main difference between the Black-Scholes [Black, Scholes 1973] and the Heston models [Heston 1993] refers to the variance of the rate of return on the underlying asset. In the Black-Scholes model volatility is fixed over time, while in the Heston model it is stochastic and given by the CIR process.

Pricing European options in the Heston model is based on the martingale approach to determining theoretical values of the contracts, i.e.:

$$\begin{aligned} C(S, \sigma^2, t) &= e^{-r(T-t)} E^{\mathbb{Q}^H}((S_T - K)^+) = \\ &= e^{-r\tau} E^{\mathbb{Q}^H}(S_T 1_{S_T > K}) - e^{-r\tau} K E^{\mathbb{Q}^H}(1_{S_T > K}) = \\ &= e^{x_t} P_1(x, \sigma^2, \tau) - e^{-r\tau} K P_2(x, \sigma^2, \tau), \end{aligned} \quad (3)$$

where: $\tau = T - t$, r is the risk-free rate, K is the exercise price, \mathbb{Q}^H is a martingale measure in the Heston model, $P_1(x, \sigma^2, \tau)$, $P_2(x, \sigma^2, \tau)$ are probabilities of expiring options in-the-money and the remaining notation is consistent with previously introduced.

According to S. L. Heston [Heston 1993], analytical formulas for $P_1(x, \sigma^2, \tau)$ and $P_2(x, \sigma^2, \tau)$ are not known. However, it can be concluded that the knowledge of the characteristic functions $\phi_1(\xi, x, \sigma^2)$ and $\phi_2(\xi, x, \sigma^2)$ corresponding to P_j , for $j = 1, 2$, allows to calculate $P_1(x, \sigma^2, \tau)$ and $P_2(x, \sigma^2, \tau)$. For this purpose, it is enough to calculate the inverse Fourier transforms according to the following formula (the method is referred to as H-H):

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-\mathbb{I}\xi \ln K} \phi_j(\xi, x, \sigma^2)}{\mathbb{I}\xi} \right) d\xi, \quad (4)$$

where: $\Re(\cdot)$ is the real part of the subintegral function, \mathbb{I} is the imaginary part of the complex number and the remaining notation is consistent with previously introduced.

In order to find the formula for the price of the European option it is necessary to introduce assumption concerning general form of the characteristic function corresponding to the probabilities P_j , for $j = 1, 2$, i.e.:

$$\phi_j(\xi, x, \sigma) = e^{C_j(\xi, \tau) + D_j(\xi, \tau)\sigma^2 + \mathbb{I}\xi x}, \quad (5)$$

where:

$$C_j(\xi, \tau) = r\mathbb{I}\xi\tau + \frac{a}{v^2} \left[(b_j - v\rho\mathbb{I}\xi + d_j)\tau - 2\ln \left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j} \right) \right], \quad (6)$$

$$D_j(\xi, \tau) = \frac{b_j - v\rho\mathbb{I}\xi + d_j}{v^2} \left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \right), \quad (7)$$

$$u_1 = \frac{1}{2}, \quad (8)$$

$$u_2 = -\frac{1}{2}, \quad (9)$$

$$a = \kappa\theta, \quad (10)$$

$$b_1 = \kappa + \lambda - v\rho, \quad (11)$$

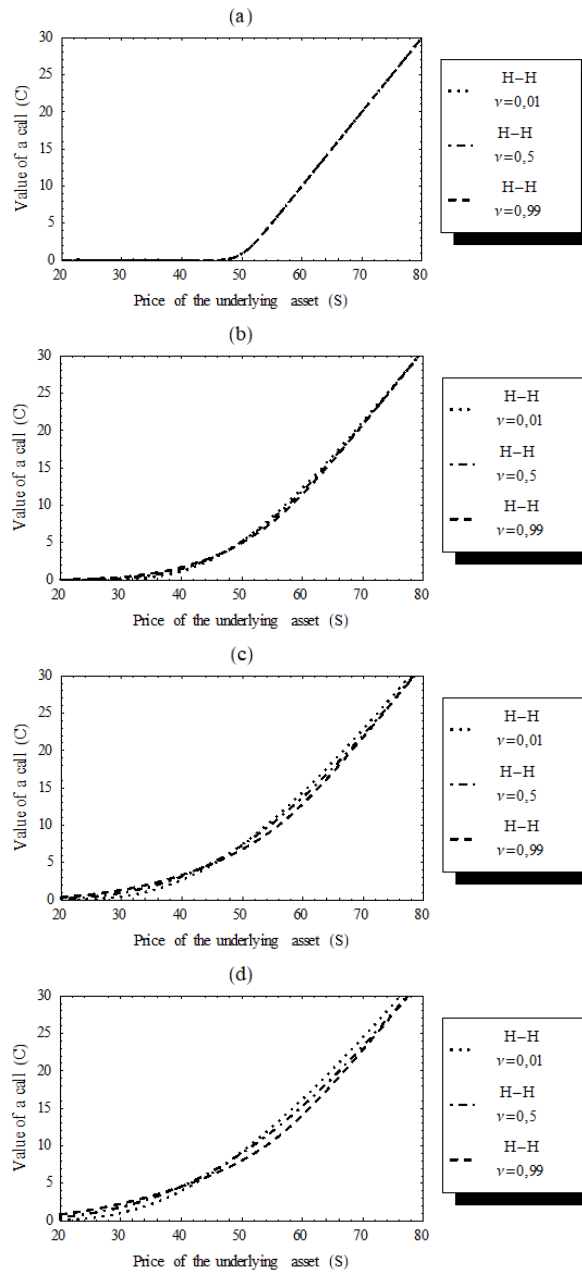
$$b_2 = \kappa + \lambda, \quad (12)$$

$$g_j = \frac{b_j - v\rho\mathbb{I}\xi + d_j}{b_j - v\rho\mathbb{I}\xi - d_j}, \quad (13)$$

$$d_j = \sqrt{(v\rho\mathbb{I}\xi - b_j)^2 - v^2(2u_j\mathbb{I}\xi - \xi^2)}. \quad (14)$$

The payoff functions of the analyzed derivatives are presented in Figure 1. The figure is prepared assuming that: $S \in [20, 80]$, $K = 50$, $\sigma^2 = 0.2$, $r = 5\%$, $\frac{T-t}{T} \in \{0.01, 0.3, 0.6, 0.9\}$, $\kappa = 0.05$, $\lambda = 0.08$, $\theta = 0.15$, and $\rho = 0.8$.

Figure 1. Payoff functions of the European call in the Heston model assuming that: $\kappa = 0.05$, $\lambda = 0.08$, $\theta = 0.15$, $\rho = 0.8$ and v takes various values for: (a) $\frac{T-t}{T} = 0.01$, (b) $\frac{T-t}{T} = 0.3$, (c) $\frac{T-t}{T} = 0.6$ i (d) $\frac{T-t}{T} = 0.9$



Source: own elaboration

SCHEMES OF THE FOURIER TRANSFORM

There are many methods of calculating the Fourier and inverse Fourier transforms. In consequence, there are many ways of determining value of the European options in the Heston model. In the article special attention in this matter will be directed to the formulas developed by P. Carr and D. Madan [Carr, Madan 1999], G. Bakshi and D. Madan [Bakshi, Madan 2000], M. Attari [Attari 2004], and A. Orzechowski [Orzechowski 2018], i.e.:

1. P. Carr and D. Madan [Carr, Madan 1999] for $\alpha = 1$ (the method is referred to as H-CM):

$$C(S_t, t) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \Re \left(e^{-\mathbb{I}\xi k} \frac{e^{-r\tau} \phi_2(\xi - (\alpha+1)\mathbb{I}, x, \sigma^2)}{\alpha^2 + \alpha - \xi^2 + \mathbb{I}(2\alpha+1)\xi} \right) d\xi. \quad (15)$$

2. P. Carr and D. Madan [Carr, Madan 1999] for $\alpha = 1$ (the method is referred to as H-CMTV):

$$C(S_t, t) = \frac{1}{\sinh(\alpha k)} \frac{1}{\pi} \int_0^\infty \Re \left(e^{-\mathbb{I}\xi k} \frac{\zeta(\xi - \mathbb{I}\alpha, x, \sigma^2) + \zeta(\xi + \mathbb{I}\alpha, x, \sigma^2)}{2} \right) d\xi, \quad (16)$$

$$\text{where: } \zeta(\xi, x, \sigma^2) = e^{-r\tau} \left(\frac{1}{1 + \mathbb{I}\xi} - \frac{\phi_2(-\mathbb{I}, x, \sigma^2)}{\mathbb{I}\xi} + \frac{\phi_2(\xi - \mathbb{I}, x, \sigma^2)}{\mathbb{I}\xi(1 + \mathbb{I}\xi)} \right).$$

3. G. Bakshi and D. Madan [Bakshi, Madan 2000] (the method is referred to as H-BM):

$$C(S_t, t) = \frac{1}{2}(S_t - Ke^{-r\tau}) + \frac{S_t}{\pi} \int_0^\infty \Re \left(\frac{e^{-\mathbb{I}\xi k} \phi_1(\xi, x, \sigma^2)}{\mathbb{I}\xi} \right) d\xi + \\ -Ke^{-r\tau} \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-\mathbb{I}\xi k} \phi_2(\xi, x, \sigma^2)}{\mathbb{I}\xi} \right) d\xi. \quad (17)$$

4. M. Attari [Attari 2004] (the method is referred to as H-A):

$$C(S_t, t) = S_t \left(1 + \frac{e^l}{\pi} \int_0^\infty \Re \left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}(\xi + \mathbb{I})} \phi_3(\xi, x, \sigma^2) \right) d\xi \right) + \\ -e^{-r\tau} K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}\xi} \phi_3(\xi, x, \sigma^2) \right) d\xi \right), \quad (18)$$

$$\text{where: } \phi_3(\xi, x, v) = e^{C_2(\xi, \tau) + D_2(\xi, \tau)\sigma^2 + \mathbb{I}\xi r\tau}, \text{ and } l = \frac{K}{S_t e^{r\tau}}.$$

5. A. Orzechowski [Orzechowski 2018] (the method is referred to as H-Au1):

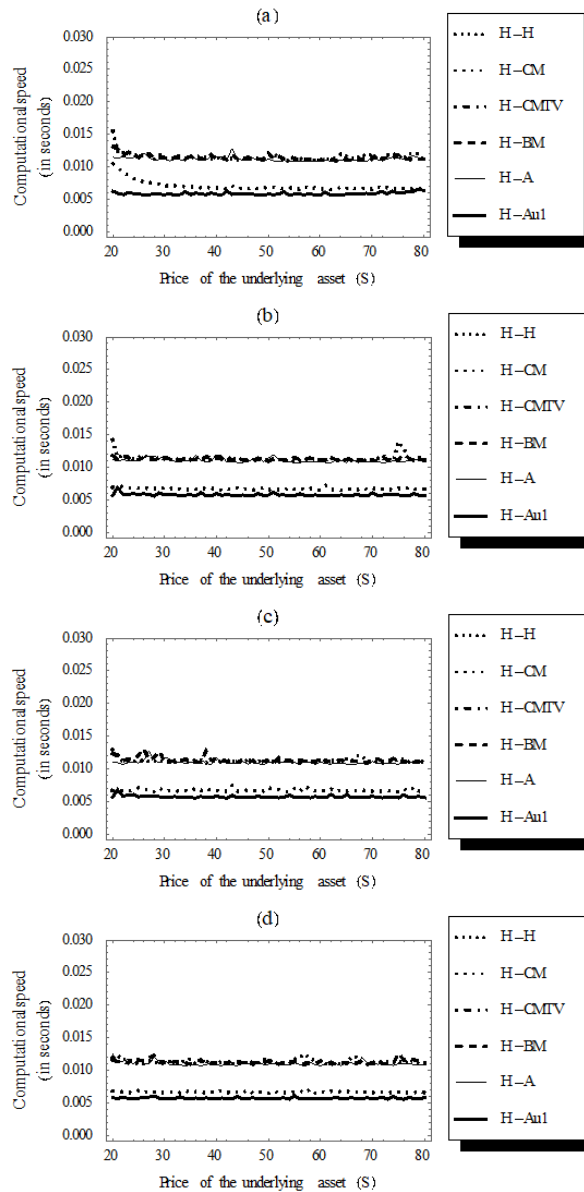
$$C(S_t, t) = \frac{1}{2}S_t - e^{-r\tau} \frac{1}{\pi} \int_0^\infty \Re \left(e^{-\mathbb{I}\xi k} \frac{\phi_2(\xi - \mathbb{I}, x, \sigma^2)}{\mathbb{I}\xi(\mathbb{I}\xi + 1)} \right) d\xi. \quad (19)$$

It is worth noting that above mentioned schemes can be applied to pricing options in diffusion, jump-diffusion as well as pure jump models.

In order to investigate which of the methods described above is the best in terms of computational speed, appropriate codes are developed in the Mathematica 10.2. The package being used is run on a computer with Intel i5-4210U CPU @ 1.70 GHz processor with RAM memory of 6 GB. It is important that each time, before the codes are started, cache memory is deleted. It is done in

order to force the written blocks of commands to be restarted by the computer. The results of the tests carried out are expressed in the graphical form - see Figure 2.

Figure 2. Computational speed in the Heston model assuming that: $\kappa = 0.05$, $\lambda = 0.08$, $\theta = 0.15$, $\rho = 0.8$ and $v = 0.2$ for: (a) $\frac{T-t}{T} = 0.01$, (b) $\frac{T-t}{T} = 0.3$, (c) $\frac{T-t}{T} = 0.6$ i (d) $\frac{T-t}{T} = 0.9$



Source: own elaboration

Based on obtained results it can be easily seen that the slowest methods of pricing European options in the Heston model are H-H, H-CMTV, H-BM and H-A. It should be noted that in the case of H-CMTV the number of bisections in the implemented numerical scheme is reduced in relation to other methods. The H-CM, and H-Au1 methods are among the fastest methods of determining theoretical values of the European options (the second one is slightly faster) regardless of the the time remaining to expiration and the moneyness of the contracts. It is impossible to disregard the fact that the fastest methods of pricing European options have two common features, i.e.: one characteristic function of the variable S_t in the formula for the price of the option and the value ξ in the denominator of the subintegral function squared.

SUMMARY


The main purpose of this article was to show that the pricing of the European options can be speeded up. Information of how to do it can be important for the possibility of developing high-frequency and algorithmic trading strategies. What is especially important is that the increase in the computational speed of the pricing of the European options is obtained not via technological advances in the computer hardware, information processing or telecommunications but by developing new method of calculating the Fourier and inverse Fourier transforms. It is also worth noting that the results can be used to develop numerical schemes based of the Fourier transform, i.e. the discrete and fast Fourier transforms.

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PRICING EUROPEAN OPTIONS IN THE VARIANCE GAMMA MODEL

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Abstract: The purpose of the article was to investigate if it is possible to speed up the process of pricing European options in the variance gamma setting. The analysis carried out for this purpose refers to the choice of the Fourier transform scheme, which allows to obtain accurately and fast the final result (theoretical value of the European option). The issues being discussed that refer to other methods of pricing options via Fourier transform are also briefly discussed.

Keywords: European options, the variance gamma model, Fourier transform

JEL classification: C02, G13

INTRODUCTION

The world's most well-known options pricing model is the one discovered by F. Black and M. Scholes [Black, Scholes 1973]. The model is based on the assumption of existence of two types of financial assets. The first one is riskless, which means that its purchase guarantees achievement of predetermined benefits. With this in mind, it can be stated that the modeling of the price path of such an instrument (in practice it is a government bond) is possible with the use of a first order ordinary differential equation. The second type of assets are stocks which prices follow geometric Brownian motion. In consequence, their prices are assumed to be unpredictable (upward and downward movements in value of the securities are described by a stochastic differential equation).

As part of the proposed theory, the following assumptions are introduced:

- short-term interest rate is known and does not change over time,
- shares' prices follow random walk,

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- variance of the returns on risky assets is constant,
- conditional distribution of the market prices of risky assets is lognormal,
- stocks do not generate dividends,
- only European options are subject to analysis,
- stocks are perfectly divisible,
- short sale of both risky and risk-free assets is not restricted,
- money can be borrowed and invested at the short-term risk-free interest rate.

In order to better explain changes in the value of options, it is often assumed that the process responsible for movements of the underlying asset's prices is discontinuous. In the literature [Tankov, Voltchkova 2009], extension of the diffusion process by a jump component is explained in many ways. According to the approach of F. Black and M. Scholes [Black, Scholes 1973], the probability of achieving large profits or incurring significant losses in short time intervals is much lower than it results from the analysis of empirical data. In consequence, on the basis of historical observations, it can be concluded that out-of-the-money contracts which are close to expiration remain underestimated. Moreover, the diffusion model of option pricing is based on the concept of perfect hedging. In reality, however, perfect hedging is implausible, and one of the reasons may be the occurrence of jumps in the quotations of financial instruments. Among existing arguments supporting the inclusion of discontinuities into the models of option pricing are empirical observations confirming the occurrence of such phenomena on various segments of the financial market.

Over time, jump-diffusion models were transformed into pure jump models in which diffusion component disappears and only the one referring to jumps remains. In consequence, nowadays, all models of option pricing with discontinuities can be divided into two groups: (1) finite activity models in which continuous changes in prices of the underlying assets are occasionally disturbed by jumps [Merton 1976, Kou 2002] and (2) infinite activity models in which only discrete changes (in values of the underlying asset) in a finite time interval appear [Barndorff et al. 1991, Carr et al. 2002, Eberlein et al. 1998, Madan et al. 1998, Madan et al. 1991]. Such classification is partly confirmed by W. Schoutens [2003].

The aim of the article is to show that properly chosen Fourier transform scheme can increase efficiency of the European option pricing in the variance gamma model. The article is organized as follows. In the first section definitions of the characteristic function and the Fourier transform are given. Moreover, characteristic functions of two pure jump models are presented. In the second section, the application of the variance gamma model to the valuation of the European options is briefly discussed. In the third section, efficiency of the valuation of the contracts in the variance gamma framework is analyzed. Finally, in fourth section, the article is summarized and major conclusions are drawn.

THE FOURIER TRANSFORM AND CHARACTERISTIC FUNCTION

If $f(x)$ is a piecewise continuous real-valued function defined over the domain of real numbers which satisfies the following condition, then the Fourier transform of $f(x)$ is defined by:

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} e^{\mathbb{I}\xi x} f(x) dx, \quad (1)$$

where: \mathbb{I} is the imaginary part of the complex number and $\xi \in \mathbb{R}$.

If X is a random variable having the density function $g(x)$, then the characteristic function of X is defines by:

$$\phi_X(\xi) = \int_{-\infty}^{\infty} e^{\mathbb{I}\xi x} g(x) dx, \quad (2)$$

where the notation is consistent with previously introduced.

If G_t is a gamma process with parameters $\alpha = \beta = \frac{1}{\nu}$, where ν is the volatility of the time change, then X_t is the variance gamma process with infinite number of jumps in the finite time interval, i.e.:

$$X_t = \theta G_t + \sigma B_{G_t}, \quad (3)$$

where θ is a skewness parameter, σ is a variance, B_{G_t} is a subordinated Brownian motion and the remaining notation is consistent with previously introduced.

The distribution of the process is negatively and positively skewed when $\theta < 0$ and $\theta > 0$ respectively. Moreover, it is infinitely divisible and has stationary and independent increments.

The characteristic function of X is given by:

$$\phi_X(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{1}{\nu}} \quad (4)$$

and the characteristic function of X_t is expressed by the formula:

$$\phi_{X_t}(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{t}{\nu}}, \quad (5)$$

where the notation is consistent with previously introduced.

If X_t is a normal inverse Gaussian process with parametrs $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$, then it can be treated as an inverse Gaussian subordinated Brownian motion with a drift. If I_t is an inverse Gaussian process with parameters $\alpha = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, then X_t can be expressed by the following equation

$$X_t = \beta\delta^2 I_t + \delta B_{I_t}. \quad (6)$$

The characteristic function of X is given by:

$$\phi_X(\xi) = e^{-\delta(\sqrt{\alpha^2 - (\beta + \mathbb{I}\xi)^2} - \sqrt{\alpha^2 - \beta^2})} \quad (7)$$

and the characteristic function of X_t is expressed by the formula:

$$\phi_{X_t}(\xi) = e^{-\delta t(\sqrt{\alpha^2 - (\beta + \mathbb{I}\xi)^2} - \sqrt{\alpha^2 - \beta^2})}, \quad (8)$$

where the notation is consistent with previously introduced.

Although normal inverse Gaussian processes can be applied to pricing option they are left for later analysis. In the remaining part of the article only variance gamma processes are of special interest.

PRICING EUROPEAN OPTIONS VIA FOURIER TRANSFORM

If the general form of the characteristic function of a variable identified with log-price of the underlying asset is known, then the price of the European option can be easily determined. As there are many approaches to the calculation of the Fourier and inverse Fourier transforms final formulas for the theoretical price of the European option can take different forms. Assuming that T is the moment of option's expiration and t is the moment of option's pricing characteristic function of the natural logarithm of the spot price of the underlying asset is given by:

$$\phi(\xi) = \left(1 - \mathbb{I}\theta v\xi + \frac{1}{2}\sigma^2 v\xi^2\right)^{-\frac{(T-t)}{v}} e^{\mathbb{I}\xi\left(s_t + (T-t)\left(r + \frac{1}{v}\ln\left(1 - \theta v - \frac{1}{2}\sigma^2 v\right)\right)\right)}, \quad (9)$$

where: $s_t = \ln(S_t)$ and the remaining notation is consistent with previously introduced.

Assuming k is the natural logarithm the exercise price K , S_0 is the price of the underlying asset at time $t = 0$ and r is the risk-free rate of return, the formulas for the price of the European call in the methods of P. Carr and D. Madan [Carr, Madan 1999], G. Bakshi and D. Madan [Bakshi, Madan 2000], M. Attari [Attari 2004], and A. Orzechowski [Orzechowski 2018] are the following:

1. P. Carr and D. Madan [1999] for $\alpha = 1$ (the method is referred to as VG-CM):

$$C(S_0, 0) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{e^{-rT} \phi(\xi - (\alpha+1)\mathbb{I})}{\alpha^2 + \alpha - \xi^2 + \mathbb{I}(2\alpha+1)\xi}\right) d\xi. \quad (10)$$

2. P. Carr and D. Madan [1999] for $\alpha = 1$ (the method is referred to as VG-CMTV):

$$C(S_0, 0) = \frac{1}{\sinh(\alpha k)} \frac{1}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{\zeta(\xi - \mathbb{I}\alpha) + \zeta(\xi + \mathbb{I}\alpha)}{2}\right) d\xi, \quad (11)$$

where: $\zeta(\xi) = e^{-rT} \left(\frac{1}{1 + \mathbb{I}\xi} - \frac{\phi(-\mathbb{I})}{\mathbb{I}\xi} + \frac{\phi(\xi - \mathbb{I})}{\mathbb{I}\xi(1 + \mathbb{I}\xi)}\right)$.

3. G. Bakshi and D. Madan [2000] (the method is referred to as VG-BM):

$$C(S_0, 0) = \frac{1}{2}(S_0 - Ke^{-rT}) + \frac{S_0}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k} \phi(\xi - \mathbb{I})}{\mathbb{I}\xi \phi(-\mathbb{I})}\right) d\xi + \\ - Ke^{-rT} \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k} \phi(\xi)}{\mathbb{I}\xi}\right) d\xi. \quad (12)$$

4. M. Attari [2004] (the method is referred to as VG-A):

$$C(S_0, 0) = S_0 \left(1 + \frac{e^l}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}(\xi + \mathbb{I})} \phi_1(\xi)\right) d\xi\right) + \\ - e^{-rT} K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi l}}{\mathbb{I}\xi} \phi_1(\xi)\right) d\xi\right), \quad (13)$$

where: $\phi_1(\xi) = \left(1 - \mathbb{I}\theta\nu\xi + \frac{1}{2}\sigma^2\nu\xi^2\right)^{-\frac{t}{\nu}} e^{\mathbb{I}\xi\left(t\left(\frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)\right)\right)}$, and $l = \frac{K}{S_t e^{rT}}$.

5. D. S. Bates [2006] (the method is referred to as VG-B):

$$C(S_0, 0) = S_0 - e^{-rT}K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-\mathbb{I}\xi k}}{\mathbb{I}\xi(1-\mathbb{I}\xi)}\right) \phi(\xi) d\xi\right). \tag{14}$$

6. A. Lewis [2001] and A. Lipton [2002] (the method is referred to as VG-LL):

$$C(S_0, 0) = S_0 - \frac{\sqrt{S_0 K} e^{-\frac{rT}{2}}}{\pi} \int_0^\infty \Re\left(\phi\left(\xi - \frac{\mathbb{I}}{2}\right) \frac{e^{\mathbb{I}u(s_0 - k + rT)}}{\xi^2 + \frac{1}{4}}\right) d\xi. \tag{15}$$

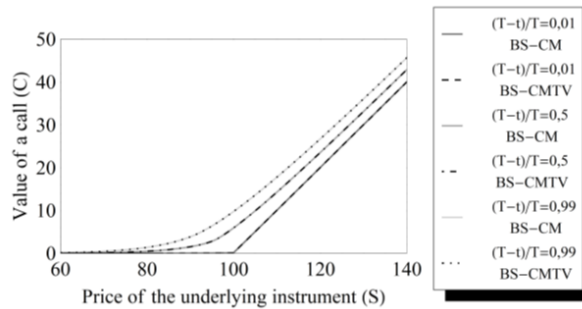
7. A. Orzechowski [2018] (the method is referred to as VG-Au):

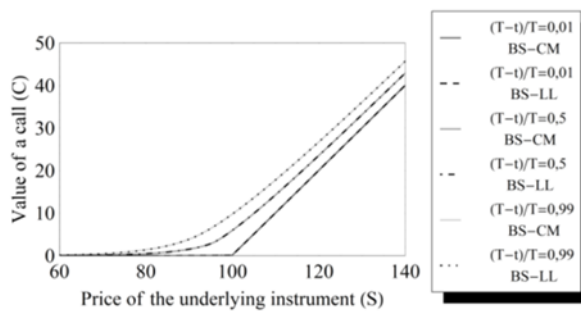
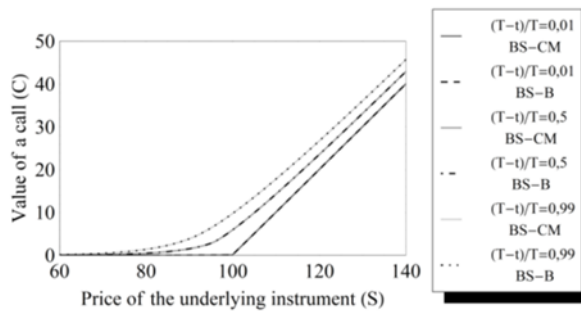
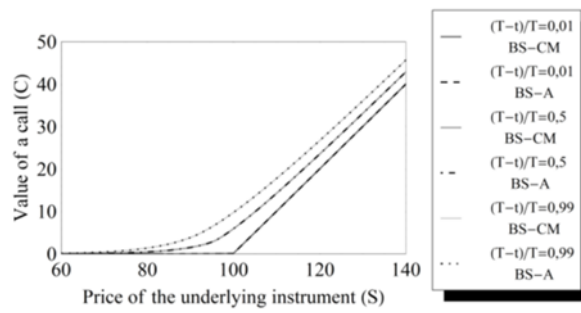
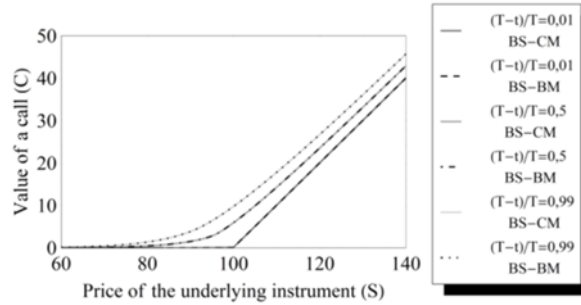
$$C(S_0, 0) = \frac{1}{2}S_0 - e^{-rT} \frac{1}{\pi} \int_0^\infty \Re\left(e^{-\mathbb{I}\xi k} \frac{\phi(\xi - \mathbb{I})}{\mathbb{I}\xi(\mathbb{I}\xi + 1)}\right) d\xi. \tag{16}$$

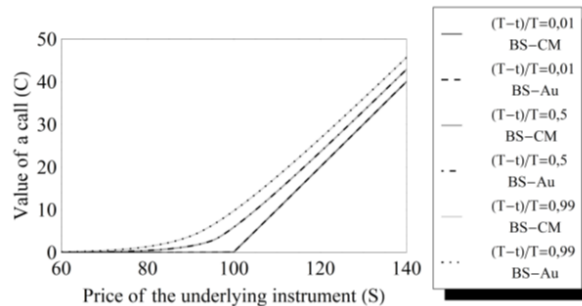
As no analytical methods of the European options' pricing were included in the article computational speed and accuracy are investigated with respect to the VG-CM method.

Computational accuracy is analyzed by comparing deviations of the theoretical prices of the European calls in the VG-CMTV, VG-BM, VG-A, VG-B, VG-LL, VG-Au methods from the theoretical values of the contracts in the VG-CMTV method. In every case it is assumed that: $K = 100$, $r = 5\%$, $\sigma = 20\%$, $\theta = -0.06$, $\nu = 1.44$ and $S \in [60, 140]$. Obtained results are presented in Figure 1.

Figure 1. Computational accuracy in the variance gamma model assuming that: $\theta = -0.06$, $\nu = 1.44$ in VG-CMTV, VG-BM, VG-A, VG-B, VG-LL, VG-Au methods comparing to VG-CMTV method





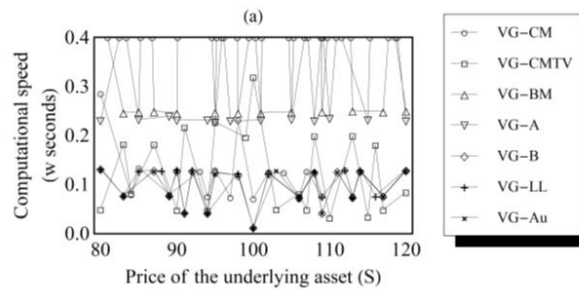


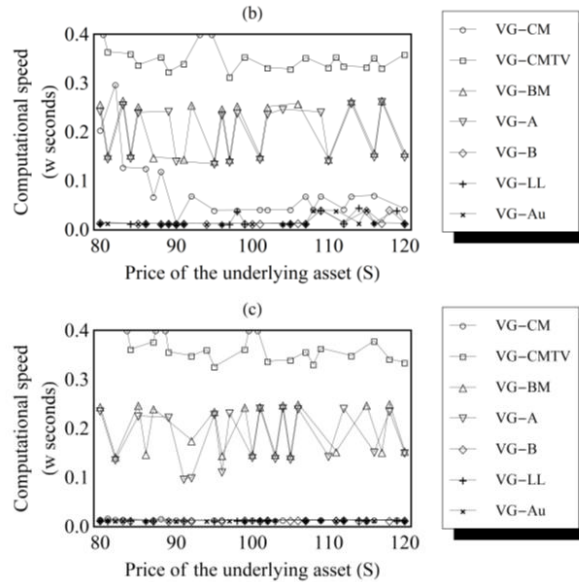
Source: own elaboration

From Figure 1 it can be easily concluded that all methods of pricing European options based on the Fourier transform are relatively well convergent to the VG-CM method. It means that all approaches to the valuation of options are comparable in terms of computational accuracy.

In order to investigate computational speed of the contracts' valuation, appropriate codes are developed in Mathematica 10.2. The package being used is run on a computer with Intel i5-4210U CPU @ 1.70 GHz processor with RAM memory of 6 GB. Each time, before the codes are started, cache memory is deleted. It is done in order to force the written blocks of commands to be restarted by the computer. The results of the tests carried out are expressed in the graphic form - see Figure 2.

Figure 2. Computational accuracy in the variance gamma model assuming that: $\theta = -0.06$, $\nu = 1.44$ for: (a) $\frac{T-t}{T} = 0.01$, (b) $\frac{T-t}{T} = 0.5$ i (c) $\frac{T-t}{T} = 0.99$





Source: own elaboration

Obtained results are ambiguous for the contracts which are close to expiration. In other cases, however, it can be easily concluded that the slowest methods of pricing European options in the variance gamma setting are: VG-CMTV, VG-BM and VG-A. A little faster is the VG-CM method and the fastest methods of pricing European options are: VG-B, VG-LL and VG-Au. In their cases parameter ξ in the denominator of the subintegral function is squared and there is only one characteristic function of the variable S_t in the subintegral function of the final formula for the price of the European call.

SUMMARY

In the finance industry more and more important is the speed of valuation of all instruments listed on the stock exchanges. This happens because developing computer technologies strongly affect the time of obtaining and processing market information. In consequence, investment strategies that are implemented by investors to a greater extent are focused on searching and immediately discounting every relevant piece of news appearing on the market. Sometimes the strategies are implemented to the high-frequency or algorithmic trading.


In the article, it was shown that the right choice of the Fourier transform scheme is needed to price the European options in variance gamma model. The scheme of the Fourier transform plays a key role in the effectiveness of the valuation of the analyzed contracts. Moreover, it was proved that the approach proposed by the author of the article, i.e. VG-Au, belongs to the group of methods

that are equally accurate as VG-CM, VG-CMTV, VG-BM, VG-A, VG-B, and VG-LL methods. At the same time, the method allows to speed up calculations comparing to some of the existing approaches to the valuation of the European options.

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ASSESSMENT OF DIGITAL EXCLUSION OF POLISH HOUSEHOLDS

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Abstract: In the article author attempts to assess the level of digital exclusion of Polish households in the years 2003-2015. This assessment was based on the author's synthetic digital divide index. For its creation it was used 40 variables describing 3 levels of digital exclusion: possession of ICT, use of ICT and digital competences. In the analyzed period, the phenomenon of digital divide systematically decreases, however the problem of digital exclusion is still significant in Polish society.

Keywords: ICT, digital divide, inequalities, digital divide index

JEL classification: O33

INTRODUCTION

In Poland, as a result of the development of the information society, inequalities in access to information and communication technologies appeared, as well as related problems. Earlier, other more developed countries also faced such difficulties [Bangemann 1994, Chmielarz 2007]. Inequalities in access to ICT are usually referred to as digital exclusion [van Dijk et al. 2003], which is defined as „the difference between persons, households, enterprises and geographic areas at different socioeconomic levels, both with regard to their opportunities for access to information and communication technologies, as well as their use on the Internet in a wide range of activities” [OECD 2001].

Van Dijk and Hacker [2003] distinguished three dimensions of digital exclusion: having ICT, using ICT and digital competences. The impact of individual dimensions on the scale of digital exclusion depending on the phase of development of the information society was different. In the first phase of

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information society development, the phenomenon of exclusion was to a large extent related to the aspect of ICT technology possession, in the second aspect to the aspect of using it, and in the third to digital competences.

In the 21st century, the problem of inequality is not only diminishing, but is gaining momentum. Lack of access to digital technology is now the same exclusion as slavery, lack of access to education or work before. One of the reasons for the increase in inequality is the development of information and communication technologies occurring at an unprecedented scale because it concerns both the sphere of enterprises, administration and households. However, this development is not evenly one of the main threats to sustainable economic development and social order [Karwińska 2004, DiMaggio et al. 2004, Budziński et al. 2009].

In the digital exclusion studies, two approaches are used: lenticular and holistic. In the lenticular approach, the phenomenon of digital exclusion is analyzed independently for each of the dimensions (e.g. the dimension: access, use, skills), which leads to difficulties in the overall assessment of the scale of digital exclusion. However, in the holistic approach, all dimensions are taken into account at the same time. The most often researched dimensions in the holistic approach are: infrastructure, accessibility and the use. In the case of some indicators, the following conditions are additionally taken into account: political, economic and socio-demographic. The most popular holistic measures include: DIDIX (Digital Divide Index), NRI (Network Readiness Index), IDI (ICT Development Index), and DDI (Digital Divide Index). The mentioned measures of digital exclusion are determined on the basis of regional data, which prevents their direct use to assess the level of digital exclusion of individuals in society. The second very important methodological problem of the above measures are constant weights for particular areas, which may lead to distorting the obtained results. Due to the above and due to the nature of the Social Diagnosis and Household Budget data which constituted the empirical material in the conducted research, it was not possible to apply any of the approximate holistic indicators directly.

The purpose of this article is to present a proposal for a synthetic digital exclusion index that will allow the assessment of digital exclusion in Polish households in the years 2003-2015.

METHODS

Due to the possibility of presenting the phenomenon of digital exclusion with one coherent and understandable value, the author decided to apply a holistic approach, to develop a synthetic digital exclusion index. 40 variables describing the three dimensions of digital exclusion were adopted for its construction: possession (3 variables), use (5 variables), and competences (32 variables). Then selected variables were used to create 19 features of digital exclusion, which were categorized in three groups representing these dimensions:

- P - possession of:
 - P1 - a computer,
 - P2 - an Internet connection,
 - P3 - a mobile phone;
- K - use:
 - K1 - computer,
 - K2 - Internet,
 - K3 - mobile phone,
 - K4 - the intensity of computer use in the last week (in hours),
 - K5 - the intensity of Internet use in the last week (in hours);
- C - digital competences (purpose):
 - C1 - basic computer skills (copying files, duplication of files),
 - C2 - intermediate computer skills (using spreadsheets, creating presentations, installing software),
 - C3 - advanced computer skills (using programming languages),
 - C4 - communication (mail, instant messaging, chat, forums, internet TV, social networks),
 - C5 - creating own content (running a blog, publishing own graphics or music on the Internet),
 - C6 - raising qualifications/work (training, looking for a job, for professional purposes),
 - C7 - purchases (auctions, online stores),
 - C8 - entertainment (games, listening, watching, booking, reading)
 - C9 - downloading data (music, films),
 - C10 - electronic banking,
 - C11 - e-administration (BIP, e-government).

Selected variables were neither quasi-constant (volatility factor above 10%) nor excessively correlated (correlation coefficient below 0.70). In addition to two quantitative features representing the intensity of computer use (K4) and the Internet use (K5), other features were expressed on an ordinal scale.

In order to enable to compare all the features with each other, the process of unitarisation (1) was carried out, on the basis of which for each of the features the weights were determined according to the formula (2):

$$z_{ij} = \frac{x_{ij} - \min_i \{x_{ij}\}}{\max_i \{x_{ij}\} - \min_i \{x_{ij}\}}, \quad (1)$$

where: x_{ij} - the value of the i -th feature for the j -th observation

$$w_i = \frac{V_i}{\sum_{i=1}^n V_i}, \quad (2)$$

where: $V_i = \frac{s(z)_i}{z_i}$, $s(z)_i$ - standard deviation for the i-th feature after the

normalization process, \bar{z}_i - the mean value for the i-th feature after the normalization process.

The obtained weights were used to determine for each adult the digital exclusion index calculated in accordance with the formula (3):

$$wwc_j = \sum_{i=1}^n w_i \cdot x_{ij}. \quad (3)$$

Depending on the level of digital exclusion, each adult person was assigned to one of four groups according to the division method used by Nowak (1990) and according to formulas (4) (5) (6) (7):

- GWC I – digitally excluded person: $[\min wwc, \overline{wwc} - s(wwc)]$, (4)

- GWC II – a person at risk of digital exclusion: $[\overline{wwc} - s(wwc), \overline{wwc}]$ (5)

- GWC III - a person partially using the latest ICT solutions: $[\overline{wwc}, \overline{wwc} + s(wwc)]$, (6)

- GWC IV - a person fully using the latest ICT solutions: $[\overline{wwc} + s(wwc), \max wwc]$. (7)

On the basis of the obtained results, the percentage shares of digital exclusion groups were determined in accordance with the formula (8):

$$y_j = \frac{x_i}{\sum_{i=1}^4 x_i}, \quad (8)$$

where: x_i - the number of people in the i-th group from the 4 digital exclusion groups.

The designated percentage shares created a table containing the profile of digital exclusion groups in subsequent years of the study. Then, the average values and dynamics of changes for the examined period were assessed. To determine the average value of the examined feature observed at different moments, the chronological average was calculated based on formula (9) [Witkowska 2001, Sobczyk 2007]:

$$\bar{y} = \frac{1}{n+1} \sum_{t=1}^n y_t, \quad (9)$$

where: y_t - the value of the tested feature in a year t ; n – number of years.

In addition to determining the average value of the tested feature observed at different times, its change of dynamics was also assessed. The most commonly used measures to assess changes are absolute increases, relative increments and dynamics measures (indexes). Due to the inability to use absolute increments to compare phenomena expressed in different units, chain indexes calculated according to formula (10) were used in the conducted research [Sobczyk 2007, Kukuła 2007]:

$$i_t = \frac{y_t}{y_{t-1}} \quad \text{for} \quad t = 1, 2, \dots, n. \quad (10)$$

Determined chain indexes of the examined feature were used to assess its Average Rate of Change calculated in accordance with formula (11). The obtained values of the average rate of change meant the average percentage by which the value of the examined feature changed on average every year during the analyzed period:

$$arc = \sqrt[n-1]{\prod_{t=1}^n i_t} - 1 = \sqrt[n-1]{\prod_{t=1}^n \frac{y_t}{y_{t-1}}} - 1, \quad (11)$$

where: $\min(i_t) \leq \text{arc} \leq \max(i_t)$.

The assessment of the degree of differentiation of digital exclusion was made on the basis of the Gini coefficient [Anand 1983] (12) and the classic coefficient of variation [Kukuła 2007] (13) in accordance with the formulas:

$$G = \frac{2}{n^2 \bar{x}} \sum_{i=1}^n p_i x_i - \frac{n+1}{n}, \quad (12)$$

where: n – number of persons, x_i - the value of the exclusion index for a given person, \bar{x} - the mean value of the digital exclusion index, p_i – the average position of the person in the rank in non-descending order according to the values x_i .

$$V = \frac{s(x)}{\bar{x}}, \quad (13)$$

where: $s(x)$ – standard deviation, \bar{x} - the mean value of the digital exclusion index.

In order to present the obtained results and identifying interdependencies, Gradual data analysis was used, which belongs to the methods of multidimensional data analysis [Borkowski et al. 2005]. A detailed description of this method of data exploration includes, among others: Szczesny [2002] and Kowalczyk, Pleszczyńska and Ruland [2004].

RESULTS

The use of the author's method of measuring digital exclusion allowed us to obtain the following results. For all people in general, digitally excluded persons were the dominant group (the average value of the exclusion level achieved 49.3%). The dominance of the above group was particularly observed in the first years of the study, in which the percentage of people excluded in the structure of digital exclusion was respectively: 69.0% in 2003 and 61.3% in 2005. In Poland, along with the development of the information society, the dominance of the group of people excluded has been gradually reduced (an average decrease of 5.0%) for the parallel increase in other groups of digital exclusion (average increase from 6.7% to 10.0%). In 2015, the excluded group remained the dominant group, however, the share in the digital exclusion structure of other groups was increasing. In 2015, people who fully and partially benefited from the latest ICT solutions constituted nearly 35% of the total (see Table 1).

Table 1. Digital exclusion profile for all people in total in 2003-2015

	2003	2005	2007	2009	2011	2013	2015	pwpw	arc
GWC I	69.0%	61.3%	58.9%	53.4%	49.8%	51.3%	50.6%	49.3%	-5.0%
GWC II	8.5%	10.3%	9.6%	12.0%	11.7%	12.6%	14.7%	9.9%	9.5%
GWC III	5.7%	7.8%	8.5%	9.3%	11.2%	10.3%	9.9%	7.8%	10.0%
GWC IV	16.8%	20.6%	23.0%	25.3%	27.3%	25.8%	24.8%	20.5%	6.7%

Explanations for the table: pwpw average value of the exclusion level, dc - average rate of change.

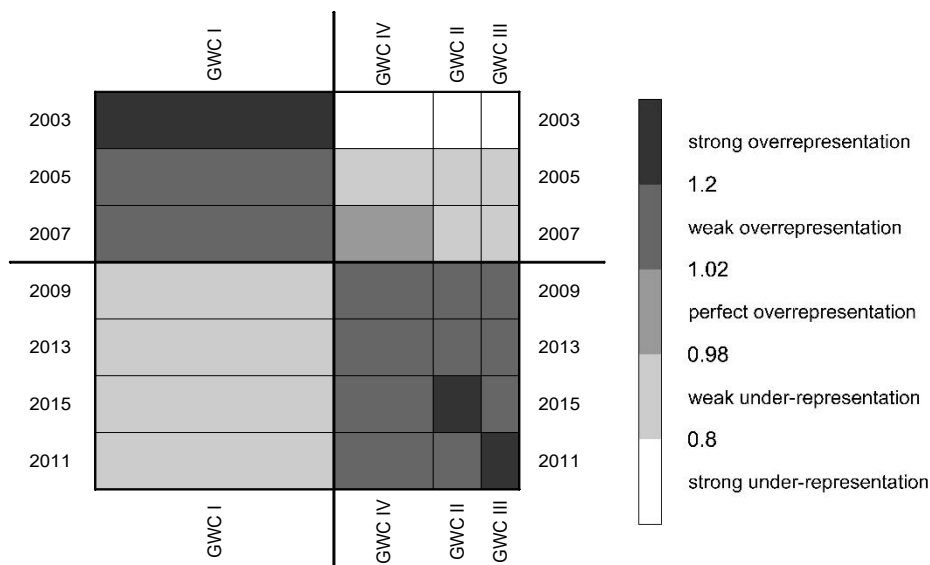
Source: own elaboration based on data Czapiński and Panek [2003-2015]

The conducted research also showed a very high diversity of digital exclusion, which in subsequent years was characterized by a downward trend. This phenomenon has been confirmed by assessment of both the Gini coefficient and the classical coefficient of variation. The Gini coefficient valued from 0.832 (in 2003) to 0.645 (in 2015), while the classic coefficient of variation from 1.591 (in 2003) to 0.907 in (2015). The explanation of such a large diversity was the domination of two antagonistic groups of digital exclusion in the society.

The above results regarding the profile and diversity are also confirmed by the overrepresentation map (see Figure 1). The largest share in the structure of digital exclusion was characterized by the first group of excluded people (GWC I, the boundary value $w. b. = 0.563$). There were successive groups of digital exclusion: the fourth (GWC IV, $w. b. = 0.233$), the second (GWC II, $w. b. = 0.114$) and the third (GWC III, $w. b. = 0.090$). In the first years of the study, a larger share, in relation to the average structure of digital exclusion, was the group of digitally excluded persons (GWC I), and the smaller share of groups (GWC II, GWC III, GWC IV) was characteristic. In the following years, this dependence was reversed. Research of the level of digital exclusion using graduated data analysis has also

shown some similarities between digital insertion groups. Two main categories were distinguished: the first - composed of excluded people; the second one - constructed from other remaining groups (GWC II, GWC III, GWC IV).

Figure 1. Map of overrepresentation of digital divide groups for total population in years 2003-2015



Source: own elaboration based on data Czapiński and Panek [2003-2015]

SUMMARY

The conducted research indicates the existence of a large diversity in the access and use of ICT. In the analyzed period of 2003-2015 the phenomenon of digital exclusion is systematically decreasing, however the problem of digital inequalities is still significant in Polish society. Although nearly 80% of households were equipped with a computer and connection to the Internet, the share of the group of people digitally excluded in 2015 still accounted for over 50%. On the other hand, the share of people fully using the latest ICT solutions was only 24.8%. The obtained results confirm that the scale of digital exclusion is increasingly influenced by the digital competence dimension, in turn with the increasingly smaller influence of ICT technology possession.

The current stratification of society and its increasingly important impact on the economy at the micro and macro scale causes the need for continuous monitoring and explanation of processes related to the phenomenon of digital exclusion. Further research based on the digital exclusion assessment method

proposed in this paper would allow for a better understanding of the existing connections and dependencies.

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ASYMMETRIC SQUARE ROOT OPTIONS – CAN WE PRICE THEM VIA THE FOURIER TRANSFORM?

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Abstract: The aim of this article is to investigate computational speed and convergence of asymmetric square root options' pricing in the F. Black and M. Scholes setting. The methodology of the conducted research is based on the comparison of pricing efficiency of the contracts with the use of BS, FT-BM and FT-B methods (including two different numerical schemes). Based on obtained results, it can be concluded that the BS method is better than methods based on the Fourier transform. However, it can be used only in the F. Black and M. Scholes setting. When analyzing other models, e.g. stochastic volatility models, one should use model based on the Fourier transform. Among those described in the article, the most efficient is FT-B with Clenshaw-Curtis numerical rule.

Keywords: asymmetric square root options, Fourier transform, Black-Scholes model

JEL classification: C02, G13

INTRODUCTION

Asymmetric square root options are exotic derivatives which payoff functions depend on the path followed by the difference between the square roots of the underlying asset's price and the exercise price at the moment of the contract's expiration (for the call option). Such instruments allow investors to implement conservative speculative or hedging strategies at a low cost - asymmetric square root options are cheaper than corresponding plain vanilla contracts or power options (assuming that the power to which the payoff function is raised is greater than one).

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There are many methods of calculating theoretical values of this kind of derivatives. All of them can be divided into three main groups:

1. methods based on partial integro-differential equations and numerical schemes dedicated to these equations [Cont, Voltchkova 2005],
2. Monte Carlo methods and other simulation techniques [Brandimarte 2014],
3. methods based on stochastic differential equations.

Taking into account the above mentioned classification, it can be easily seen that the further part of the considerations to the largest extent concerns the third group of methods. Only in their case, apart from the methods presented in the article, approaches referring to the discrete and the fast Fourier transforms [Carr, Madan 1999] as well as the Fourier series [Fang, Oosterlee 2008] can be distinguished.

The aim of this article is to investigate computational speed and accuracy of the valuation of asymmetric square root options in the F. Black and M. Scholes setting [Black, Scholes 1973]. As a part of the conducted research analytical formula for the theoretical values of the analyzed derivatives is determined. Then the Fourier transform theory is applied to the valuation of asymmetric square root options. Finally, the role of selected numerical schemes in the pricing of the analyzed contracts is discussed. In the summary, main conclusions are drawn and the best method of the valuation of asymmetric square root options is chosen.

VALUATION OF ASYMMETRIC SQUARE ROOT OPTIONS

In order to achieve the objective defined in the introduction to this article, simulation experiments were performed. In consequence, the best method of the valuation of the analyzed derivatives, out of the martingale approach [Black, Scholes 1973] and two approaches based on the Fourier transform, was selected. It is worth noting that the determination of the theoretical values of the analyzed derivatives via the Fourier transform is limited only to methodologies proposed by [Bakshi, Madan 2000] and [Bates 2006]. It can not be overlooked that square root options, as well as many other exotic options [Orzechowski 2018a, Orzechowski 2018b, Orzechowski 2018c, Orzechowski 2018d, Orzechowski 2017], according to the best knowledge of the author, are not traded on any of the exchanges. It causes great difficulty in getting access to empirical data. As a result, it is not possible to investigate how close to each other are theoretical and empirical prices of the contracts.

Martingale approach

The first method of the valuation of square root options is identified with the F. Black and M. Scholes model [Black, Scholes 1973]. The approach presented in the article, however, is more similar to the martingale method, which assumes that the option is worth as much as the expected value (calculated with respect to

a certain martingale measure) of the profits generated by the contract discounted to the moment of valuation. Taking into account payoff function of asymmetric square root options, it can be stated that the following formula is correct:

$$C(S_0, 0) = e^{-rT} E^{\mathbb{Q}} \left((\sqrt{S_T} - \sqrt{K}, 0)^+ | \Omega_0 \right), \quad (1)$$

where: S_0 and S_T are the market values of the underlying asset at the time of writing and the expiration of the option, K is the exercise price, r is the risk-free interest rate, e^{-rT} is the discount factor, $E^{\mathbb{Q}}$ is the operator of the expected value with respect to a certain martingale measure \mathbb{Q} , and Ω_0 is the price history of the underlying instrument until the moment of the valuation.

Formula (1) can be transformed to the following form:

$$C(S_0, 0) = E^{\mathbb{Q}} \left(e^{-rT} \sqrt{S_T} \mathbb{1}_{\{S_T > K\}} \right) - e^{-rT} \sqrt{K} E^{\mathbb{Q}} \left(\mathbb{1}_{\{S_T > K\}} \right), \quad (2)$$

where: $\mathbb{1}_{\{S_T > K\}}$ is the indicator function taking value of 1 when $S_T > K$ and 0 otherwise, and the remaining notation is the same as previously introduced.

Changing the martingale measure of $E^{\mathbb{Q}} \left(e^{-rT} \sqrt{S_T} \mathbb{1}_{\{S_T > K\}} \right)$ and applying the Itô lemma, first for the square root of the price of the underlying asset, then for the natural logarithm of S_T allows to obtain the theoretical price of asymmetric square root options (the method is referred to as BS):

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2}rT - \frac{1}{8}\sigma^2 T} \mathcal{N}(d_1) - \sqrt{K} e^{-rT} \mathcal{N}(d_2), \quad (3)$$

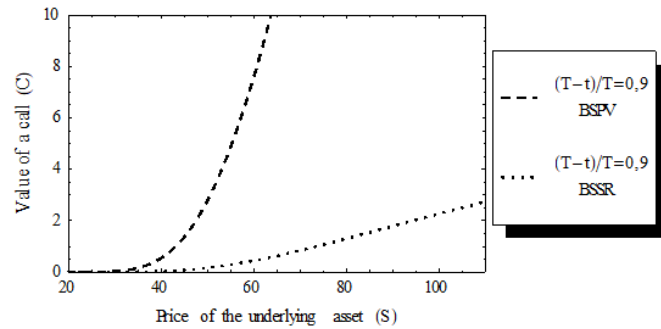
where: σ is the standard deviation of the rates of return of the underlying asset, and $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standardized normal distribution with parameters d_1 and d_2 defined by the following formulas:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + rT}{\sigma\sqrt{T}}, \quad (4)$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \quad (5)$$

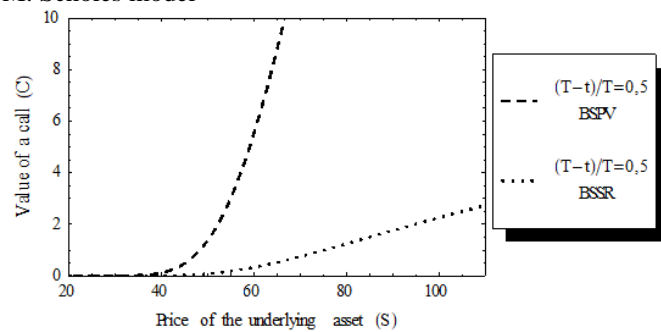
Based on equations (3) - (5) it is easy to determine payoff functions of the European square root options (BSSR) and compare them with the payoff functions of the plain vanilla European contracts (BSPV) - see Figures 1 - 3. Figures 1 - 3 are prepared assuming that the prices of the underlying asset are between 20 to 110, standard deviation of the rates of return of the underlying asset is equal to 29%, risk-free rate amounts to 4 %, exercise price of the option is equal to 60, and the relative time to expiration takes the following values: 0.9, 0.5 and 0.01.

Figure 1. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(T - t)/T = 0.9$ in the F. Black and M. Scholes model



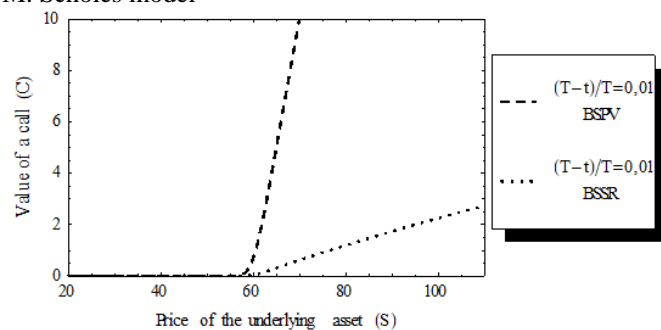
Source: own preparation

Figure 2. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(T - t)/T = 0.5$ in the F. Black and M. Scholes model



Source: own preparation

Figure 3. Payoff functions of European plain vanilla call options (BSPV) and European asymmetric square root call options (BBSR) for $(T - t)/T = 0.01$ in the F. Black and M. Scholes model



Source: own preparation

Approach based on the Fourier transform

Another method of the valuation of options is based on the Fourier transform. This approach is applicable to jump-diffusion models by [Metron 1976, Kou 2002], pure jump models, such as variance-gamma model [Madan et al. 1998], NIG model [Rydberg 1997], CGMY model [Carr et al. 2002] and stochastic volatility models [Heston 1997, Stein et al. 1991].

One of the earliest methods of the valuation of options via the Fourier transform is the one developed by [Bakshi, Madan 2000]. The method (referred to as FT-BM) consists of several steps. At the beginning, logarithmic transformations of the underlying asset's price and the exercise price are performed, i.e. $s_T = \ln S_T$, $k = \ln K$. This allows to write the formula for the price of asymmetric square root options in the following way:

$$C(S_0, 0) = e^{-rT} \int_k^\infty \left(e^{\frac{1}{2}s_T} - e^{\frac{1}{2}k} \right) \mathbb{Q}(s_T | \Omega_0) ds_T, \quad (6)$$

where: $\mathbb{Q}(s_T | \Omega_0)$ is the probability density function of the variable s_T with filtration Ω_0 , and the remaining notation is consistent with previously introduced.

Formula (6) is divided into two parts and for each of them Fourier transforms are determined. Obtained results are presented in formulas (7) and (8):

$$\chi_T^1(\xi) = \frac{e^{-rT} E(\sqrt{s_T}) \phi\left(\xi - \frac{1}{2}\mathbb{I}\right)}{i\xi \phi\left(-\frac{1}{2}\mathbb{I}\right)}, \quad (7)$$

$$\chi_T^2(\xi) = \sqrt{K} e^{-rT} \frac{\phi(\xi)}{\mathbb{I}\xi}, \quad (8)$$

where: $E(\cdot)$ is the operator of expected value, $\phi(\xi)$ is the characteristic function of the variable s_T , \mathbb{I} is the imaginary part of the complex number, and the remaining notation is consistent with previously introduced.

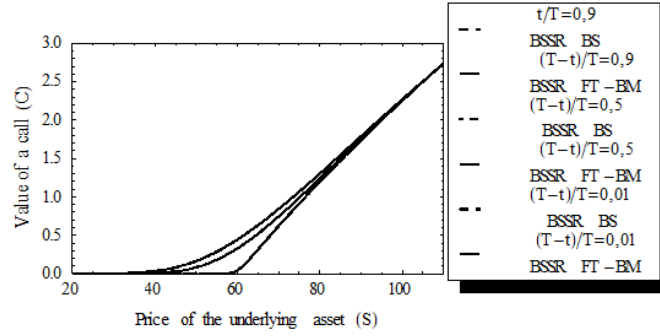
Finally, for the purpose of the valuation of analyzed derivatives, inverse Fourier transforms are calculated. As a consequence, the following equation is obtained:

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2}rT + \frac{1}{8}\sigma^2 T} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-\mathbb{I}\xi k} \phi\left(\xi - \frac{1}{2}\mathbb{I}\right)}{\mathbb{I}\xi \phi\left(-\frac{1}{2}\mathbb{I}\right)} \right] d\xi \right) + \\ -\sqrt{K} e^{-rT} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-\mathbb{I}\xi k} \phi(\xi)}{\mathbb{I}\xi} \right] d\xi \right), \quad (9)$$

where: $\Re[\cdot]$ is the real part of the subintegral function, and the remaining notation is consistent with previously introduced.

It is worth noting that formula (9) can be used to price square root options almost exactly in the same way as in the Black-Scholes model (possible differences result from the fact that the integrals in formula (9) are calculated numerically). This is confirmed by Figure 4.

Figure 4. Payoff functions of European asymmetric square root call options (BBSR) for $(T - t)/T = 0.9$, $(T - t)/T = 0.5$ and $(T - t)/T = 0.01$ determined with the use of BS and FT-BM methods



Source: own preparation

It can be easily seen that determination of the theoretical values of square root options with the use of the FT-BM method is not efficient. From the construction of the formula (9) it can be concluded that the FT-BM method requires greater calculation effort than in the BS method. For this reason alternative concept allowing for faster and more accurate valuation of analyzed options should be considered.

From the set of available methods particular attention should be paid to the proposal of [Bates 2006]. This method is based on the transformation of formula (6) to the following form:

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2}rT - \frac{1}{8}\sigma^2 T} - e^{-rT} \int_{\infty}^k e^{\frac{1}{2}S_T} \mathbb{Q}(S_T | \Omega_0) ds_T + e^{-rT} e^{\frac{1}{2}k} \int_k^{\infty} \mathbb{Q}(S_T | \Omega_0) ds_T, \quad (10)$$

where proposed notation is consistent with previously introduced.

For the second and third terms on the right-hand side of equation (10), the inverse Fourier transforms are determined. After appropriate simplifications it can be concluded that:

$$\chi_T^1(\xi) = e^{-rT} \frac{1}{2\pi} e^{\frac{1}{2}k} \int_{-\infty}^{\infty} \frac{e^{-\mathbb{I}\xi k}}{\frac{1}{2} - \mathbb{I}\xi} \phi(\xi) d\xi, \quad (11)$$

$$\chi_T^2(\xi) = e^{-rT} \frac{1}{2\pi} e^{\frac{1}{2}k} \int_{-\infty}^{\infty} \frac{e^{-\mathbb{I}\xi k}}{\mathbb{I}\xi} \phi(\xi) d\xi, \quad (12)$$

where proposed notation is consistent with previously introduced.

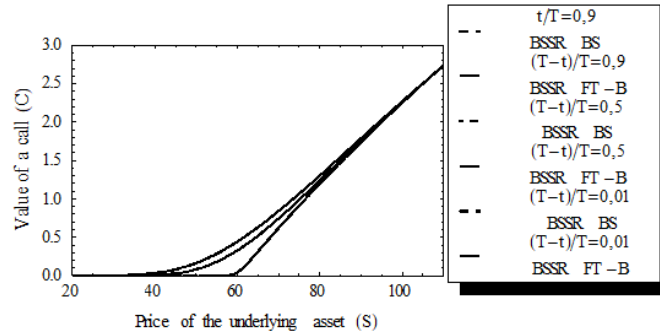
Finally, the theoretical value of asymmetric square root options is calculated as follows (the method is referred to as FT-B):

$$C(S_0, 0) = \sqrt{S_0} e^{-\frac{1}{2}rT + \frac{1}{8}\sigma^2 T} - \frac{1}{2} e^{-rT} \sqrt{K} + e^{-rT} \sqrt{K} \frac{1}{2\pi} \int_0^{\infty} \Re \left[\frac{e^{-\mathbb{I}\xi k} \phi(\xi)}{\mathbb{I}\xi \left(\mathbb{I}\xi - \frac{1}{2} \right)} \right] d\xi, \quad (13)$$

where proposed notation is consistent with previously introduced.

Similarity of the valuation of asymmetric square root options in the BS and the FT-B methods is confirmed by Figure 5.

Figure 5. Payoff functions of European asymmetric square root call options (BBSR) for $(T-t)/T = 0.9$, $(T-t)/T = 0.5$ and $(T-t)/T = 0.01$ determined with the use of BS and FT-B methods



Source: own preparation

It is relatively easy to see that the FT-B method is better than the FT-BM in at least two aspects, i.e. computational speed and accuracy. This is due to the fact that in formula (13), in contrast to formula (9), there is only one characteristic function and in the denominator of the subintegral function ξ is squared. It increases speed of the valuation of asymmetric square root options and improves computational accuracy of obtained results (it should be remembered that the inverse Fourier transforms in formulas (9) and (13) are calculated numerically).

Having formulas (9) and (13) one can proceed to examine how much time is required to obtain the final result of the calculations. It is assumed that the input data needed to price asymmetric square root options is identical as in the case of Figure 1-3. An important element of this part of the research is the selection of numerical schemes used for the calculation of integrals in formulas (9) and (13). For the needs of simulation experiments, the following numerical methods were selected: trapezoidal and Clenshaw-Curtis rules.

All calculations were performed in the Mathematica 8.0 run on a computer with Intel i5-4210U CPU @ 1.70 GHz processor and RAM memory of 6 GB. It is worth noting that in the proposed procedure cache memory is deleted every time before the option pricing process begins. Obtained results are presented in Tables 1-3.

Table 1. Computational speed of asymmetric square root options in seconds
 $((T - t)/T = 0.9)$

	OTM ($S_T=50$)	ATM ($S_T=60$)	ITM ($S_T=70$)
BS	0	0	0
FT-BM (trapezoidal rule)	0.031	0.031	0.032
FT-BM (Clenshaw-Curtis rule)	0.015	0.015	0.015
FT-B (trapezoidal rule)	0.016	0.015	0.015
FT-B (Clenshaw-Curtis rule)	0.015	0.015	0.015

Source: own preparation

Table 2. Computational speed of asymmetric square root options in seconds
 $((T - t)/T = 0.5)$

	OTM ($S_T=50$)	ATM ($S_T=60$)	ITM ($S_T=70$)
BS	0	0	0
FT-BM (trapezoidal rule)	0.031	0.031	0.031
FT-BM (Clenshaw-Curtis rule)	0.015	0.016	0.016
FT-B (trapezoidal rule)	0.016	0.016	0.016
FT-B (Clenshaw-Curtis rule)	0.016	0.015	0.015

Source: own preparation

Table 3. Computational speed of asymmetric square root options in seconds
 $((T - t)/T = 0.01)$

	OTM ($S_T=50$)	ATM ($S_T=60$)	ITM ($S_T=70$)
BS	0	0	0
FT-BM (trapezoidal rule)	0.031	0.031	0.031
FT-BM (Clenshaw-Curtis rule)	0.015	0.016	0.016
FT-B (trapezoidal rule)	0.015	0.016	0.016
FT-B (Clenshaw-Curtis rule)	0.015	0.015	0.015

Source: own preparation

Based on Tables 1-3 it can be concluded that the BS method allows for the fastest pricing of asymmetric square root options (irrespective of the moneyness of the contracts). Such statement, however, does not mean that the BS method is the best method of pricing analyzed contracts. Introducing assumptions concerning existence of discontinuities or stochastic volatility of the standard deviation of returns of the underlying assets makes it practically impossible to use discussed model. In consequence, there is a need to refer to alternative methods, including those based on the Fourier transform. In the case of methods in which characteristic functions are used there appear two criteria on the basis of which the best approach should be chosen:

- the number of inverse Fourier transforms (the less the better),
- the power to which ξ in the denominator of the subintegral function is raised (the higher, the faster the convergence of the method).

Moreover, the numerical scheme used to approximate theoretical values of the contracts in proposed formulas is also important. Among the methods based on the Fourier transform, the FT-B method with the Clenshaw-Curtis rule seems to be the best, regardless of the moneyness of the option and the period remaining to expiration. The worst method is the FT-BM with a trapezoidal rule.

SUMMARY

Asymmetric square root options can be priced in many ways. The choice of the appropriate method of pricing analyzed contracts depends to a large extent on the assumptions regarding the process of generating prices of underlying assets.

This article analyzes computational speed and accuracy of derivatives in the Black-Scholes setting. The research carried out indicates that the BS method is the most efficient. From the group of methods based on the Fourier transform the FT-B method with the Clenshaw-Curtis rule should be chosen. It is important to note that the drawn conclusions are true when the variance of return on underlying assets is constant and no jumps in the prices of the underlying instruments appear. At the same time, it should be remembered that the choice of the optimal method of determining the Fourier transform and the inverse Fourier transform, as well as the selection of the best numerical scheme, depends to a large extent on the assumptions under which the valuation of derivatives is carried out.

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THE RISK OF THE POLISH EQUITY FUNDS IN THE YEARS 2004-2018 DETERMINED USING THE VAR AND CVAR MEASURES

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Abstract: The work analyzed Polish equity funds due to the risk associated with investing in this type of instruments. The study was conducted using risk measures, ie Value at Risk (VaR) and conditional Value at Risk (CVaR). Similarity was measured using the tau-Kendall coefficient. The study examined 15 equity funds that existed in Poland since 2004. The entire research period (2004-2018) was divided into shorter periods ie 2-, 3-, 4- and 5-year periods. The fund rankings based on VaR and CVaR risks gave similar results. In periods of big changes in the economic situation the measures were not able to properly estimate the risk.

Keywords: investment risk, open-end mutual funds, Value at Risk (VaR), conditional Value at Risk (CVaR)

JEL classification: G11, G14

INTRODUCTION

Investment portfolio management is connected with the continuous analysis of changes occurring on the market and correcting the composition of the portfolio. The investor, when entrusting his financial excess institutions of collective investment, decides that such diversification was made by the investment fund managers, of course, within a given type of fund. Often the investor's fear of a potential loss is greater than the prospect of a possible profit. That is why it is so important to focus on the probable risk of loss and to easily calculate it. Such tools are provided by the Value at Risk (VaR) measure. VaR was originally used to evaluate credit risk and then to determine the risk associated with different types of

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investments. Increasingly, VaR is also the basis for measures of investment efficiency.

Value at Risk was introduced in 1986 by Ken Garbade of Banker's Trust, when Ken Garbade of Banker's Trust presented a VaR measure calculated for the portfolio of bonds with a different maturity. In the following years, many financial institutions developed the methodology of this measure, presenting various ways of measuring it. Value at Risk was disseminated by JP Morgan, who in 1994 published a model based on the VaR measure. The frequency of applying the Value at Risk was also influenced by the references of the Basel Committee on Banking Supervision, which recommends VaR as a measure of market risk. [Kulczycki 2015].

There is a lot of literature related to the Value at Risk. Many of them deal with risk estimation using different methods and their evaluation due to the results achieved. e.g. [Wiener 1997], [Bohdalova 2007] or [Miletic, Miletic 2015]. VaR estimation for different types of funds can be found, among others in the works [Grau-Carles et al. 2009], [Fedor 2010]. [Weng, Trueck 2011], [Ardia, Boudt 2016], or [Liang, Park 2007]. Research [Grau-Carles et al. 2009] related to monthly returns of 239 UK investment funds over 11 years, from January 1995 to December 2005. Studies focused on risk assessment for loss using Value at Risk (VaR). 10 funds with the highest and lowest monthly rate of return were selected for the research. The results obtained on the basis of the VaR measure and its modifications turned out to be similar. There was also a strong correlation of rankings obtained on the basis of a modified Sharpe index using Value at Risk as a measure of risk. Another VaR estimate can be found in [Tehrani et al. 2014], [Gallali, Guesmi 2008] or [Deb, Barenje 2009]. Tehrani used parametric and non-parametric methods. Gall and Guesmi used a parametric method, historical simulation and Monte Carlo to the portfolios of 14 funds of the Tunisian market. Deb and Barenjee estimated VaR using parametric and non-parametric methods for equity funds of the Indian market.

From Polish literature, it is necessary to mention works on the estimation of Value at Risk for investment funds, i.e. [Olbryś 2009], [Filipowicz 2011], [Rutkowska-Ziarko, Garsztka 2015] or [Rutkowska-Ziarko, Sobieska 2016]. Olbryś calculated VaR for the period 2007-2009 divided into 17 sub-periods. Rutkowska-Ziarko and Sobieska estimated VaR for funds belonging to different risk classes. Filipowicz [2011] evaluated VaR for a portfolio of two equity funds.

The aim of the work is to assess Polish equity funds in terms of risk related to investing in these instruments and answer the question whether it is important to choose the right measure. VaR and CVaR risk measures will be used, which will be analyzed in terms of investment length. The work is the introduction to further research using risk measures to study the effectiveness of investments.

DATA AND METHODOLOGY OF THE STUDY

Value at Risk (VaR) is the value of a potential loss that will be achieved with a certain probability in a fixed time horizon t [Jajuga 2008]:

$$P(W \leq W_0 - VaR_{\alpha,t}) = \alpha, \quad (1)$$

where:

$VaR_{\alpha,t}$ - Value at Risk,

α - level of significance,

t - time horizon,

W_0 - initial value,

W - value at the end of the period.

If the rates of return are taken into account, the following Value at Risk is used:

$$P(z_t \leq -VaR_{\alpha,t}) = \alpha, \quad (2)$$

where:

z_t - quantile of the distribution of returns.

One of the methods of estimating Value at Risk is the method of variance-covariance¹, which requires assumption as to the distribution of return rates.

Assuming that the distribution of rates of return is a normal distribution, the Value at Risk is calculated using the formula [Jorion 2006]:

$$VaR = -(r_p + z\delta), \quad (3)$$

where:

r_p - average rate of return,

z - a negative value of the quantile of standard normal distribution,

δ - standard deviation of return rates.

Value at Risk (VaR) can be applied to various types of individual assets as well as portfolios composed of many assets. However, it is not a coherent measure. It is also impossible to use it to estimate the loss value if it exceeds the VaR level [Angelidis, Benos 2008]². Based on the negation of the VaR measure, a conditional Value at Risk (CVaR) was created. CVaR is also called Expected Shortfall (ES). It is described by the following relationship:

$$CVaR_{\alpha,t} = E(W | W > VaR_{\alpha,t}), \quad (4)$$

¹ Other methods are also used, including Monte Carlo method or historical simulation (Kuziak 2003).

² In addition, Artzner et al. (1997) gave four axioms that a risk measure must meet.

CVaR estimates the value of the loss that exceeds the VaR level. Assuming that the distribution of rates of return is consistent with the normal distribution, CVaR can be determined from the formula [Albrecht, Koryciarz 2003]:

$$CVaR = r_p - \frac{\varphi_z}{\alpha} \delta, \quad (5)$$

where:

φ_z - the density of a standardized normal distribution.

The research concerned 15 equity funds that existed on the Polish market since 2004. These were funds: Aviva Investors Polskich Akcji, Esaliens Akcji, Investor Akcji, Investor Akcji Spółek Dywidendowych, Investor Top 25 Małych Spółek, Millennium Akcji, NN Akcji, Novo Akcji, Pekao Akcji Polskich, PZU Akcji Krakowiak, Rockbridge Akcji, Rockbridge Małych I Średnich Spółek, Santander Akcji, Skarbiec Akcja, UniKorona Akcje. The period 2004-2018 was divided into two-year sub-periods (2004-2005, 2006-2007, 2008-2009, 2010-2011, 2012-2013, 2014-2015, 2016-2017), three-year (2004-2006, 2007-2009, 2010-2012, 2013-2015, 2016-2018), four-year (2004-2007, 2008-2011, 2012-2015) and five years (2004-2008, 2009-2013, 2014-2018).

Fund share units values provided the basis for determining the monthly normal returns using the formula:

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}}, \quad (6)$$

where:

p_t – the value of fund units during the period t .

Values of VaR and CVaR risk measures were estimated at the 0.05 significance level.

In order to answer the question whether the choice of a measure matters, first the Wilcoxon test was verified [Domański 1979].

Having two series of values determined in one sub-period: VaR ($x_i, i=1, \dots, m$) and CvaR ($y_j, j=1, \dots, m$) measures, an ordered sequence is created, consisting of both subseries. Then the elements are sorted in ascending order and give them rank. The value of the statistics is as follows:

$$W = \sum_{i=1}^m R(x_i), \quad (7)$$

where $R(x)$ -ranks of the first series.

The value of the statistic (7) is compared with the critical values $W_{-\alpha}$ and $E(W)$ for the Wilcoxon test read out for n and m with the adopted level of alpha significance. There is no reason to reject the null hypothesis about the lack of differences between subseries if $W_{-\alpha} < W < 2E(W) - W_{-\alpha}$. If there is no difference between the series of VaR and CVaR, the coefficient of similarity of tau-Kendall was determined and its significance was examined.

The null hypothesis is that the correlation coefficient of r_i is statistically insignificant to the alternative hypothesis that the correlation coefficient of r_i is statistically significant. A significance level of 0.05 was assumed in the research. Statistica software was used for the research.

MARKET OF THE MUTUAL FUNDS IN POLAND

The first open-end mutual fund was established in Poland in 1992 (the First Polish American Trust Fund Pioneer), and another one only three years later. From 20 funds in 1997, the number of funds in 2004 increased more than seven times. The development of the investment funds market was influenced by further legal regulations and an increase in investment awareness of Poles.

The Act on Trust Funds of August 28, 1997 changed the terminology of funds (the trust fund was changed into an open-end mutual fund) and gave the fund legal personality. In 1999, the first closed fund was created. The amendment to the Act in 2000 enabled the sale of fund units outside brokerage houses and through individuals. In this year, funds were created that invest outside of Poland and global funds (investing in Poland and abroad). In 2001, the market for bond funds and money market funds developed.

The entry into force of the Act on investment funds in 2004 had a major impact on the current shape of the investment fund market. The adaptation of Polish law to the regulations in force in the European Union unified the principles of fund management, information obligations of open-end funds and the rules for selling shares by foreign funds based in the EU. In 2004, the Chamber of Fund and Asset Management was also set up to carry out activities related to the operation of the funds and the standards of their operation.

At the end of 2004, the net assets of investment funds amounted to 37.43 (PLN bn), and at the end of December, the net asset value of the share-based funds alone was 29.63 (PLN bn). Comparing the Polish and European funds market, the share of fund assets registered in Poland in European Fund assets is small (around 0.5%). However, this is an understandable situation considering the twenty-year period of the capital market functioning and the much shorter development period of the fund market. Nevertheless, despite the small share in the European market, the assets of the funds increase, their share in relation to GDP increases (the exception is the period of the financial crisis), the market structure from the point of view of their division into equity, bonds, money market, etc., reflects the structure of the European market.

ANALYSIS OF FUNDS DUE TO INVESTMENT RISKS

The average rates of return and standard deviation of the return rates of each fund allowed the VaR to be estimated. Value at Risk results for two-year periods are presented in Table 1.

Table 1. VaR values for equity funds in two-year periods

Fund	2004-2005	2006-2007	2008-2009	2010-2011	2012-2013	2014-2015	2016-2017
Aviva Investors Polskich Akcji	0.036	0.082	0.154	0.089	0.042	0.050	0.031
Esaliens Akcji	0.044	0.074	0.135	0.071	0.055	0.058	0.029
Investor Akcji	0.037	0.075	0.144	0.094	0.051	0.052	0.022
Investor Akcji Spółek Dywidendowych	0.043	0.073	0.148	0.088	0.056	0.047	0.034
Investor Top 25 Małych Spółek	0.043	0.087	0.169	0.100	0.045	0.060	0.033
Millennium Akcji	0.050	0.077	0.130	0.086	0.053	0.052	0.041
NN Akcji	0.047	0.078	0.136	0.095	0.060	0.046	0.045
Novo Akcji	0.044	0.072	0.155	0.079	0.071	0.080	0.060
Pekao Akcji Polskich	0.048	0.084	0.182	0.128	0.063	0.059	0.037
PZU Akcji Krakowiak	0.043	0.069	0.141	0.086	0.067	0.067	0.044
Rockbridge Akcji	0.037	0.077	0.135	0.089	0.059	0.056	0.043
Rockbridge Małych i Średnich Spółek	0.058	0.107	0.162	0.089	0.049	0.053	0.036
Santander Akcji	0.038	0.071	0.158	0.092	0.059	0.048	0.037
Skarbiec Akcja	0.045	0.062	0.128	0.088	0.063	0.062	0.059
UniKorona Akcje	0.043	0.071	0.129	0.086	0.061	0.046	0.035

Source: own study

The highest loss of investments was observed for the years 2008-2009, ie the period of the financial crisis. It was often four times higher than the value from the period 2004-2005, in which VaR turned out to be the lowest. The highest risk was characteristic for the following funds: Rockbridge Małych i Średnich Spółek (in 2004-2005, 2006-2007). Pekao Akcji Polskich (in 2008-2009, 2010-2011) and Novo Akcji (in 2012-2013, 2014-2015, 2016-2017).

Dividing the whole period of research into three-year sub-periods, the highest VaR values occurred in the 2007-2009 subperiod, and the lowest in the 2004-2006 period. The highest risk was characteristic for the following funds: Rockbridge Małych i Średnich Spółek (in 2004-2006). Pekao Akcji Polskich (in the years 2007-2009, 2010-2012) and Novo Akcji (in the years 2013-2015, 2016-2018). Among the four-year sub-periods, the largest loss of value occurred in 2008-2011, and among the five-year sub-period - in 2004-2008. The highest risk was for

funds: Rockbridge Małych i Średnich Spółek (in the period 2004-2007), Pekao Akcji Polskich (2008-2011) and Novo Akcji (2012-2015).

In the next step, CVaR values were determined in 2-, 3-, 4- and 5-year periods. All CVaR values turned out to be higher than VaR, both for two-year periods as well as for the remaining years studied. This is justified by the definition of this measure. The largest loss in two-year sub-periods was noted, similarly to the Value at Risk, in the period 2008-2009 (CVaR values were between 0.16 to 0.23) and the lowest loss in the period 2004-2005. Slightly lower loss values than for the period 2008-2009 were observed for the years 2007-2009, which turned out to be the period with the highest values in the three-year periods. Of the four-year periods, the largest loss was recorded in 2008-2011, when CVaR ranged from 0.13 to 0.19. The highest values in the five-year subperiods occurred in 2004-2008, ranging from 0.11 to 0.15. The highest risk among specific funds in all sub-periods was characterized by the same values as in the case of the VaR measure.

The VaR and CVaR values obtained were the basis for creating rankings in individual sub-periods. Ranking positions created on the basis of Value at Risk for two-year periods are presented in Table 2. The last position was shaded grey, while the first position was grayed out with a frame. Some VaR values turned out to be the same in individual sub-periods. In such cases, the ranking positions were therefore designated as the arithmetic average of the subsequent items.

Table 2. Ranking positions obtained based on VaR determined in two-year periods

Fund	2004-2005	2006-2007	2008-2009	2010-2011	2012-2013	2014-2015	2016-2017
Aviva Investors Polskich Akcji	15	4	6	7	15	11	13
Esaliens Akcji	6.5	9	11.5	15	10	6	14
Investor Akcji	13.5	8	8	4	12	9.5	15
Investor Akcji Spółek Dywidendowych	9.5	10	7	9.5	9	13	11
Investor Top 25 Małych Spółek	9.5	2	2	2	14	4	12
Millennium Akcji	2	6.5	13	12	11	9.5	6
NN Akcji	4	5	10	3	6	14.5	3
Novo Akcji	6.5	11	5	14	1	1	1
Pekao Akcji Polskich	3	3	1	1	3.5	5	7.5
PZU Akcji Krakowiak	9.5	14	9	12	2	2	4
Rockbridge Akcji	13.5	6.5	11.5	7	7.5	7	5
Rockbridge Małych i Średnich Spółek	1	1	3	7	13	8	9
Santander Akcji	12	12.5	4	5	7.5	12	7.5
Skarbiec Akcja	5	15	15	9.5	3.5	3	2
UniKorona Akcje	9.5	12.5	14	12	5	14.5	10

Source: own study

In division into sub-periods, two-year individual funds in different sub-periods occupied very different positions. The riskiest fund in the years 2004-2005 and 2006-2007 (Rockbridge Małych i Średnich Spółek) in subsequent sub-periods occupied the middle positions and the second half of the ranking. Pekao Akcji Polskich in all sub-periods was at the forefront of the ranking. Similarly, the situation of fund rankings appeared in a longer time horizon, i.e. three-year sub-periods. The extension of the time horizon to both the four- and five-year subperiods did not affect the stability of the results (Table 3).

Table 3. Ranking positions obtained based on VaR determined in four-year and five-year periods

Fund	2004-2007	2008-2011	2012-2015	2004-2008	2009-2013	2014-2018
Aviva Investors Polskich Akcji	6	5	15	5	11	15
Esaliens Akcji	7	15	5	10.5	15	13
Investor Akcji	10	7.5	13	7.5	11	4
Investor Akcji Spółek Dywidendowych	8	7.5	11.5	7.5	11	10.5
Investor Top 25 Małych Spółek	2	2	8	2	6.5	6
Millennium Akcji	4	13	11.5	10.5	14	7.5
NN Akcji	5	9	10	9	4	9
Novo Akcji	10	6	1	6	2	1
Pekao Akcji Polskich	3	1	4	3	1	7.5
PZU Akcji Krakowiak	12.5	10	2	12	6.5	2
Rockbridge Akcji	10	11	6	13	6.5	5
Rockbridge Małych i Średnich Spółek	1	3.5	14	1	6.5	13
Santander Akcji	14	3.5	8	4	3	13
Skarbiec Akcja	15	13	3	15	11	3
UniKorona Akcje	12.5	13	8	14	11	10.5

Source: own study

Rankings obtained on the basis of CVaR were similar to the rankings obtained on the basis of VaR. They differed most often by several positions although there were cases where this difference was higher.

To test the similarity of results, the Wilcoxon test was used first. In all the sub-periods examined, there are no grounds to reject the null hypothesis about the equality of for the considered measures. In the next step, the tau-Kendall coefficient was determined. Similar rankings obtained using measures based on Value at Risk were confirmed by high values of the tau-Kendall coefficient. The values of the coefficient for two-year sub-periods are shown in Table 4.

Table 4. The tau-Kendall coefficient determined in two-year periods

2004-2005		2006-2007		2008-2009		2010-2011		2012-2013		2014-2015		2016-2017	
	CVaR		CVaR		CVaR		CVaR		CVaR		CVaR		CVaR
VaR	0.956	VaR	0.951	VaR	0.981	VaR	0.939	VaR	0.99	VaR	0.976	VaR	0.995

Source: own study

The results obtained indicate a strong similarity between the measures set for two-year sub-periods. A similar situation occurs for the other sub-periods.

SUMMARY

Considering the results achieved by the funds in terms of their effectiveness, the risk associated with investing in this type of instruments is often forgotten. Analysis of funds due to the risk they achieve is therefore very important from the point of view of the potential loss of the investor. Due to the existence of different types of risk, the work focused on one of it - measures of potential loss, ie VaR and CVaR. Despite the fact that the VaR risk measure is not a coherent measure, the rankings created on the basis of VaR and CVaR give similar results, although CVaR values themselves are higher than VaR.

Earlier author's research on the drawdown measures, which include the Ulcer and Pain indices, showed a similar order of individual sub-periods in terms of the risk being examined [Żebrowska-Suchodolska 2017]. However, in the period of the financial crisis, the average values of maximum decreases were significantly higher than the average VaR or CVaR values. Value at Risk and Conditional Value at Risk, therefore, were not able to overestimate the risks that occurred in that period.

The riskiest funds turned out to be Rockbridge Małych i Średnich Spółek, Pekao Akcji Polskich and Novo Akcji. The research also showed a strong similarity between the measures examined. In all of the sub-periods, the tau-Kendall coefficients turned out to be statistically significant.

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