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CONTENTS

AN APPLICATION OF RADAR CHARTS TO GEOMETRICAL MEASURES OF STRUCTURES' OF CONFORMABILITY

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Abstract: In the following work we presented a method of using radar charts to calculate measures of conformability of two objects according to formulas given by, among others, Dice, Jaccard, Tanimoto and Tversky. This method incorporates another one presented by the authors of this study [Binderman, Borkowski, Szczesny 2010]. Presented methods can be also utilized to define similarities between given objects, as well as to order and group objects. Measures described in this work satisfy the condition of stability as they do not depend on the order of studied features.

Key words: radar method, radar measure of conformability, Dice's, Jaccard's measure of similarity, synthetic measures, classification, cluster analysis.

CONSTRUCTION OF RADAR MEASURES OF CONFORMABILITY

In previous works authors used methods that have a simple interpretation in the form of a radar chart to order, classify and measure similarity of objects [Binderman, Borkowski, Szczesny 2008, 2009, 2009a, 2010, 2010a, b, c, d, Binderman, Szczesny 2009, 2011, Binderman 2009, 2009a]. Those methods do not depend on the way the features of a given object are ordered. In the following work authors attempted to utilize those methods in other, widely known means of measuring similarity between two objects. Comparing structures of objects is chosen here as an example. Coefficients of Jaccard, Dice and Tanimoto, Tversky index and cosine similarity are all exemplary geometrical measures of similarity.

The methods presented here may seem numerically complicated but in the age of computers this problem is of little significance.

Numerous studies conducted in many different fields of science: economics, statistics, computer science, chemistry, biology, ecology, psychology, culture and tourism have proven the usefulness of those methods [Binderman 2009a, Binderman, Borkowski, Szczesny 2010b, c, Ciok, Kowalczyk, Pleszczyńska, Szczesny 1995, Deza E., Deza M.M. 2006, Duda, Hart, Stork 2000, Gordon 1999, Hubalek 1982, Kukuła 2000, 2010, Legendre P., Legendre L. 1998, Monev 2004, Szczesny 2002, Tan, Steinbach, Kumar 2006, Warrens 2008].

Let Q and P be two objects that are described by a set of n (n>2) features. Assume that objects Q and P are described by two vectors $\mathbf{x}, \mathbf{y} \in \mathfrak{R}_+^n$, where:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \ \ \mathbf{y} = (y_1, y_2, \ldots, y_n); \ \ x_i, y_i \geq 0; \ \ i = 1, 2, \ldots, n$$

and

$$\sum_{i=1}^{n} x_i = 1, \ \ \sum_{i=1}^{n} y_i = 1.$$

In order to graphically represent the methods we inscribe a regular n-gon into a unit circle (with a radius of 1) with a centre in the origin of a polar coordinate system $0uv$ and we will connect the vertices of this n-gon with the origin of the coordinate system. Thus, one constructs line segments of length 1, we will denote, in sequence, $01, 02, \ldots, 0n,$ starting, for definitiveness, with the line segment covering w axis. Assume that at least two coordinates of each of the vectors $\mathbf{x}$ and $\mathbf{y}$ are non-zero. Because features of objects $\mathbf{x}$ and $\mathbf{y}$ take on values from an interval <0,1>, that is $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \equiv 0 \leq x_i \leq 1$, $\mathbf{0} \leq \mathbf{y} \leq \mathbf{1} \equiv 0 \leq y_i \leq 1$, i=1,2,...n, where $\mathbf{0} := (0,0,\ldots,0)$, $\mathbf{1} := (1,1,\ldots,1)$, we can represent the values of those features as a radar chart. To do so, let x_i (y_i) denote those points on the $0i$ axis that came into being by intersecting the $0i$ axis with a circle with the centre at the origin of the coordinate system and radius of x_i (y_i), i=1,2,...,n. By connecting the points: x_1 with x_2, x_2 with x_3, ..., x_n with x_1 (y_1 with y_2, y_2 with y_3, ..., y_n with y_1) we get n-gons S_Q and S_P, where its areas $|S_Q|$ and $|S_P|$, are given by formulas:

$$\left| S_Q \right| = \left| S_\mathbf{x} \right| = \sum_{i=1}^{n} \frac{1}{2} x_i x_{i+1} \sin \frac{2\pi}{n} = \frac{1}{2} \sin \frac{2\pi}{n} \sum_{i=1}^{n} x_i x_{i+1}, \ \ \text{gdzie} \ \ x_{n+1} := x_1 \ ,$$

$$\left| S_P \right| = \left| S_\mathbf{y} \right| = \sum_{i=1}^{n} \frac{1}{2} y_i y_{i+1} \sin \frac{2\pi}{n} = \frac{1}{2} \sin \frac{2\pi}{n} \sum_{i=1}^{n} y_i y_{i+1}, \ \ \text{gdzie} \ \ y_{n+1} := y_1 \ .$$

(1)

The formula for the area of the intersection of those n-gons, which we will denote by $S_{\mathbf{x} \cap \mathbf{y}} := S_\mathbf{x} \cap S_\mathbf{y}$ has a more complicated form. Its form and detailed determination can be found in [Binderman, Borkowski, Szczesny 2010]. Using those formulae we can denote the area of the union of n-gons S_x and S_y as

$$\left| S_x \cup S_y \right| = \left| S_x \right| + \left| S_y \right| - \left| S_x \cap S_y \right|, \tag{2}$$

where the areas $\left| S_x \right|, \left| S_y \right|$ are defined by formulae (1).

Figure 1 presents two graphical illustrations of vectors $\mathbf{x}$=(0,2, 0,2, 0,3, 0,15, 0,1, 0,05) and $\mathbf{y}$ =(0,15, 0,15, 0,2, 0,25, 0,15, 0,1) that describe two exemplary demographical structures (for age ranges: 0-14, 15-24, 25-49, 50-64, 65-79, >80), while Fig. 1A and 1B differ only by the order of axes (meaning the permutation of the coordinates).

Fig. 1. Radar charts for vectors **x** and **y,** which coordinates present two exemplary demographical structures, by different ordering of axes.

Source: own work
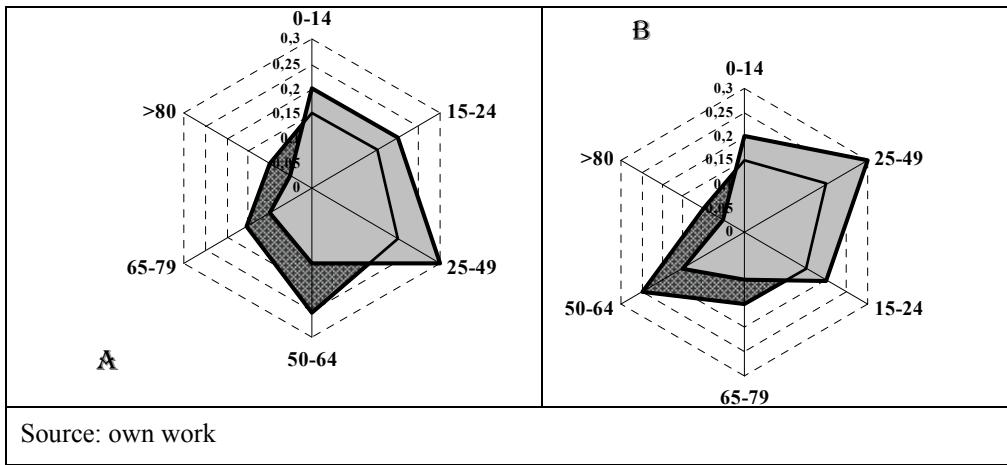
From the figure it is clear that areas of n-gons $\left| S_x \right|, \left| S_y \right|$ and their unions on figures

1A and 1B differ in size. They are: 0,076; 0,074; 0,051 and 0,075; 0,069; 0,047, respectively.

In works [Binderman Borkowski, Szczesny 2008, 2010] authors proposed a measure of conformability of objects that uses a geometrical interpretation in the form of radar charts and is defined as follows:

$$R_{xy} = \begin{cases} \sqrt{\dfrac{\left| S_x \cap S_y \right|}{\sigma_{xy}}} & \text{for } n=3 \\[4mm] \sqrt{\dfrac{\left| S_x \cap S_y \right|}{\omega_{xy}}} & \text{for } n \geq 4 \end{cases}, \tag{3}$$

where

$$\sigma_{\mathbf{xy}} := \begin{cases} \min(|S_{\mathbf{x}}|, |S_{\mathbf{y}}|) & \text{gdy } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 1 & \text{gdy } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}, \quad \omega_{\mathbf{xy}} := \begin{cases} \max(|S_{\mathbf{x}}|, |S_{\mathbf{y}}|) & \text{gdy } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 1 & \text{gdy } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}.$$

Note that such a measure of conformability (similarity) has the property of: $0 \leq \mu_{x,y} \leq 1$ and depends on the ordering of features [cf. Binderman Borkowski, Szczesny 2008].

To define a measure of conformability of objects that does not depend on the ordering of features, let us denote by π_j – a j-th permutation of numbers 1,2,…,n. It is known that the number of all such permutations is equal to *n!* [Mostowski, Stark 1977]. Each permutation of coordinates of vectors x and y corresponds to one permutation π_j. Let vectors x_j, y_j denote the j-th permutation of coordinates of vectors x and y, respectively, assuming that $x_1 := x$, $y_1 := y$.

For example, if n=3, $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ and $\pi_1=(1,2,3)$, $\pi_2=(1,3,2)$, $\pi_3=(2,1,3)$, $\pi_4=(2,3,1)$, $\pi_5=(3,1,2)$, $\pi_6=(3,2,1)$ then: $x_1=(x_1,x_2,x_3)$, $y_1=(y_1,y_2,y_3)$, $x_2=(x_1,x_3,x_2)$, $y_2=(y_1,y_3,y_2)$ $x_3=(x_2,x_1,x_3)$, $y_3=(y_2,y_1,y_3)$, $x_4=(x_2,x_3,x_1)$, $y_4=(y_2,y_3,y_1)$, $x_5=(x_3,x_1,x_2)$, $y_5=(y_3,y_1,y_2)$, $x_6=(x_3,x_2,x_1)$, $y_6=(y_3,y_2,y_1)$.

A result from our earlier works is that a coefficient of conformability of structures corresponds to each *j*-th permutation x_j, y_j of coordinates of vectors **x** and **y**

$$R_{Q,P}^j = R_{\mathbf{x}_j \mathbf{y}_j}, \tag{4}$$

where naturally $R_{Q,P}^1 = R_{\mathbf{xy}}$.

Therefore, we can assume that the following designations of three different measures of conformability of considered objects Q and P. Naturally, those measures are invariant under the ordering of coordinates for vectors **x** and **y.**

$$R_{Q,P}^M = R_{\mathbf{xy}}^M = \max_{1 \leq j \leq n!} R_{Q,P}^j,$$

$$R_{Q,P}^m = R_{\mathbf{xy}}^m = \min_{1 \leq j \leq n!} R_{Q,P}^j, \tag{5}$$

$$R_{Q,P}^s = R_{\mathbf{xy}}^s = \frac{1}{n!} \sum_{j=1}^{n!} R_{Q,P}^j.$$

Other well-known in literature techniques that use geometrical interpretations, such as radar charts, may be used to compare two structures $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$. Most well known among them are:

cosine similarity [Deza, Deza 2006]

$$\hat{c}_{\mathbf{xy}} = \begin{cases} \dfrac{|S_{\mathbf{x}} \cap S_{\mathbf{y}}|}{\sqrt{|S_{\mathbf{x}}||S_{\mathbf{y}}|}} & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 0 & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}, \tag{6}$$

Jaccard coefficient [Jaccard 1901, 1902, 1908]

$$\hat{J}_{\mathbf{xy}} = \begin{cases} \dfrac{|S_{\mathbf{x}} \cap S_{\mathbf{y}}|}{|S_{\mathbf{x}} \cup S_{\mathbf{y}}|} & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 0 & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}, \tag{7}$$

Dice's coefficient [Dice 1945]

$$\widehat{D}_{\mathbf{xy}} = \begin{cases} \dfrac{2|S_{\mathbf{x}} \cap S_{\mathbf{y}}|}{|S_{\mathbf{x}}| + |S_{\mathbf{y}}|} & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 0 & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}, \tag{8}$$

Tanimoto coefficient [Tanimoto 1957, 1959]

$$\widehat{\mathfrak{I}}_{\mathbf{xy}} = \begin{cases} \dfrac{|S_{\mathbf{x}} \cap S_{\mathbf{y}}|}{|S_{\mathbf{x}}| + |S_{\mathbf{y}}| - |S_{\mathbf{x}} \cap S_{\mathbf{y}}|} & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| > 0 \\ 0 & \text{for } |S_{\mathbf{x}}||S_{\mathbf{y}}| = 0 \end{cases}, \tag{9}$$

Tversky index [Tversky 1957]

$$\widehat{T}_{\mathbf{xy}} = \begin{cases} \dfrac{\left|S_{\mathbf{x}} \cap S_{\mathbf{y}}\right|}{\left|S_{\mathbf{x}} \cap S_{\mathbf{y}}\right| + \alpha\left|S_{\mathbf{x}} \setminus S_{\mathbf{y}}\right| + \beta\left|S_{\mathbf{y}} \setminus S_{\mathbf{x}}\right|} & \text{for } \left|S_{\mathbf{x}}\right|\left|S_{\mathbf{y}}\right| > 0 \\ 0 & \text{for } \left|S_{\mathbf{x}}\right|\left|S_{\mathbf{y}}\right| = 0 \end{cases} \quad \alpha, \beta \geq 0. \qquad (10)$$

Let us note that if in the above formula the coefficients fulfill $\alpha=\beta=1$ then we get Tanimoto's formula and if $\alpha=\beta=\dfrac{1}{2}$ then we get Dice's formula. Here and in the sequel we shall assume that $\alpha=\beta=\dfrac{1}{4}$.

Note that the defined above measures of similarity, take a value between [0, 1], **are dependent on the ordering of features** in case once represents the object by a radar chart.

Another simple way of visualizing the structure $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a bar graph, in which each coordinate is represented as a rectangle of width 1 and height x_i (for i=1,…,n). The area of such graph is equal to 1 and one of the most popular indicators of similarity of two structures $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ is defined as [Malina 2004]:

$$W_{\mathbf{xy}} := \sum_{i=1}^{n} \min(x_i, y_i) , \qquad (11)$$

It is clear that its value is independent of the ordering of features and, in the case of such graphical representation of structure, takes a value identical to the values of coefficients defined in (6) and (8).

In every situation when the indicator of similarity of two structures that uses a graphical interpretation is not invariant under the permutation of coordinates, we may modify its definition, in a way shown above (see formula (5)). Thus, to define a measure of conformability that would be independent of the ordering of features, let us denote by p_j – the j-th permutation of numbers 1,2,…,n. Naturally, each permutation of coordinates of vectors x and y corresponds to one permutation p_j. Let vectors x_j, y_j denote j-th permutation of coordinates of vectors x and y, respectively. Assume that $x_1=x$, $y_1=y$, for each j-th permutation x_j, y_j of coordinates of vectors x and y corresponds a coefficient of conformability of structures

$$c_{Q,P}^{j} = \widehat{c}_{\mathbf{x}_j\mathbf{y}_j} \quad , \qquad (12)$$

where naturally $c_{Q,P}^1 = \overset{\frown}{c}_{\mathbf{xy}}$, and the cosine similarity $\overset{\frown}{c}_{\mathbf{xy}}$ is defined as in formula (6).

With regard to the above, let us assume the following definitions of three different measures of conformability for objects Q and P

$$c_{Q,P}^M = c_{\mathbf{x,y}}^M = \max_{1 \le j \le n!} c_{Q,P}^j,$$

$$c_{Q,P}^m = c_{\mathbf{x,y}}^m = \min_{1 \le j \le n!} c_{Q,P}^j, \tag{13}$$

$$c_{Q,P}^s = c_{\mathbf{x,y}}^s = \frac{1}{n!} \sum_{j=1}^{n!} c_{Q,P}^j.$$

In a similar manner we can define other coefficients $J_{\mathbf{xy}}^M, J_{\mathbf{xy}}^m, J_{\mathbf{xy}}^s; D_{\mathbf{xy}}^M, D_{\mathbf{xy}}^m, D_{\mathbf{xy}}^s; \mathscr{I}_{\mathbf{xy}}^M, \mathscr{I}_{\mathbf{xy}}^m, \mathscr{I}_{\mathbf{xy}}^s; T_{\mathbf{xy}}^M, T_{\mathbf{xy}}^m T_{\mathbf{xy}}^s$.

In order to demonstrate the presented above method of comparing structures, let us consider a simple example.

Example 1. Let $Q = \mathbf{x} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $R = \mathbf{y} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Let us assume the following denotations:

$$\mathbf{x}_1 := \mathbf{x}_4 := \mathbf{x}, \ \ \mathbf{x}_2 := \mathbf{x}_5 := \left(\frac{1}{2}, 0, \frac{1}{2}\right), \ \mathbf{x}_3 := \mathbf{x}_6 := \left(0, \frac{1}{2}, \frac{1}{2}\right),$$

$$\mathbf{y}_1 := \mathbf{y}_2 := \mathbf{y}_3 := \mathbf{y}_4 := \mathbf{y}_5 := \mathbf{y}_6 = \mathbf{y}.$$

Thus we have:

$$\left|S_{\mathbf{x}_i}\right| = \frac{1}{2}\frac{1}{2}\frac{1}{2}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{16}, \quad \left|S_{\mathbf{y}_i}\right| = 3\frac{1}{2}\frac{1}{3}\frac{1}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{12},$$

$$\left|S_{\mathbf{x}_i} \cap S_{\mathbf{y}_i}\right| = \frac{1}{2}\frac{1}{3}\frac{1}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{36}, \quad \left|S_{\mathbf{x}_i} \cup S_{\mathbf{y}_i}\right| = \frac{17\sqrt{3}}{144},$$

$$R_{\mathbf{x}_i \mathbf{y}_i} = \sqrt{\frac{\sqrt{3}}{36} \Big/ \frac{\sqrt{3}}{16}} = \frac{2}{3}, \text{ dla } i = 1, 2, ..., 6.$$

So $R_{\mathbf{x,y}}^M = R_{\mathbf{x,y}}^m = R_{\mathbf{x,y}}^s = \dfrac{2}{3} =\sim 0,666$, where coefficients $R_{Q,P}^M, R_{Q,P}^s, R_{Q,P}^m$

are defined as in formulae (5). It can be easily verified that coefficients of conformability of structures: cosine (formula (13)), Jaccard, Dice's are equal to:

$$c_{\mathbf{x,y}}^M = c_{\mathbf{x,y}}^m = c_{\mathbf{x,y}}^s = 0,385; \quad J_{\mathbf{x,y}}^M = J_{\mathbf{x,y}}^m = J_{\mathbf{x,y}}^s = 0,236; \quad D_{\mathbf{x,y}}^M = D_{\mathbf{x,y}}^m = D_{\mathbf{x,y}}^s = 0,381.$$

Note that

$$\sqrt{c_{\mathbf{x,y}}^M} = 0,620; \quad \sqrt{J_{\mathbf{x,y}}^M} = 0,486; \quad \sqrt{D_{\mathbf{x,y}}^M} = 0,617.$$

It is also noteworthy that in this case the coefficient of conformability of structures (defined by formula (11)) is $W_{\mathbf{xy}} = \dfrac{1}{3} + \dfrac{1}{3} + 0 = \dfrac{2}{3}$. The value of the coefficient define by formula (7) or (9) that uses an interpretation of the structure as a bar graph is equal to $0,5 = 0,6(6)/1,3(3)$.

The above example shows that measures of similarity of two objects calculated by different methods (e.g. a method that includes the manner of the graphical representation of the structure or a method of normalizing, which, when applied, causes the measure of the area of the union of faces to take a value between [0, 1]), can be significantly different. A single measure of similarity of objects can be far from optimal in the understanding of a given expert. Furthermore, experts can disagree on the meanings of individual measures. Thus it is safer to use, in the analysis of structures, a measure that is, for example, an average of several different measures of similarity [see: Breiman 1994].

EMPIRICAL RESULTS

In order to verify the approach described in the previous section, we present an evaluation of the size of changes in demographical structures of European countries between the years 1999 and 2000, using the discussed above coefficients.

The following Tables 1 and 2 contain values of indicators evaluating the change of demographical structures for 27 countries between years 1999 and 2010; with an indication what position they occupied in the ranking of values of individual measures as well as two partitions of countries into 4 groups (columns **C1** and **C2**). The partition is made based on the values of indicator M (arithmetic mean of values of indicators **R**, **C**, **J**, **D** and **T**) and indicator W**,** while the thresholds were defined as: *A-d*, *A*, *A+d,* where *A* denotes an average and *d* – standard deviation.

Table 1. Values and rankings of indicators evaluating the similarity of demographical structures of 27 European countries in the years 1999 and 2010. Indicators are defined on the grounds of formulas: **R** - (3), **C** - (6), **J** - (7), **D** - (8), **T**-(10), **M**=(**R+C+J+D+T**)/5**, W** - (11) for the following ordering of age ranges: 0-14, 15-24, 25-49, 50-64, 65-79, >80. The last two columns contain information about the partition into 4 groups, according to values of indicators **M** and **W**, respectively.

No.	country	R	C	J	D	T	M	W	R	C	J	D	T	M	W	C1	C2
1	Austria	0,9676	0,9536	0,9110	0,9534	0,9762	0,9524	0,9630	5	5	5	5	5	5	6	**1**	**1**
2	Belgium	0,9616	0,9427	0,8913	0,9425	0,9704	0,9417	0,9590	9	7	7	7	7	7	8	**2**	**2**
3	Bulgaria	0,9455	0,9038	0,8243	0,9037	0,9494	0,9054	0,9480	14	15	15	15	15	15	11	**3**	**2**
4	Cyprus	0,9319	0,8967	0,8120	0,8963	0,9453	0,8964	0,9320	21	16	16	16	16	17	24	**3**	**4**
5	Czech Republic	0,9347	0,8776	0,7818	0,8776	0,9348	0,8813	0,9370	19	23	23	23	23	23	21	**3**	**3**
6	Denmark	0,9781	0,9606	0,9242	0,9606	0,9799	0,9607	0,9700	1	3	3	3	3	2	3	**1**	**1**
7	Estonia	0,9634	0,9438	0,8933	0,9437	0,9710	0,9430	0,9600	6	6	6	6	6	6	7	**2**	**2**
8	Finland	0,9469	0,9123	0,8385	0,9121	0,9540	0,9128	0,9400	13	14	14	14	14	14	19	**2**	**3**
9	France	0,9504	0,9180	0,8482	0,9178	0,9572	0,9183	0,9480	12	13	13	13	13	13	11	**2**	**2**
10	Germany	0,9624	0,9340	0,8762	0,9340	0,9659	0,9345	0,9470	7	10	10	10	10	10	15	**2**	**3**
11	Greece	0,9364	0,8900	0,8016	0,8899	0,9417	0,8919	0,9450	17	18	18	18	18	18	16	**3**	**3**
12	Hungary	0,9441	0,8950	0,8100	0,8950	0,9446	0,8978	0,9480	16	17	17	17	17	16	10	**3**	**2**
13	Ireland	0,9225	0,8754	0,7779	0,8751	0,9334	0,8769	0,9400	25	24	24	24	24	24	19	**4**	**3**
14	Italy	0,9620	0,9346	0,8771	0,9345	0,9662	0,9349	0,9710	8	9	9	9	9	9	2	**2**	**1**
15	Latvia	0,9587	0,9386	0,8839	0,9384	0,9682	0,9375	0,9520	10	8	8	8	8	8	9	**2**	**2**
16	Lithuania	0,9451	0,9239	0,8577	0,9234	0,9602	0,9220	0,9440	15	12	12	12	12	12	17	**2**	**3**
17	Luxembourg	0,9716	0,9650	0,9318	0,9647	0,9820	0,9630	0,9720	2	1	1	1	1	1	1	**1**	**1**
18	Malta	0,9263	0,8809	0,7867	0,8806	0,9365	0,8822	0,9260	24	22	22	22	22	22	25	**3**	**4**
19	Netherlands	0,9531	0,9276	0,8646	0,9274	0,9623	0,9270	0,9470	11	11	11	11	11	11	13	**2**	**3**
20	Poland	0,9061	0,8464	0,7331	0,8460	0,9166	0,8497	0,9220	27	27	27	27	27	27	26	**4**	**4**
21	Portugal	0,9357	0,8873	0,7973	0,8872	0,9402	0,8895	0,9470	18	19	19	19	19	19	13	**3**	**3**
22	Romania	0,9346	0,8820	0,7888	0,8820	0,9373	0,8849	0,9360	20	21	21	21	21	21	22	**3**	**3**
23	Slovakia	0,9131	0,8531	0,7435	0,8529	0,9206	0,8566	0,9220	26	26	26	26	26	26	27	**4**	**4**
24	Slovenia	0,9288	0,8667	0,7647	0,8667	0,9286	0,8711	0,9330	23	25	25	25	25	25	23	**4**	**4**
25	Spain	0,9299	0,8860	0,7949	0,8857	0,9394	0,8872	0,9440	22	20	20	20	20	20	17	**3**	**3**
26	Sweden	0,9689	0,9594	0,9215	0,9592	0,9792	0,9576	0,9670	4	4	4	4	4	4	5	**1**	**1**
27	United Kingdom	0,9700	0,9625	0,9272	0,9622	0,9808	0,9605	0,9690	3	2	2	2	2	3	4	**1**	**1**

Source: own work

Note that each of the first 5 indicators presented in Table 1, has an identical geometrical interpretation of similarity of structures, an intersection of two hexagons that represent those structures. They differ only by the method used to normalize that area, so that the value of the indicator of similarity is between [0, 1]. That is why all the indicators, with the exception of indicator **R**, they give the same ordering of European countries, according to the similarity of structures for the years 1999 and 2010. Small differences are visible only in the case of indicator **R**. The results do not change if we modify the indicator so that its value is independent of the ordering of coordinates of the vector representing the structure (see. Table 2). On the other hand, differences between the ordering by the value of indicator **W** (based on a different visualization of structures that the rest), and the

ordering by the value of indicator **M** are noticeable. Even more so in the last two columns of Table 2, which represent the partition of countries into 4 groups, according to the similarity of structures for years 1999 and 2010. In case of Spain, we can observe a substantial difference in the assignment to a group depending on the used indicator.

Table 2. Description is similar to that of Table 1. The calculations of individual indicators where performed based on the first formulas (maximum) from (5), (13) and analogous modifications freeing the value of an indicator from the ordering of coordinates of a vector describing a given structure.

No.	country	R^M	C^M	J^M	D^M	T^M	M^M	W	R^M	C^M	J^M	D^M	T^M	M^M	W	C1	C2
1	Austria	0,9737	0,9572	0,9177	0,9571	0,9781	0,9567	0,9630	6	6	6	6	6	6	6	**1**	**1**
2	Belgium	0,9671	0,9427	0,8913	0,9425	0,9704	0,9428	0,9590	10	9	9	9	9	10	8	**1**	**2**
3	Bulgaria	0,9658	0,9360	0,8796	0,9360	0,9669	0,9369	0,9480	13	15	15	15	15	15	11	**2**	**2**
4	Cyprus	0,9592	0,9239	0,8575	0,9233	0,9601	0,9248	0,9320	19	20	21	21	21	19	24	**2**	**4**
5	Czech Republic	0,9582	0,9216	0,8545	0,9215	0,9592	0,9230	0,9370	21	22	22	22	22	22	21	**2**	**3**
6	Denmark	0,9812	0,9658	0,9338	0,9658	0,9826	0,9658	0,9700	2	2	2	2	2	2	3	**1**	**1**
7	Estonia	0,9702	0,9512	0,9066	0,9510	0,9749	0,9508	0,9600	8	7	7	7	7	7	7	**1**	**2**
8	Finland	0,9605	0,9294	0,8673	0,9290	0,9632	0,9299	0,9400	18	18	18	18	18	18	19	**2**	**3**
9	France	0,9590	0,9234	0,8577	0,9234	0,9602	0,9247	0,9480	20	21	20	20	20	20	11	**2**	**2**
10	Germany	0,9668	0,9386	0,8842	0,9385	0,9683	0,9393	0,9470	11	12	12	12	12	12	15	**1**	**3**
11	Greece	0,9629	0,9316	0,8718	0,9315	0,9646	0,9325	0,9450	17	17	17	17	17	17	16	**2**	**3**
12	Hungary	0,9635	0,9318	0,8722	0,9317	0,9647	0,9328	0,9480	15	16	16	16	16	16	10	**2**	**2**
13	Ireland	0,9665	0,9429	0,8918	0,9428	0,9705	0,9429	0,9400	12	8	8	8	8	9	19	**1**	**3**
14	Italy	0,9804	0,9669	0,9357	0,9668	0,9831	0,9666	0,9710	3	1	1	1	1	1	2	**1**	**1**
15	Latvia	0,9632	0,9416	0,8890	0,9412	0,9697	0,9410	0,9520	16	11	11	11	11	11	9	**1**	**2**
16	Lithuania	0,9527	0,9254	0,8598	0,9246	0,9608	0,9247	0,9440	24	19	19	19	19	21	17	**2**	**3**
17	Luxembourg	0,9791	0,9650	0,9318	0,9647	0,9820	0,9645	0,9720	4	3	4	4	4	4	1	**1**	**1**
18	Malta	0,9491	0,9097	0,8340	0,9095	0,9526	0,9110	0,9260	25	26	26	26	26	26	25	**3**	**4**
19	Netherlands	0,9653	0,9379	0,8831	0,9379	0,9680	0,9384	0,9470	14	13	13	13	13	14	13	**1**	**3**
20	Poland	0,9468	0,9031	0,8233	0,9031	0,9491	0,9051	0,9220	27	27	27	27	27	27	26	**3**	**4**
21	Portugal	0,9680	0,9376	0,8826	0,9376	0,9678	0,9387	0,9470	9	14	14	14	14	13	13	**1**	**3**
22	Romania	0,9548	0,9172	0,8470	0,9171	0,9568	0,9186	0,9360	22	23	23	23	23	23	22	**3**	**3**
23	Slovakia	0,9544	0,9118	0,8378	0,9118	0,9538	0,9139	0,9220	23	25	25	25	25	25	27	**3**	**4**
24	Slovenia	0,9482	0,9151	0,8416	0,9140	0,9551	0,9148	0,9330	26	24	24	24	24	24	23	**3**	**4**
25	Spain	0,9705	0,9425	0,8913	0,9425	0,9704	0,9434	0,9440	7	10	10	10	10	8	17	**1**	**3**
26	Sweden	0,9814	0,9649	0,9322	0,9649	0,9822	0,9651	0,9670	1	4	3	3	3	3	5	**1**	**1**
27	United Kingdom	0,9790	0,9634	0,9293	0,9634	0,9813	0,9633	0,9690	5	5	5	5	5	5	4	**1**	**1**

Source: own work

Tables 1 and 2 show that the greatest stability of the demographical structure between 1999 and 2010 was possessed by: Austria, Denmark, Luxembourg, Sweden and United Kingdom. On the other hand, the greatest changes were observed in: Cyprus, Malta, Poland, Slovakia and Slovenia. The greatest change occurred in Poland, and the smallest one in Luxembourg.

SUMMARY

Means for defining the values of indicators of similarity that use geometrical interpretations in the form of a value of an area and are described in this work can also be used in other geometrical ways of studying the similarity of structures as well as objects. These ways are an example of applying geometrical methods that are introduced by the authors using radar charts [Binderman, Borkowski, Szczesny 2008, 2010]. The empirical analysis shows that when structures are not subject to large changes then the values of individual indicators, based on the same geometrical interpretation, they order the structures similarly. However, if we change the way of visualizing the similarity (the geometrical interpretation) then we see changes in ordering. That is the reason why it is advisable to use several different indicators that use different means of visualization.

Furthermore, it is worth noting that by using geometrical interpretation as a basis to construct an indicator of similarity we can obtain an indicator that is very sensitive to changes in the ordering of coordinates of a vector that numerically represents a given structure. In practice there may be situations in which a researcher desires such quality in an indicator so it may visibly highlight even small differences between structures, but for a given ordering of their components. However, one needs to remember that methods of constructing indicators of similarity that use a geometrical interpretation are often applied mainly because of the ease of visualization of multidimensional data. Then an unseasoned researcher may misuse them. It must be highlighted that indicators based solely on those illustrations do not satisfy – often posed in the literature on this subject – the basic requirement of stability of the used method [see Jackson 1970], that means the independence of the ordering of features. Techniques presented by the authors show how a definition of an indicator must be modified (the method of measurement) to remove this flaw. Techniques that were pointed out may seem numerically complex; nevertheless, in the age of computers that problem became insignificant. On the other hand, this simple and stable empirical example shows that by applying modifications, that is making the measurement of similarity independent of the ordering of individual components of the structure, we obtain different results (see Tables 1 and 2, e.g., Spain).

The measurement of similarity of structures based on geometrical interpretation becomes even more complicated when a researcher is interested in changes that occurred in a given structures during the whole studied period and not only between the beginning and the end of the sample. Further works on this subject can be found in the work Binderman and Szczesny 2011.

REFERENCES

Binderman Z., Borkowski B., Szczesny W. (2008) O pewnej metodzie porządkowania obiektów na przykładzie regionalnego zróżnicowania rolnictwa, Metody ilościowe w badaniach ekonomicznych, IX, wyd. SGGW, 39-48.

Binderman Z., Borkowski B., Szczesny W (2009) O pewnych metodach porządkowych w analizie polskiego rolnictwa wykorzystujących funkcje użyteczności, Roczniki Nauk Rolniczych PAN, Seria G, Ekonomika Rolnictwa , T. 96, z. 2, 77-90.

Binderman Z., Borkowski B., Szczesny W. (2009a) Tendencies in changes of regional differentiation of farms structure and area Quantitative methods in regional and sectored analysis / sc. ed. D. Witkowska D., Łatuszyńska M., U.S., Szczecin, 33-50.

Binderman Z., Borkowski B., Szczesny W. (2010) Radar measures of structures' conformabilty, Quantitative methods in economy, v. XI, No.1, 45-59.

Binderman Z., Borkowski B., Szczesny W. (2010a) The tendencies in regional differentiation changes of agricultural production structure in Poland, Quantitative methods in regional and sectored analysis , U.S., Szczecin, 67-103.

Binderman Z., Borkowski B., Szczesny W. (2010b) Wykorzystanie metod geometrycznych do analizy regionalnego zróżnicowania kultury na wsi, Seria T. XII, z. 5, 25-31.

Binderman, Z., Borkowski B., Szczesny W. (2010c) Regionalne zróżnicowanie turystyki w Polsce w latach 2002–2008, Oeconomia 9 (3) 71-82.

Binderman Z., Borkowski B., Szczesny W. (2010d) Regionalnego zróżnicowanie kultury między wsią a miastem w latach 2003-2008, Między dawnym a nowym – na szlakach humanizmu, wyd. SGGW, 345-360.

Binderman Z., Szczesny W. (2009) Arrange methods of tradesmen of software with a help of graphic representations Computer algebra systems in teaching and research, Siedlce, Wyd. WSFiZ, 117-131.

Binderman Z., Szczesny W., (2011) Comparative Analysis of Computer Techniques for Visualization Multidimensional Data, Computer algebra systems in teaching and research, Siedlce, wyd. Collegium Mazovia, 243-254.

Binderman Z. (2009) Syntetyczne mierniki elastyczności przedsiębiorstw, Kwartalnik Prace i Materiały Wydziału Zarządzania Uniwersytetu Gdańskiego, nr 4/2, 257-268.

Binderman, Z. (2009a) Ocena regionalnego zróżnicowania kultury i turystyki w Polsce w 2007 roku Roczniki Wydziału Nauk Humanistycznych, SGGW, T XII, 335-351.

Borkowski B., Szczesny W. (2002) Metody taksonomiczne w badaniach przestrzennego zróżnicowania rolnictwa. Warszawa, RNR, Seria G, T 89, z. 2. 42-53.

Breiman L. (1994) Bagging predictors, Technical Report 420, Departament of Statistics, University of California, CA, USA, September 1994.

Ciok A., Kowalczyk T., Pleszczyńska E., Szczesny W. (1995) Algorithms of grade correspondence-cluster analysis. The Collected Papers on Theoretical and Aplied Computer Science, 7, 5-22.

Deza E., Deza M.M. (2006) Dictionary of Distances, Elsevier.

Dice Lee R. (1945) Measures of the Amount of Ecologic Association Between Species, Ecology, Ecological Society of America, Vol. 26, No. 3, 297-302.

Duda R. O., Hart P. E., Stork D. G. (2000) Pattern Classification. John Wiley & Sons, Inc., 2nd ed.

Gordon, A.D. (1999) Classification, 2nd edition, London-New York.

Hubalek Z. (1982) Coefficients of Association and Similarity, Based on Binary (Presence-Absence) Data: An Evaluation, Biological Reviews, Vol.57-4, 669-689.

Jaccard P. (1901) Étude comparative de la distribution florale dans une portion des Alpes et des Jura. Bulletin del la Société Vaudoise des Sciences Naturelles 37, 547-579.

Jaccard, P. (1902) Lois de distribution florale dans la zone alpine. Bull. Soc. Vaud. Sci. Nat. 38, 69-130.

Jaccard, P. (1908) Nouvel les recherches sur la distribution floral e. Bull. Soc. Vaud. Sci. Nat. 44, 223-270.

Jackson D. M. (1970) The stability of classifications of binary attribute data, Technical Report 70-65, Cornell University, 1-13.

Kukuła K. (2000) Metoda unitaryzacji zerowej, PWN, Warszawa.

Kukuła K. (red) (2010) Statystyczne studium struktury agrarnej w Polsce, PWN, Warszawa.

Legendre P., Legendre L. (1998) Numerical Ecology, Second English Edition Ed., Elsevier.

Malina A. (2004) Wielowymiarowa analiza przestrzennego zróżnicowania struktury gospodarki Polski według województw. AE, Seria Monografie nr 162, Kraków.

Monev V. (2004) Introduction to Similarity Searching in Chemistry, MATCH Commun. Math. Comput. Chem. 51, 7-38.

Mostowski A., Stark M.: Elementy algebry wyższej, PWN, Warszawa.

Szczesny W. (2002) Grade correspondence analysis applied to contingency tables and questionnaire data, Intelligent Data Analysis, vol. 6, 17-51.

Tan P., Steinbach M., Kumar V. (2006) Introduction to Data Mining, Pearson Education, Inc.

Tanimoto, T.T. (1957) IBM Internal Report 17th Nov. 1957

Tanimoto T.T. (1959) An Elementary Mathematical Theory of Classification and Prediction, IBM Program IBCFL.

Tversky A. (1957) Features of similarity. Psychological Review, 84(4) 327-352.

Warrens M. J. (2008) On Association Coefficients for 2×2 Tables and Properties that do not depend on the Marginal Distributions, Psychometrika Vol. 73, n. 4, 777–789.

SOME REMARKS ON APLICATIONS
OF ALGEBRAIC ANALYSIS TO ECONOMICS

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Abstract: In this paper, the author continues the investigations started in his earlier work [Binderman 2009]. Here, problems of linear equation Dx=y with the difference operator D are studied. The work is an introduction to applications of the theory of right invertible operators to economics. As an example, quotations of KGHM on Warsaw Stock Exchange are considered.

Key words: algebraic analysis, right invertible operator, difference operator, quotations of Stock Exchange, Jacobian matrix

In memory of Professor Krystyna Twardowska

INTRODUCTION

In mathematics the term Algebraic Analysis is used in two completely different senses [cf. Przeworska - Rolewicz 2000]. Here, meaning of Algebraic Analysis is closely connected with theory of right invertible operators [cf. Przeworska - Rolewicz 1988]. In the earlier work of the author [Binderman 2009] a new definition of elasticity operators in algebras with right invertible operators was proposed. The definition uses logarithmic mappings of algebraic analysis [cf. Przeworska-Rolewicz 1998]. The obtained results were applied to economics in order to find a function if elasticity of this function is given.

Here, possibilities of applications of algebraic analysis to economics, on the simple example of the difference operator D and the linear equation Dx=y are presented. The paper is an introduction in this range.

Throughout this work $\mathbb{F}$ will denote either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$. Let X and Y be a linear space over $\mathbb{F}$. The set of all linear operators domains contained in X and ranges contained in Y will be denoted by $\mathfrak{L}(X,Y)$. We shall write:

$$\mathfrak{L}_0(X, Y) := \{ A \in \mathfrak{L}(X, Y) : \mathrm{dom}\, A = X \}, \quad \mathfrak{L}(X) := \mathfrak{L}(X, X),$$

$$\mathfrak{L}_0(X) := \mathfrak{L}_0(X, X), \quad \ker A : \{ x \in \mathrm{dom}\, A : Ax = 0 \} \text{ for } A \in \mathfrak{L}(X, Y).$$

Following D. Przeworska - Rolewicz [c.f. Przeworska - Rolewicz 1988], an operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in \mathfrak{L} L_0(X)$ such that $RX \subset \mathrm{dom}\, D$ and $DR=I$. The operator R is called a *right inverse* of D.

We shall consider in $L(X)$ the following sets:

- the set $\mathfrak{R}(X)$ of all right invertible operators belonging to $L(X)$;
- the set $\mathfrak{R}_D := \{ R \in \mathfrak{L}_0(X) : DRx = x \text{ for all } x \in X \}$;
- the set $\mathfrak{F}_D := \{ F \in \mathfrak{L}_0(X) : F^2 = F, FX = \ker D \text{ and } \exists R \in R_D : FR=0 \}$ of all initial operators for a $D \in \mathfrak{R}(X)$.

We note, if $D \in \mathfrak{R}(X)$, $R \in \mathfrak{R}_D$ and $\ker D \neq \{0\}$, then the operator D is right invertible, but not invertible. We have

$$DRx = x \text{ for all } x \in X \quad \text{and} \quad \exists x \in \mathrm{dom}\, D: RDx \neq x.$$

Here, the invertibility of an operator $A \in \mathfrak{L}(X)$ means that the equation $Ax=y$ has the unique solution for every $y \in X$. If $D \in \mathfrak{R}(X)$ and $0 \neq z \in \ker D$ and x_1 is a solution of the equation $DX=y$ then the element $x_1 + z$ is also the solution of this equation.

If F is an initial operator for D corresponding to R then

$$Fx = x - RDx = (I-RD)x \text{ for } x \in \mathrm{dom}\, D \text{ and } Fz=z \text{ for } z \in \ker D. \tag{1}$$

We note, a different approach to the definition of right invertible linear operators is presented in the work [Binderman 2009].

In the sequel we shall assume that $D \in \mathfrak{R}(X)$, $R \in \mathfrak{R}_D$, $F \in \mathfrak{F}_D$ is an initial operator for D corresponding to R and $\dim \ker D > 0$, i.e. D is right invertible but not invertible.

We observe, that if we know one right inverse of D then the sets [c.f. Przeworska - Rolewicz 1988]

$$\mathfrak{R}_D = \{ R + FA : A \in \mathfrak{L}_0(X) \}; \tag{2}$$

$$\mathfrak{F}_D = \{ F(I-AD) : A \in \mathfrak{L}_0(X) \}. \tag{3}$$

We shall need the two following theorems [c.f. Przeworska - Rolewicz 1988].

Theorem 1. The general solution of the equation

$$Dx = y, \quad y \in X, \tag{4}$$

is given by the formula

$$x = z + Wy, \tag{5}$$

where $z \in \ker D$ is arbitrary and $W \in \mathcal{R}_D$ is arbitrarily fixed.

Theorem 2. The initial value problem

$$Dx = y, \quad y \in X, \tag{4}$$

$$Fx = z_0, \quad z_0 \in \ker D, \tag{6}$$

has the unique solution of the form

$$x = z_0 + Ry, \tag{7}$$

where $R \in \mathcal{R}_D$ is the right inverse of operator D, corresponding to F.

We consider the several examples of operators which are often present in economics.

EXAMPLE 1. [cf. Przeworska – Rolewicz 1988, Bittner 1974, Levy, Lessman 1959, Gelfond 1957]. We suppose that X is the set of all sequences $\mathbf{x} = \{x_n\}$, where $x_n \in \mathbb{R}$, $n \in \mathbb{N} = \{1, 2, \dots\}$ with addition and multiplication by scalars defined as follows: if $\mathbf{x} = \{x_n\}$, $\mathbf{y} = \{y_n\}$, $\lambda \in \mathbb{R}$ then $\mathbf{x} + \mathbf{y} = \{x_n + y_n\}$, $\lambda \mathbf{x} = \{\lambda x_n\}$. Define the difference operators by the equalities:

$$D\mathbf{x} = \{x_{n+1} - x_n\}, \ \mathbf{x} = \{x_n\},$$

$$R\mathbf{x} = \{0, x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots\}.$$

For $\mathbf{x} = \{x_n\} \in X$ we have:

$$DR\mathbf{x} = D\{0, x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots\} = \{x_1 - 0, x_1 + x_2 - x_1, x_1 + x_2 + x_3 - x_2 - x_1, \dots\} =$$
$$= \{x_1, x_2, x_3, \dots\} = \mathbf{x}.$$
$$RD\mathbf{x} = R\{x_2 - x_1, x_3 - x_2, x_4 - x_3, \dots\} = \{0, x_2 - x_1, x_2 - x_1 + x_3 - x_2, x_2 - x_1 + x_3 - x_2, x_2 -$$
$$x_1 + x_3 - x_2 + x_4 - x_3, \dots\} = \mathbf{x} - x_1 \mathbf{e} \neq \mathbf{x} \text{ for } x_1 \neq 0,$$

where $\mathbf{e} := \{1, 1, 1, 1, \dots\}$.

The above shows that D is not invertible. The kernel of the operator D has the form:

$$\ker D = \{\mathbf{z} = \{z_n\} : z_n = c, n \in \mathbb{N}, c \in \mathbb{R}\}.$$

By Theorem 1, the general solution of the equation (4) is of the form

$$\mathbf{x} = \{x_n\} = \{c, c + x_1, c + x_1 + x_2, c + x_1 + x_2 + x_3, \dots\}, \tag{8}$$

where $c \in \mathbb{R}$ is arbitrarily fixed.

We observe that the initial operator F for D corresponding to R is defined by the formula:

$$F\mathbf{x} = (I - RD)\mathbf{x} = \mathbf{x} - (\mathbf{x} - x_1 \mathbf{e}) = x_1 \mathbf{e} \in \ker D, \ \mathbf{x} = \{x_n\} \in X.$$

EXAMPLE 2. [cf. Przeworska-Rolewicz 1988]. We denote by X=C[a,b] the set of all real-valued function defined and continuous on a closed interval [a,b]. The set X is a linear space over the field of real numbers $\mathbb{R}$ if the addition and multiplication by a number are defined as follows: (x+y)(t)=x(t)+y(t); $(\alpha x)(t)=$ $\alpha x(t)$ for x,y∈X, t∈[a,b], $\alpha \in \mathbb{R}$. Similarly properties has the set $C^1[a,b] \subset X$ of all real-valued functions defined on a closed interval [a,b] and having continuous derivative in (a,b). Suppose that we are given a point $t_0 \in$[a,b] and c is an arbitrary fixed real number. We define operators as follows:

(Dx)(t):= x'(t) for x∈C^1[a,b]⊂X; t∈[a,b],

$$(Rx)(t)=\int_{t_0}^{t} x(\tau)d\tau \text{ for x∈X; t∈[a,b].}$$

The definition of D and R implies that R∈ $\mathscr{R}_D$ and

(Fx)(t)=[(I-RD)x](t) =x(t_0) on dom D= C^1[a,b].

EXAMPLE 3. [cf. Binderman 1992, 1993, 2000] Suppose that X is defined as in Example 2, where 0∈(a,b). Let $C_0^1[a,b] \subset X$ denotes the set of all real-valued function defined on a closed interval [a,b] and having continuous derivative in the point 0. We define operators as follows:

$$(Dx)(t) = \begin{cases} \dfrac{x(t)-x(0)}{t} & \text{for } t \in [a,b] \\[2mm] x'(0) & \text{for } t = 0 \end{cases}, x \in C_0^1[a,b].$$

(Rx)(t)=tx(t), x∈X; t∈[a, b]

The operator D is called a Pommiez operator or a backward shift operator [Dimovski 1990, Dimovski 2005, Douglas, Shapiro, Shields 1970, Fage, Nagnibida 1987, Linchuk 1988].The definition of D, R implies that R∈ $\mathscr{R}_D$ and

(Fx)(t) =[(I-RD)x](t) =x(0).

EXAMPLE 4. [cf. Binderman 2009]. In similar way as in Example 2 we denote by X=C[a,b], where a>0, the set of all real-valued function defined and continuous on

a closed interval [a,b]. Suppose that we are given a point $t_0 \in$[a,b]. We define the operators D as follows:

$$(Dx)(t):=tx'(t) \quad \text{for } x \in C^1[a,b] \subset X; \ t \in [a,b].$$

$$(Rx)(t) = \int_{t_0}^{t} \frac{x(\tau)}{\tau} d\tau \quad \text{for } x \in X; \ t \in [a,b].$$

The operator R is well-defined for all continuous functions. The definition of R implies that $RX \subset$ dom D= C^1[a,b] and

$$(DRx)(t) = D(Rx(t)) = t\frac{d}{dt}\left[\int_{t_0}^{t} \frac{x(\tau)}{\tau} d\tau \right] = t \ \frac{x(t)}{t} = x(t), \quad \text{for all } x \in X \ ; \ t \in [a,b].$$

The operator D is right invertible but not invertible since

$$(RDx)(t) = \int_{t_0}^{t} \frac{(Dx)(\tau)}{\tau} \ d\tau = \int_{t_0}^{t} \frac{\tau x'(\tau)}{\tau} \ d\tau = \ x(t) - x(t_0) \quad \text{for } x \in \text{ dom D.}$$

In this case, the operator defined as follows

$$(Fx)(t):= x(t) - (RDx)(t) = ((I - RD)x)(t) = x(t_0) \text{ for } x \in C^1[a,b]; \ t \in [a,b]$$

is an initial operator of D corresponding to the right inverse R of D.

We note, in the work of the author [Binderman 2009] the operator D was used to construct an operator of elasticity.

EXAMPLE 5. [cf. Binderman 2000]. Suppose that X is defined as in Example 2. We define the family of operators D_h as follows:

$$(D_h x)(t) := \begin{cases} h\dfrac{x(t) - x(h)}{t - h} \ \text{for } t \neq h, \\ \\ \qquad\qquad\qquad\qquad h \in (a,b), x \in X, \\ hx'(h) \qquad \text{for } t = h, \end{cases}$$

and

$$(R_h x)(t) := \frac{t - h}{h}, 0 \neq h \in (a,b).$$

We can prove, the operators D_h are right invertible, $R_h \in \mathfrak{R}_{D_h}$ and

$$(F_h x)(t) = [(I - R_h D_h)x](t) = x(h) \quad \text{for all} \quad 0 \neq h \in (a,b), \; x \in X.$$

DIFFERENCE EQUATIONS

PROBLEM. Suppose that X, D, R is defined as in Example 1. Let a sequence $\mathbf{x} = \{x_n\} \in X$ denotes a point series or time series. We set the problem of finding elements of $\mathbf{x}$ if we know the members: $y_1, y_2, ..., y_m$ of the sequence $\mathbf{y} = \{y_n\}$, under the condition $D\mathbf{x} = \mathbf{y}$.

If x_1 is given then by the definition of the operator D, only we receive:

$$x_2 = y_1 + x_1, \; x_3 = y_2 + y_1 + x_1, .., \; x_{m+1} = x_1 + \sum_{i=1}^{m} y_i \, .$$

Clearly, by Theorem 2 we obtain the same result

$$\mathbf{x} = x_1 \mathbf{e} + R\mathbf{y} = \{x_1, x_1, ..., x_1, ...\} + \{0, y_1, y_1 + y_2, y_1 + y_2 + y_3, ...\}.$$

Let us consider the following problem: find a solution of the equation

$$D\mathbf{x} = \mathbf{y}, \tag{4}$$

which satisfies the linear condition

$$(\mathbf{a}, \mathbf{x}) = b, \tag{9}$$

where $(\mathbf{a}, \mathbf{x}) := \sum_{i=1} a_i x_i$, $\mathbf{a}, \mathbf{x}, \mathbf{y} \in X$; $b \in \mathbb{R}$.

In order to determine the members $x_1, x_2, ..., x_m$ of a sequence $\mathbf{x}$, we assume that elements $a_1, a_2, ..., a_m$; $y_1, y_2, ... y_{m-1}$ of the sequences $\mathbf{a}$, $\mathbf{y}$, respectively are known and

$$\sum_{i=1}^{m} a_i \neq 0; \; a_j = 0 \quad \text{for} \quad j > m \in \mathbb{N} \, .$$

We can prove that the problem is equivalent to the initial value problem

$$D\mathbf{x} = \mathbf{y}, \tag{4}$$

with the condition

$$F\mathbf{x} = \mathbf{z}_0, \text{ where } \quad \mathbf{z}_0 := \frac{b}{\sum\limits_{i=1}^{m} a_i} \mathbf{e} \in \ker D, \; \mathbf{e} = \{1, 1, ...\}. \tag{10}$$

It is easy to check that F is the initial operator for D corresponding to the operator

$$R\mathbf{y} = R\{y_n\} = \{u_n\} := \begin{cases} u_1 = \dfrac{-1}{\displaystyle\sum_{i=1}^{m} a_i} \sum_{i=2}^{m} \left(\sum_{j=i}^{m} a_j \right) y_{i-1} \\ \\ u_n = u_1 + \displaystyle\sum_{j=1}^{n-1} y_j \ \text{for}\ n > 1 \end{cases}. \tag{11}$$

By the Theorem 2 we receive that $\mathbf{x} = \mathbf{z}_0 + R\mathbf{y}$. Hence we receive the following theorem.

Theorem 3. The unique solution of the problem (4), (9) is determined by the following formula:

$$x_1 = \dfrac{b - \displaystyle\sum_{i=2}^{m} \left(\sum_{j=i}^{m} a_j \right) y_{i-1}}{\displaystyle\sum_{i=1}^{m} a_i}, \ x_n = x_1 + \sum_{j=1}^{n-1} y_j \ \text{for}\ n = 2,3,...,m. \tag{12}$$

We consider special cases of the problem (4), (9). The above result implies the following conclusions.

Remark 1. [see also Przeworska-Rolewicz 1988] Let the condition (9) has the form: $a_p x_p = b, \ a_p \neq 0, \ p \in [1, m] \cap \mathbb{N}$. By formulas (10), (11) we obtain that the operator $F_{(p)}\mathbf{x} := \{b/a_p\}$; $\mathbf{x} \in X$ is the initial operator for D corresponding to the operator

$$R_{(p)}\mathbf{x} = R_{(p)}\{x_n\} := \begin{cases} -\displaystyle\sum_{i=n}^{p-1} x_i & \text{for}\ n < p \\ 0 & \text{for}\ n = p, \ \mathbf{x} \in X. \\ \displaystyle\sum_{i=1}^{n-p} x_{p+i-1} & \text{for}\ n > p \end{cases}$$

By Theorem 3 we obtain that the solution of the considering problem is determined by the formula

$$\mathbf{x} = \frac{b}{a_p}\mathbf{e} + R_{(p)}\mathbf{y}.$$

Hence, the first m members of the sequence $\mathbf{x} = \{x_n\} \in X$ is determined by the following formula:

$$
x_n = \begin{cases}
\dfrac{b}{a_p} - \displaystyle\sum_{i=n}^{p-1} y_i & \text{for } n = 1, 2..., p-1, \\[2em]
\dfrac{b}{a_p} & \text{for } n = p, \\[2em]
\dfrac{b}{a_p} + \displaystyle\sum_{i=1}^{n-p} y_{p+i-1} & \text{for } n = p+1, p+2, ..., m
\end{cases}
$$

Remark 2. [see also Przeworska - Rolewicz 1988] Let the condition (9) has the form: $\displaystyle\sum_{i=1}^{k} x_i = k \bar{x}_k, k \in [2, m]$, where $\bar{x}_k \in \mathbb{R}$ is given. As it was pointed out in the introduction, if $D \in \mathfrak{R}(X)$, then there is the set of right inverses, determined by Formula (2). Let $\mathbf{x} = \{x_n\} \in X$, we define the operator

$$
R_k \mathbf{x} := \mathbf{y} = \{y_n\} = \begin{cases}
\dfrac{-1}{k} \displaystyle\sum_{i=1}^{k} (k-i) x_i, & \text{for } n = 1, \\[2em]
y_1 + \displaystyle\sum_{i=1}^{n-1} x_i & \text{for } n > 1,
\end{cases}
$$

It is easy to check that $R_k \in \mathfrak{R}_D$ and the operator

$$
F_k \mathbf{x} := \left\{ \frac{1}{k} \sum_{i=1}^{k} x_i \right\} = \bar{x}_k \mathbf{e} \in \ker D,
$$

is the initial operator for D corresponding to the operator R_k. Hence, the first k members of the sequence $\mathbf{x} = \{x_n\} \in X$ are determined by the following formula:

$$
x_n = \begin{cases}
\bar{x}_k - \dfrac{1}{k} \displaystyle\sum_{i=1}^{k} (k-i) y_i & \text{for } n = 1, \\[2em]
x_1 + \displaystyle\sum_{i=1}^{n-1} y_i & \text{for } n = 2, 3, ..., k,
\end{cases}
\tag{13}
$$

EXAMPLE 6. We consider the quotations (in pln) of KGHM (KGHM Polska Miedź S.A.) on Warsaw Stock Exchange for the period 1 August 2011 – 12 August 2011. Let $x_1, x_2, ..., x_{10}$ be unknown daily quotations of KGHM in 1, 2, 3, 4, 5, 8, 9, 10, 11, 12 of August 2011, respectively. We can check that the average of the

quotations $\overline{x}_{10} := \dfrac{1}{10} \displaystyle\sum_{i=1}^{10} x_i = 170{,}7;$ the differences between the quotations $y_i = x_{i+1} - x_i,$ i=1,2,…,9 are presented in Figure 1.

Figure 1. Differences between the daily quotations of KGHM on 2,…, 12.08.2011 (in pln)
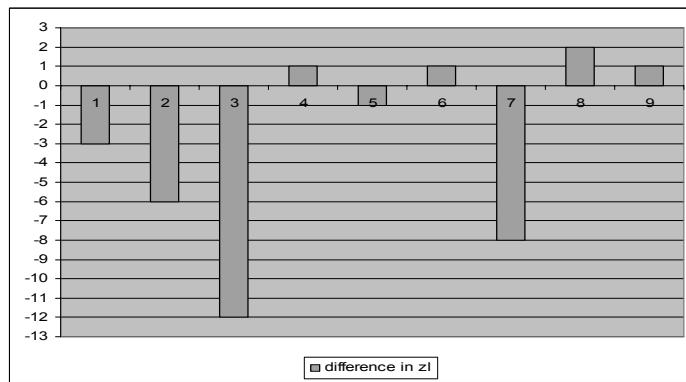
 Source: own calculations

Using Theorem 3, we receive by Formula 12 the daily quotations of KGHM in August 2011, which is presented in the last row of Table 1.

Table 1. Daily quotations of KGHM on 1, 2,…, 12.08.2011 (in pln)

	1	2	3	4	5	6	7	8	9	10
	1.08	2.08	3.08	4.08	5.08	8.08	9.08	10.08	11.08	12.08.
y		-3	-6	-12	1	-1	1	-8	2	1
x	188	185	179	167	168	167	168	160	162	163

Source: own calculations

Remark 3. We consider the equation (4) D**x**=**y** with the condition

$$\varphi(\mathbf{x}) = \varphi(x_1, x_2, \ldots, x_m) = 0, \qquad\qquad (14)$$

 where the differentiable mapping $\varphi\colon \mathbb{R}^m \to \mathbb{R}, m \in \mathbb{N}$ and $y_1, y_2, \ldots, y_{m-1} \in \mathbb{R}$ are given. It is easy to observe that the considered problem is equivalent to the following system of equations

$$\begin{cases} \varphi(x_1, x_2, \ldots, x_m) = 0 \\ x_2 - x_1 = y_1 \\ ----- \\ ----- \\ x_m - x_{m-1} = y_{m-1} \end{cases} \qquad . \qquad\qquad (15)$$

The Jacobian matrix of the above system is of the form

$$J(x_1, x_2, \ldots, x_m) = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_1} & \dfrac{\partial \varphi}{\partial x_2} & - & - & - & \dfrac{\partial \varphi}{\partial x_m} \\ -1 & 1 & 0 & - & - & 0 \\ 0 & -1 & 1 & 0 & - & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & -1 & 1 \end{bmatrix} .$$

We can check that the determinant of J - Jacobian is equal

$$|J| = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}.$$

Under some additional conditions, we can prove [cf. Sikorski 1969] the theorem.
If the Jacobian $|J|$ does not vanish at a point $\mathbf{x}_0 \in \mathbb{R}^m$, then in a neighborhood of
the point $\mathbf{x}_0$ there exists an unique solution of the system (15).
      We observe that in special case, when

$$\varphi(x_1, x_2, \ldots, x_m) \equiv m\sigma^2 - \sum_{i=1}^{m} \left(x - \overline{x}_m \right)^2 = 0 ,$$

where $\sigma > 0$, $\overline{x}_m = \dfrac{1}{m} \sum_{j=1}^{m} x_j$ are given, then the problem has not a solution or

has the solutions determined by the formula:

$$x_n = \begin{cases} c & \text{for } n = 1, \\ c + \sum_{i=1}^{n-1} y_i & \text{for } n = 2, 3, \ldots, m, \end{cases} \qquad (16)$$

where $c \in \mathbb{R}$ is arbitrarily fixed.
Indeed, the Jacobian

$$|J| = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} = \sum_{i=1}^{m} 2\left(x_i - \bar{x}_m\right)\left(1 - \frac{1}{m}\right) = \frac{2(m-1)}{m}\left(\sum_{i=1}^{m} x_i - m\bar{x}_m\right) = 2(m-1)\left(\frac{1}{m}\sum_{i=1}^{m} x_i - \bar{x}_m\right) = 0.$$

On the other hand, the members $y_1, y_2, \ldots, y_{m-1}$ determine the standard deviation σ of the elements $x_1, x_2, \ldots, x_m$. Indeed, by the formula (8) we have

$$x_1 - \bar{x}_m = x_1 - \frac{1}{m}\sum_{i=1}^{m} x_i = x_1 - \frac{1}{m}x_1 - \frac{1}{m}\left(\sum_{i=2}^{m}\left(x_1 + \sum_{j=1}^{i-1} y_j\right)\right) = \frac{-1}{m}\sum_{i=2}^{m}\sum_{j=1}^{i-1} y_j,$$

$$x_i - \bar{x}_m = x_1 + \sum_{j=1}^{i-1} y_j - \frac{1}{m}\sum_{j=1}^{m} x_j = \sum_{j=1}^{i-1} y_j - \frac{1}{m}\sum_{j=2}^{m}\sum_{k=1}^{j-1} y_k.$$

The above shows that standard deviation σ is entirely determined by the known elements $y_1, y_2, \ldots, y_{m-1}$.

We observe, the minor of the (m-1)th order of the element $\partial \varphi / \partial x_1$ of the determinant $|J|$

$$\begin{vmatrix} 1 & 0 & 0 & - & - & 0 \\ -1 & 1 & 0 & 0 & - & 0 \\ 0 & -1 & 1 & 0 & - & - \\ 0 & 0 & -1 & 1 & - & 0 \\ - & - & - & - & - & - \\ 0 & 0 & - & 0 & -1 & 1 \end{vmatrix} = 1 \neq 0.$$

By the Kronecker–Cappelli theorem we receive that considered problem has not a solution or the problem has infinite number of solutions, determined by Formula (16).

Remark 4. If for the considered problem the members $y_1, y_2, \ldots, y_{m-1}$ of a sequence $\mathbf{y} = \{y_n\}$ are known only. For example, next members of the $\mathbf{y}$ we can determine by the one of the simplest extrapolation formulas [Ralston 1965]:

$$y_n = 2y_{n-1} - y_{n-2} \text{ - linear,}$$

$$y_n = \frac{1}{2}\left(5y_{n-1} + y_{n-3} - 4y_{n-2}\right) \text{ - quadratic,}$$

where $n = m, m+1, \ldots,$.

Clearly, we can use the Newton interpolation formula [Ralston 1965].

CONCLUSION

It seems to the author that presented here considerations will permit us to use methods of Algebraic Analysis in much more complicated cases. The author intends to show other applications to economics with another right invertible operators, equations and conditions. Mathematical theory of right invertible operators provides good tools for solving these problems.

REFERENCES

Agarwal R. P. (20000) Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York.

Binderman Z. (1992) Functional shifts induced by right invertible operators, Math. Nachr. 157, 211-224.

Binderman Z. (1993) An unified approach to shifts induced by right invertible operators, Math. Nachr. 161, 239-252.

Binderman Z. (1994) On some functional shifts induced by operators complex differentiation, Opuscula Mathematica, Vol. 14, 53-64.

Binderman Z. (2000)On right inverses for functional shifts, Demonstratio Mathematica, Vol. XXXIII, No 3, 665-678.

Binderman Z. (2009) On elasticity operators and their economical applications, Polish Journal an Environment Study, Vol. 18, no. 5B, 2009, s. 29-34.

Bittner R. (1974) Rachunek operatorów w przestrzeniach liniowych, PWN Warszawa.

Dimovski I., H. (1990) Convolutional Calculus, Kluwer Acad. Publishers, Dordrecht.

Dimovski I., H., Hristov V., Z. (2005) Commutans of the Pommiez operator, IJMMS, Hindawi Publishing Corporation, 1239-1251.

Douglas R.G., Shapiro H.S., Shields A.L. (1970) Cyclic vectors and invariant spaces for backward shift operator, Ann. 1'Inst. Fourier 20, 1, 37-76.

Fage M.K., Nagnibida N.I. (1987) An equivalence problem of ordinary linear differential operators, Nauka, Novosybirsk (in Russian)

Gelfond A.O. (1957) Calculus of finite difference, FizMatGiz, Moskva (Russian)

Levy H., Lessman F. (1959) Finite difference equations, Pitman LTD London.

Linchuk N. (1988) Representation of commutants of the Pommiez operator and their applications. Mat. Zamietki 44, 794-802 (in Russian)

Przeworska-Rolewicz D. (1988) Algebraic Analysis, D. Reidel and PWN –Polish Scientific Publishers, Dordrecht-Warszawa.

Przeworska-Rolewicz D. (1998) Logarithms and antilogarithms. An Algebraic Analysis Approach. With Appendix by Z. Binderman, Kluwer Academic Publishers, Dordrecht.

Przeworska-Rolewicz D. (2000), Two centuries of the term "Algebraic Analysis", Algebraic Analysis and related topics, Banach Center Publications, V.53, 47-70, Warsaw.

Ralston A. (1965) A First Course in Numerical Analysis, McGraw-Hill Book Company.

Sikorski R. (1969) Advanced Calculus, Function of several variables, Monografie Matematyczne, PWN Warszawa.

TRADING VOLUME AND VOLATILITY OF STOCK RETURNS: EVIDENCE FROM SOME EUROPEAN AND ASIAN STOCK MARKETS[♦]

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Abstract: This paper analyses the relationship between the daily volatility of stock returns and the trading volume using the TGARCH models for selected European and Asian stock markets. The leverage effect has been proved in all analysed cases. The logarithm of the trading volume was included into the conditional volatility equation as a proxy for information arrival time. Although in case of all analysed Asian stock returns the inclusion of the trading volume led to the moderate decline of the conditional volatility persistence, the results in case of European stock returns were not so unambiguous.

Key words: volatility, TGARCH model, trading volume, stock returns

INTRODUCTION

The analysis of the stock returns volatility has attracted the interest of investors for a long time. One of the characteristic features of stock returns is that their volatility changes over time. Although this feature has long been recognized (see e.g. [Franses et al. 2000]), the pioneering work in the area of modelling volatility was presented by Engle [Engle 1982] who introduced the autoregressive conditional heteroscedasticity model ARCH. The generalized version of this model, the GARCH model, was first published by Bollerslev [Bollerslev 1986]. The main aim of the ARCH and GARCH models is to capture the time-varying

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volatility. In order to capture some other typical features of financial time series such as asymmetric effect (i.e. the different impact of the positive and negative shocks on the conditional volatility) or long – memory (i.e. variances generated by fractionally integrated processes) a large number of modifications of the standard ARCH and GARCH models has been developed and it is almost impossible to mention all of them (see e.g. [Franses et al. 2000], [Poon et al. 2003], [Rachev et al. 2007]). To capture the asymmetric behaviour of the stock returns e.g. the Engle's [Engle 1990] asymmetric GARCH (AGARCH) model, Nelson's [Nelson 1991] exponential GARCH (EGARCH) model or Zakoian's [Zakoian 1994] threshold GARCH (TGARCH) model[1] can be used. From the long – memory models, the most popular and well known is the fractional integrated GARCH (FIGARCH) model of Baillie, Bollerslev and Mikkelsen [Baillie et al. 1996].

Though the ARCH/GARCH – class models allow the volatility shocks to persist over time, they didn't provide the economic explanation for this phenomenon. The paper [Lamoureux et al. 1990] published in the Journal of Finance offers the explanation for volatility persistence. The authors proved that the daily trading volume, used as a proxy for information flow, has a significant explanatory power regarding the variance of daily returns. For a sample of 20 US actively traded stocks they found out that the GARCH effects disappeared when the trading volume was included into the conditional variance equation. The number of studies documenting the relationship between the stock returns and trading volume is constantly growing. The survey of some empirical studies dealing with this relationship can be found e.g. in [Ghysels et al. 2000], [Girard et al. 2007], [Gursoy et al. 2008], [Poon et al. 2003]. The above mentioned approach presented in [Lamoureux et al. 1990] has been applied in various studies to both individual stocks (stock-level analysis) and stock market indices (market-level analysis). Since the conclusions of the studies applying the approach of Lamoureux and Lastrapes on individual stocks are mostly in coincidence with those presented in [Lamoureux et al. 1990], the results of the market-level analysis are not so unambiguous (see e.g. [Girard et al. 2007], [Sharma et al. 1996]). There are also some papers which proved that the inclusion of the trading volume in conditional variance equation eliminates the ARCH effect for both the individual stocks and the stock index (see e.g. [Miyakoshi 2002]). The analyses in the studies [Girard et al. 2007] and [Gursoy et al. 2008] were done also for the decomposed total volume (into its predictable and unpredictable components) to examine the role of differing trading systems on the relationship between the conditional volatility and the trading volume.

The aim of this paper is to analyse the relationship between the trading volume and the daily volatility of eight European and five Asian stock returns data (i.e. market-level analysis) using the TGARCH models and applying the approach

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[1] TGARCH model is equivalent to the GJR – GARCH model independently presented by Glosten, Jagannathan and Runkle [Glosten et al. 1993].

of Lamoureux and Lastrapes. The rest of the paper is organized as follows. The second section discusses the data and the methodology used in the paper. The third section contains the empirical results of this investigation and the final section provides the concluding remarks.

DATA AND METHODOLOGY

The paper investigates the daily close values of the eight European indices – Austrian ATX, Belgian BEL20, British FTSE100, Dutch AEX, French CAC40, German DAX, Spanish SMSI, Swiss SSMI and five Asian indices – Hong Kong HSI, Indian BSE SENSEX, Indonesian JKSE, Japanese NIKKEI225, Taiwanese TSEC and also the corresponding trading volumes for the period from October, 18 2004 to April, 28 2011[2]. The source of data is Yahoo! Finance [http://finance.yahoo.com] and the number of observations is in individual cases different and spans from 656 in case of DAX to 1671 for AEX. Trading volume is the number of shares traded on a particular day.

The whole analysis was done on logarithmic transformation of daily index returns and daily trading volume. The logarithmic stock returns are calculated as the logarithmic first difference of the daily closing values of the stock indices, i.e.

$$r_t = d(\ln(P_t)) = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

where P_t is the closing value of the stock index at time t and r_t denotes logarithm of the corresponding stock return. The descriptive statistics for the logarithmic stock returns together with values of the Jarque-Bera statistics (J-B) testing the normality, values of the Ljung-Box $Q(k)$ and $Q^2(k)$ statistics testing the uncorrelatedness of the return series and the squared return series till the lag k respectively, and the Augmented Dickey – Fuller (ADF) test statistics testing the existence of the unit root are in Table 1.

Table 1 shows that the mean of logarithmic daily returns ranges between $-8,9.10^{-5}$ (SMSI) and $9,6.10^{-4}$ (JKSE), and the standard deviation between 1,2% (SSMI) and 1,9% (DAX). The calculated Jarque-Bera statistics (taking into account the skewness and the kurtosis of the tested distribution) reject the null hypothesis of normality at the 1% significance level for all analysed return series. The values of the skewness and kurtosis statistics furthermore indicate that the underlying data are leptokurtic, or fat-tailed and sharply peaked about the mean when compared with the normal distribution. The Ljung-Box Q-statistics Q(12) show (for majority of analysed return series) the existence of the serial correlation. Much higher

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[2] The data for trading volume of SMSI and DAX, respectively, were not available from the start of the above mentioned period and therefore in these cases the analysis was done from June, 29 2006 and from September, 24 2008, respectively.

values of the Ljung-Box statistics for squared return series, $Q^2(12)$, than those for the corresponding return series (substantially higher than the corresponding critical values of the χ^2- distribution) indicate the presence of the conditional heteroscedasticity. From the ADF test results it seems to be clear that the hypothesis of a unit root is strongly rejected for the all analysed stock returns (for more information about unit root tests see e.g. [Franses et al. 2000]), i.e. the return series are also stationary.

Table 1. Descriptive statistics of the logarithmic return series and some test results

	Mean	Std. Dev.	Skew- ness	Kurt- osis	J-B	Obs.	Q(12)	$Q^2(12)$	ADF
ATX	$1,9 \cdot 10^{-4}$	0,018	-0,256	8,913	2365,5 ***	1612	9,033	1872,8 ***	-38,255 ***
BEL20	$3 \cdot 10^{-6}$	0,014	-0,160	10,389	3803,5 ***	1669	32,629 ***	1588,4 ***	-39,798 ***
FTSE100	$1,7 \cdot 10^{-4}$	0,013	-0,120	11,671	5167,1 ***	1648	68,255 ***	1396,5 ***	-20,044 ***
AEX	$5,7 \cdot 10^{-5}$	0,015	-0,175	12,629	6459,7 ***	1670	43,838 ***	1668,2 ***	-42,124 ***
CAC40	$6,9 \cdot 10^{-5}$	0,015	0,126	11,323	4822,2 ***	1669	53,283 ***	1071,2 ***	-43,893 ***
DAX	$3,2 \cdot 10^{-4}$	0,019	0,235	9,313	1093,9 ***	655	24,783 ***	355,44 ***	-25,425 ***
SMSI	$-8,9 \cdot 10^{-5}$	0,017	0,297	10,856	3168,4 ***	1225	19,180 *	400,76 ***	-35,101 ***
SSMI	$1,1 \cdot 10^{-4}$	0,012	0,061	11,873	5394,4 ***	1644	64,426 ***	1709,3 ***	-20,017 ***
HSI	$3,8 \cdot 10^{-4}$	0,018	0,076	11,835	5218,8 ***	1604	28,687 ***	1501,3 ***	-41,420 ***
BSE SENSEX	$7,6 \cdot 10^{-4}$	0,018	0,076	9,742	3048,8 ***	1609	25,601 **	512,38 ***	-36,943 ***
JKSE	$9,6 \cdot 10^{-4}$	0,016	-0,636	9,038	2472,9 ***	1559	33,637 ***	633,16 ***	-35,111 ***
NIKKEI 225	$-6,7 \cdot 10^{-5}$	0,017	-0,579	12,124	5614,0 ***	1593	16,037 ***	1814,2 ***	-41,479 ***
TSEC	$2,8 \cdot 10^{-4}$	0,014	-0,410	6,052	670,3 ***	1611	24,439 **	571,19 ***	-37,956 ***

Note: The symbols *, ** and *** denote the rejection of the null hypothesis at the 10, 5 and 1 % significance levels respectively.

Source: own calculations in EViews 5.1

In order to capture the above mentioned characteristics of the analysed stock returns, the appropriate model from the ARCH-class models can be used. To model the asymmetric characteristics, such as a leverage effect, in which the negative

shocks increase volatility more than positive shocks of an equal magnitude, e.g. the TGARCH or GJR-GARCH models are used. So, besides the conditional mean equations we have to specify also the conditional variance equations.

The logarithmic stock returns equation, i.e. the conditional mean equation, can be in general written as a Box-Jenkins ARMA(m,n) model[3] of the form:

$$r_t = \omega_0 + \sum_{j=1}^{m} \phi_j r_{t-j} + \sum_{k=1}^{n} \theta_k \varepsilon_{t-k} + \varepsilon_t \tag{2}$$

where ω_0 is unknown constant, $\phi_j \left(j = 1, 2, \ldots m \right)$ and $\theta_k \left(k = 1, 2, \ldots n \right)$ are the parameters of the appropriate ARMA(m,n) model, ε_t is a disturbance term.

The conditional variance equation h_t in case of a TGARCH(p,q) model can be specified as:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{q} \gamma_i \varepsilon_{t-i}^2 I_{t-i}^- \tag{3}$$

where from $I_{t-i}^- = \begin{cases} 1, if \ \varepsilon_{t-i} < 0 \\ 0, if \ \varepsilon_{t-i} > 0 \end{cases}$, it is clear the different impact of the positive

shocks $\varepsilon_{t-i} > 0$ and negative shocks $\varepsilon_{t-i} < 0$ on the conditional variance. The impact of positive shocks is given by the value of α_i, the impact of negative shocks by the value $\alpha_i + \gamma_i$. If $\gamma_i > 0$, it means that the negative shocks increase volatility which confirms the presence of the leverage effect of i-th order. If $\gamma_i \neq 0$, we speak about the asymmetric impact of shocks.

To examine the effect of trading volume on stock returns volatility, the following modification of the conditional variance equation (3) is used:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{q} \gamma_i \varepsilon_{t-i}^2 I_{t-i}^- + \delta V_t \tag{4}$$

where V_t is the logarithm of the trading volume. Meaning of the remaining symbols is the same as in equation (3). According to the [Lamoureux et al. 1990] the parameter δ should be positive and the volatility persistence should become negligible.

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[3] ARMA model = Autoregressive Moving Average model

EMPIRICAL RESULTS

The analysis was done in two steps. In the first step the parameters of conditional mean equation (2) together with the conditional volatility equation without trading volume (3) were estimated and in the second step we estimated parameters of the conditional mean equation (2) together with the parameters of conditional volatility equation with included trading volume (4)[4]. The whole analysis was done in econometrical software EViews 5.1.

The appropriate ARMA(m,n) models for logarithmic stock returns were as follows: ATX – AR(1), BEL20 – ARMA((3,4),4), FTSE100 – AR(1), AEX – AR(3,4,5,7), CAC40 – ARMA(1,1), DAX – AR(2), SMSI – ARMA(1,(1,3)), SSMI – AR(2), HSI – MA(9,10), BSE SENSEX – AR(1), JKSE – AR(1), NIKKEI225 – ARMA(1,1), TSEC – AR(1,6,8). The estimation results of conditional variance equations (together with the information about the values p and q in TGARCH model) for both model 1 and model 2 are presented in table 2.

The results summarized in table 2 show quite high degree of the volatility persistence, since the sum $\sum_{i=1}^{q} \hat{\alpha}_i + \sum_{i=1}^{p} \hat{\beta}_i$ is high in all analysed cases. In model 1 (i.e. model without trading volume variable) it takes values from 0,798 (JKSE) to 0,939 (TSEC), and besides also the existence of the leverage effect ($\gamma_i > 0$) was proved in all analysed cases. This means confirmation of the fact that the negative shocks increase volatility more than positive shocks of an equal magnitude.

For model 2 (i.e. model with trading volume variable) the volatility persistence varies between 0,793 (JKSE) and 0,930 (TSEC), but it is (similarly as for model 1) less than 1, proving the stationary persistence. As it is furthermore clear from the table 2, after inclusion of the trading volume variable, the volatility persistence declined for all analysed Asian stock returns and parameter δ corresponding to the trading volume variable was statistically significant at 1% significance level. For the Japanese NIKKEI225 the surprisingly negative relationship between trading volume and conditional volatility was proved. The impact of the trading volume on the European stock returns was different – the volatility persistence declined in five cases (ATX, BEL20, FTSE100, CAC40, SSMI), for AEX remains almost the same and in two cases even rose (DAX, SMSI). The significant negative relationships between trading volume and conditional volatility were confirmed for the British FTSE100 and German DAX. For two stock returns (AEX, SMSI) the parameter estimates of trading volume were insignificant. Regarding the presence of the leverage effect in model 2 we received the similar results as for model 1.

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[4] The equations estimated in the first step we denote as model 1 and equations estimated in the second step we denote as model 2.

Table 2. Summary of the estimated parameters from the conditional variance equations (3) and (4)

	TGARCH (p,q)	Without trading volume, i.e. Model 1		With trading volume, i.e. Model 2		
		$\sum_{i=1}^{q}\hat{\alpha}_i + \sum_{i=1}^{p}\hat{\beta}_i$	$\hat{\gamma}_1$	$\sum_{i=1}^{q}\hat{\alpha}_i + \sum_{i=1}^{p}\hat{\beta}_i$	$\hat{\gamma}_1$	δ
ATX	(1,2)	0,870 ***	0,209 ***	0,850 **	0,209 ***	$6 \cdot 10^{-6}$ ***
BEL20	(1,1)	0,884 #	0,195 ***	0,850 #	0,210 ***	$3 \cdot 10^{-6}$ ***
FTSE100	(1,2)	0,895 ***	0,181 ***	0,881 ***	0,199 ***	$-3 \cdot 10^{-6}$ ***
AEX	(1,1)	0,896 ***	0,183 ***	0,896 ***	0,184 ***	$-5 \cdot 10^{-7}$
CAC40	(2,2)	0,893 ***	0,182 ***	0,859 #	0,224 ***	$5 \cdot 10^{-6}$ ***
DAX	(1,2)	0,846 ***	0,203 ***	0,857 **	0,243 ***	$-2 \cdot 10^{-5}$ ***
SMSI	(3,2)	0,923 ***	0,129 ***	0,928 ***	0,120 ***	$1 \cdot 10^{-6}$
SSMI	(1,2)	0,878 **	0,202 ***	0,862 **	0,203 ***	$3 \cdot 10^{-6}$ ***
HSI	(1,2)	0,934 ***	0,107 ***	0,873 ***	0,156 ***	$5 \cdot 10^{-6}$ ***
BSE SENSEX	(1,1)	0,914 ***	0,137 ***	0,843 *	0,198 ***	$2 \cdot 10^{-5}$ ***
JKSE	(1,1)	0,798 *	0,257 ***	0,793 *	0,258 ***	$2 \cdot 10^{-6}$ ***
NIKKEI 225	(1,2)	0,863 **	0,208 ***	0,830 ***	0,247 ***	$-6 \cdot 10^{-6}$ ***
TSEC	(1,1)	0,939 **	0,081 ***	0,930 #	0,094 ***	$6 \cdot 10^{-6}$ ***

Note: The symbols *, ** and *** indicate that the corresponding parameter is statistically significant at the 10, 5 and 1 % significance level respectively. In case of the sum of coefficients alpha and beta the mentioned symbols refer to the statistical significance of all the parameters. The symbol # means that at least one parameter alpha or beta was not statistically significant at any of the mentioned significance levels.

Source: own calculations in EViews 5.1

Finally, in order to have the information about adequacy of the estimates presented in table 2, we tested the standardized residuals. The uncorrelatedness of the standardized residuals and squared standardized residuals was tested using the

Ljung – Box Q – statistics and Q^2 – statistics, respectively till the lag 12 and the Jarque – Bera test was used to test the normality (see table 3).

Table 3 The diagnostic check statistics of the standardized residuals

	Without trading volume, i.e. Model 1			With trading volume, i.e. Model 2		
	Q(12)	Q^2(12)	J-B	Q(12)	Q^2(12)	J-B
ATX	9,935	12,497	36,868 ***	9,799	11,271	25,946 ***
BEL20	4,018	5,108	83,864 ***	4,787	5,157	55,058 ***
FTSE100	7,014	20,510 **	66,672 ***	7,557	20,402 **	65,103 ***
AEX	8,805	9,669	33,330 ***	8,953	9,774	34,226 ***
CAC40	5,226	15,127	56,880 ***	5,153	17,756 *	44,424 ***
DAX	5,104	9,706	14,322 ***	5,469	6,070	23,800 ***
SMSI	2,756	14,521	35,013 ***	2,668	12,476	30,000 ***
SSMI	12,617	15,607	62,558 ***	12,387	13,448	56,443 ***
HSI	12,765	9,712	24,851 ***	9,141	8,368	14,817 ***
BSE SENSEX	16,724	9,144	540,76 ***	16,765	7,108	784,91 ***
JKSE	5,784	5,080	471,56 ***	6,386	5,716	455,95 ***
NIKKEI 225	13,808	7,696	44,420 ***	15,008	9,940	51,398 ***
TSEC	6,192	9,106	172,56 ***	5,421	8,630	131,78 ***

Note: The symbols *, ** and *** denote the rejection of the null hypothesis at the 10, 5 and 1 % significance levels respectively.

Source: own calculations in EViews 5.1

From the results presented in table 3 it is clear that no first- and second-order dependence in standardized residual series at the significance level 1% was detected. It can be also concluded that there is no remaining heteroscedasticity in any case, i.e. the estimated TGARCH models have successfully accounted for all

linear and nonlinear dependencies in the analysed return series[5]. The normality condition was violated in all analysed cases, which means that the estimates are consistent only as quasi-maximum likelihood estimates (see e.g. [Franses et al. 2000], [Poon et al. 2003], [Rachev et al. 2007]).

CONCLUDING REMARKS

The presented paper analyses the relationship between the trading volume and the daily volatility of eight European and five Asian stock returns data. To capture the conditional volatility it uses the TGARCH models and the relationship between the trading volume and the conditional volatility is tested following the approach of [Lamoureux et al. 1990].

In coincidence with this approach, the logarithm of the trading volume was included into the conditional volatility equation in order to confirm or to reject that it is a good proxy for information arrival. The results for European and Asian stock return volatilities are different. Since the inclusion of the trading volume led in case of all analysed Asian stock returns to the moderate decline of the conditional volatility persistence, the results for European stock returns were not so unambiguous. The volatility persistence declined only in five cases (ATX, BEL20, FTSE100, CAC40, SSMI), for AEX remains almost the same and in two cases even rose (DAX, SMSI). For AEX and SMSI the logarithmic variable trading volume was even not statistically significant. Taking into account some other papers (e.g. [Girard et al. 2007], [Gursoy et al. 2008], [Sharma et al. 1996]), the results of our analysis coincide with theirs, i.e. that we can join the conclusions that the trading volume can be in general considered (in case of the market-level analysis) to be only a poor proxy for information flow.

REFERENCES

Baillie R.T., Bollerslev T., Mikkelsen H.O. (1996) Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics, Vol. 74, No. 1, pp. 3-30.
Bollerslev T. (1986) Generalized Autoregressive Conditional Heteroscedasticity, Journal of Econometrics, Vol. 31, No. 3, pp. 307-327.
Engle R.F. (1982) Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation, Econometrica, Vol. 50, No. 4, pp. 987-1008.
Engle R.F. (1990) Discussion: Stock Market Volatility and the Crash of '87, Review of Financial Studies, Vol. 3, No.1, pp. 103-106.

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[5] Although the graphical representation of the time-varying volatility for individual TGARCH models can not be presented here from space reasons (graphs can be provided by the author upon request), it clearly confirms the use of the conditional heteroscedasticity models for modelling volatility.

Engle R.F., Ng V.K. (1993) Measuring and testing the impact of news on volatility, Journal of Finance, Vol. 48, No. 5, pp. 1749 – 1778.

Franses P.H., van Dijk D. (2000) Non-Linear Time Series Models in Empirical Finance, Cambridge University Press, Cambridge, 280 p.

Ghysels E., Gouriéroux C., Jasiak J. (2000) Causality between Returns and Traded Volumes, Annales D'Économie et de Statistique, Issue 60, pp. 189-206.

Giorgioni G., Holden K., Hančlová J. (2008) A review of the current methods of econometric modelling, In: Proceedings of 26th International Conference - Mathematical Methods in Economics 2008, 1. ed. Technická univerzita Liberec, Liberec, pp. 140-148.

Girard E., Biswas R. (2007) Trading Volume and Market Volatility: Developed versus Emerging Stock Markets, The Financial Review, Vol. 42, No. 3, pp. 429 – 459.

Glosten L.R., Jagannathan R., Runkle D. (1993) Relationship between the expected value and the volatility of the nominal excess return on stocks, Journal of Finance, Vol. 48, No. 5, pp. 1779-1801.

Gursoy G., Yuksel A., Yuksel A. (2008) Trading volume and stock market volatility: evidence from emerging stock markets, Investment Management and Financial Innovations, Vol. 5, Issue 4, pp. 200 – 210.

Lamoureux C., Lastrapes N. (1990) Heteroscedasticity in stock return data: volume versus GARCH Effects, The Journal of Finance, Vol. XLV, No. 1, pp. 221-229.

Miyakoshi T., ARCH versus Information-Based Variances: Evidence from the Tokyo Stock Market, Available at: http://www.sciencedirect.com/science/article/B6VF1-454HH56-3/2/46a3bbcfbfe601d9e08ae139aaf81e92

Nelson D. (1991) Conditional heteroskedasticity in asset returns: A new approach, Econometrica, Vol. 59, No. 2, pp. 347-370.

Poon S-H., Granger C.W.J. (2003) Forecasting Volatility in Financial Markets: A Review, Journal of Economic Literature, Vol. 41, No. 2, pp. 478 – 539.

Rachev S.T. et al. (2007) Financial Econometrics. From Basics to Advanced Modeling Techniques, John Wiley&Sons, Hoboken, New Jersey, 553 p.

Sharma J.L., Mougoue M., Kamath R. (1996) Heteroscedasticity in stock market indicator return data: volume versus GARCH effects, Applied Financial Economics, Vol. 6, Issue 4, pp. 337-342.

Zakoian J.M. (1994) Threshold Heteroskedastic Models. Journal of Economic Dynamics and Control, Vol. 18, No. 5, pp. 931-955.

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THE QUANTILE ESTIMATION OF THE MAXIMA
OF SEA LEVELS

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Abstract.   The hydrological modeling has become an intensively studied subject in recent years. One of the most significant problems concerning this issue is to provide the mathematical and statistical tools, which allow to forecast extreme hydrological events, such as severe sea or river floodings. The extreme events on water have huge social and economic impact on the affected areas. Due to these reasons, each country has to protect itself against the flood danger, and consequently, the designing of reliable flood defences is of great importance to the safety of the region. For example, the sea dikes along the Dutch coastline are designed to withstand floods, which may occur once every 10 000 years. It means that the height of the dike is determined in such a way that the probability of the event that there is a flood in a given year equals 10^{-4}. The computation of such the height level requires the estimation of the corresponding quantiles of the distributions of certain maxima of sea levels. In our paper, we present the procedures, which lead to the estimation of such the quantiles. We are mainly concerned with the interval estimation; in this context, we present the frequentistic and Bayesian approaches in constructing the desired confidence intervals.

Key words: quantile estimation, frequentistic confidence interval, Bayesian confidence interval, peaks over threshold (POT)

INTRODUCTION AND PRELIMINARIES

The purpose of our work is to give the estimation procedures for the $(1-\alpha)$-quantile of the distribution of the maxima of the North Sea levels at Hook of Holland, the Netherlands. In our investigations, we apply the informations on sea levels from the period 1945-1995, available on the website http://live.waterbase.nl/.

The problem of the estimation of such a quantile was earlier considered in the papers of [Van Gelder 1996] and [Van Gelder et al. 1995]. The mentioned articles attracted our attention towards the issue of the quantile estimation of extreme hydrological events and encouraged us to undertake some research in this field, although our approach differs from the methods proposed by Van Gelder et al. There are some other valuable publications devoted to the modeling of extreme water events. We cite in this context the papers of [Katz et al. 2002], [Knox and Kundzewicz 1997] and [Shiau 2003] among others.

The set of our empirical data consists of the maximum sea levels along the Dutch coast in the area of Hook of Holland in the periods of 3 days. We will denote such the 3-day maxima by $X_1, X_2, ..., X_T$; $T = 5963$ (i.e., X_t will stand for the maximum sea level in the t-th 3-day period; we had 5963 such the 3-day periods in the years 1945-1995). The reason for such a choice of data is that the storms on the North Sea never last longer than 72 hours (actually, an average storm there lasts from 9 to 12 hours).

We observed some autocorrelations and the increasing linear trend in the sequence of the 3-day maxima of sea levels at Hook of Holland in the period 1945-1995 (see Figure 1). By using the least squares method, we obtained the model of the form $\hat{X}_t = 100.1 + 0.006507 \cdot t$ (both the parameters turned out to be statistically significant; the corresponding p-value $= 2e\text{-}16$). It means that, on average, the 3-day maxima were increasing by 0.006507 cm from one 3-day subperiod to the next 3-day subperiod, and consequently, they increased by around 39 cm within 50 years.

Therefore, we will consider the following model for the 3-day maxima of sea levels at Hook of Holland: $X_t = a + bt + \varepsilon_t$, where a, b are some constants and the random errors ε_t (all with a common marginal d.f. F_ε) may be correlated (see Figure 1), but in turn, the error terms exceeding a certain, sufficiently large threshold u are independent (this independence is a consequence of the fact that an average storm on the North Sea does not last longer than 3 days).

Figure 1. The 3-day maxima of sea levels, the residuals $x_t - \hat{x}_t$ and the corresponding autocorrelation functions

3−day max

Residuals for 3−day max

3−day max

Residuals for 3−day max
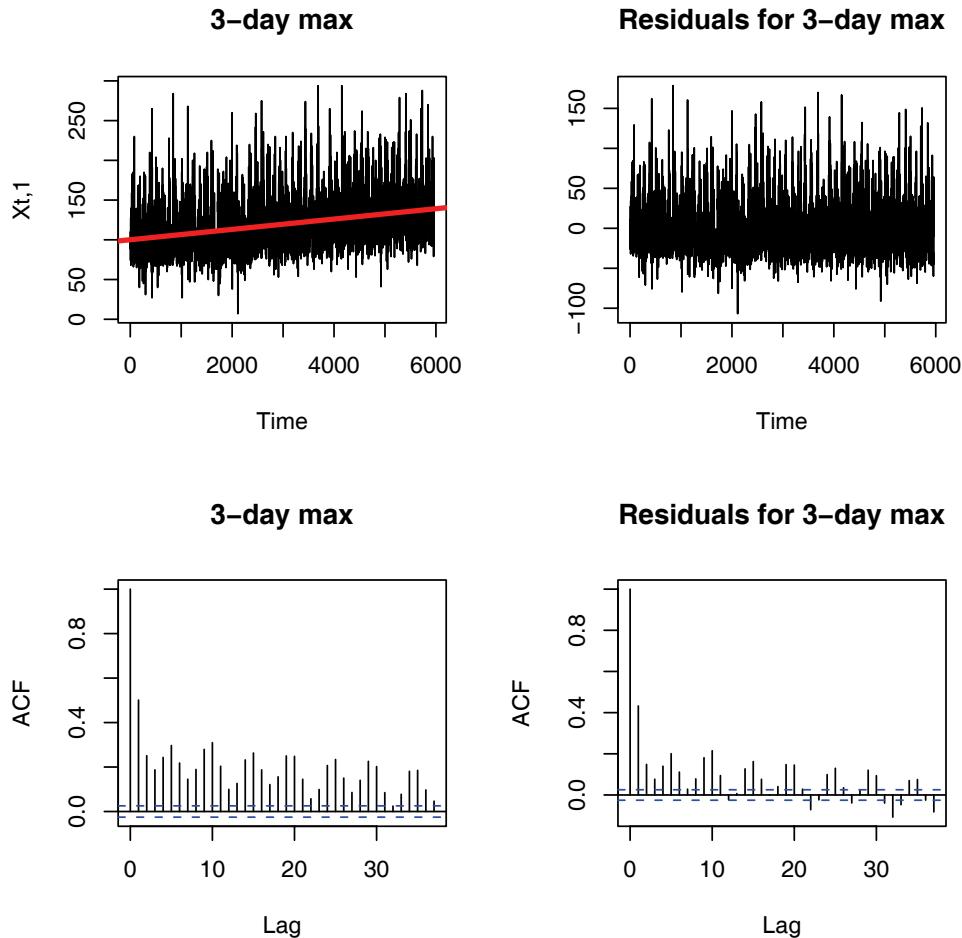
Source: own preparation

We will assume that for sufficiently large threshold u :

(A1) $\overline{F}_\varepsilon(x) = P(\varepsilon > x) \approx \exp\left(-\frac{x}{\beta}\right)$, for some $\beta > 0$, if $x > u$,

(A2)   the r.v.'s $Y_i = (\varepsilon_i - u \mid \varepsilon_i > u)$ (where $i = 1, 2, ..., K$; $K = \sum_{t=1}^{T} I(\varepsilon_t > u)$) are independent.

The assumption in (A1) means that the distribution of the model errors is (approximately) of the thin-tailed, exponential type. The validity of this condition for the errors of our model will be verified in the further stages of our study. The assumption in (A2) has its clear physical explanation since, as we have already mentioned, the storms on the North Sea never last longer than 3 days.

Our main aim is to construct the confidence intervals for $F_{X_T}^{-1}(1-\alpha)$ - the $(1-\alpha)$-quantile of the distribution of X_T. The $(1-\gamma)\cdot 100\%$ confidence interval $(L(T), R(T))$ for $F_{X_T}^{-1}(1-\alpha)$ is defined by

$$P\big(L(T)\leq F_{X_T}^{-1}(1-\alpha)\leq R(T)\big)=1-\gamma.$$

Let us denote by $F_\varepsilon^{-1}(1-\alpha)$ the $(1-\alpha)$-quantile of the distribution of the model errors ε_t. Furthermore, let $l(T)$, $r(T)$ satisfy

$$P\big(l(T)\leq F_\varepsilon^{-1}(1-\alpha)\leq r(T)\big)=1-\gamma.$$

Obviously, we have: $L(T)=a+bT+l(T)$, $R(T)=a+bT+r(T)$, where a, b are the parameters of the linear model for X_t. Thus, in order to construct the confidence interval for $F_{X_T}^{-1}(1-\alpha)$, it is sufficient to establish the confidence interval for $F_\varepsilon^{-1}(1-\alpha)$.

The remainder of the paper is organized as follows. In Section "The approximated quantile of the distribution of the model errors", we will derive the approximated formula for $F_\varepsilon^{-1}(1-\alpha)$. In Section "Interval estimation of the quantile ..." we will construct the approximated frequentistic and Bayesian confidence intervals for $F_\varepsilon^{-1}(1-\alpha)$ and $F_{X_T}^{-1}(1-\alpha)$. We will apply these formulas later in Section "The confidence intervals for the quantiles of the maxima of sea levels ..." to compute the realizations of the confidence intervals for the quantiles of the distribution of the 3-day maxima of the North Sea levels at Hook of Holland. In Section "The correctness and accuracy of the obtained estimation procedures ...", we will assess the quality of our estimation procedures. Finally, in Section "Final conclusions", we will conclude and summarize our study.

THE APPROXIMATED QUANTILE OF THE DISTRIBUTION OF THE MODEL ERRORS

Suppose that a threshold u is such as in the assumptions (A1), (A2) from the previous section. Let, for $y>0$, $F_u(y):=P\big(\varepsilon-u\leq y\mid \varepsilon>u\big)$. We have

$$F_u(x-u)=P\big(\varepsilon-u\leq x-u\mid\varepsilon>u\big)=P\big(\varepsilon\leq x\mid\varepsilon>u\big)=\frac{F_\varepsilon(x)-F_\varepsilon(u)}{1-F_\varepsilon(u)}, \qquad (1)$$

if $x - u > 0$. By (1) and the assumption (A1), we obtain that

$$F_u(x-u) = 1 - \frac{\overline{F}_\varepsilon(x)}{\overline{F}_\varepsilon(u)} \approx 1 - \frac{\exp(-x/\beta)}{\exp(-u/\beta)}, \text{ for sufficiently large } u \text{ and } x > u,$$

where $\beta = \beta(u)$ is a certain parameter, which depends on u. Therefore,

$$P(\varepsilon - u \leq x - u \mid \varepsilon > u) = F_u(x-u) \approx 1 - \exp\left(-\frac{x-u}{\beta}\right), \text{ if } x > u. \qquad (2)$$

The relation in (2) means that the conditional distribution of $\varepsilon - u$ given the event that ε exceeds a large threshold u is approximately the exponential distribution $Exp(1/\beta)$. Obviously, due to (1), we have $F_\varepsilon(x) = (1 - F_\varepsilon(u))F_u(x-u) + F_\varepsilon(u)$, for $x > u$. This and the approximation in (2) imply

$$F_\varepsilon(x) \approx (1 - F_\varepsilon(u))\left(1 - \exp\left(-\frac{x-u}{\beta}\right)\right) + F_\varepsilon(u), \text{ if } x > u.$$

Let us denote by $q_{1-\alpha}^{(\varepsilon)} > u$ the $(1-\alpha)$-quantile of F_ε. Then, $F_\varepsilon(q_{1-\alpha}^{(\varepsilon)}) = 1 - \alpha$ and, by substituting $q_{1-\alpha}^{(\varepsilon)}$ for x into the relation above, we get

$$1 - \alpha \approx (1 - F_\varepsilon(u))\left(1 - \exp\left(-\frac{q_{1-\alpha}^{(\varepsilon)} - u}{\beta}\right)\right) + F_\varepsilon(u).$$

Hence, the approximated $(1-\alpha)$-quantile of F_ε is given by

$$F_\varepsilon^{-1}(1-\alpha) = q_{1-\alpha}^{(\varepsilon)} \approx u + \beta \ln\left(\frac{1 - F_\varepsilon(u)}{\alpha}\right). \qquad (3)$$

The purpose of our next study is to construct the confidence intervals for the quantile $F_\varepsilon^{-1}(1-\alpha)$, and consequently, for the quantile $F_{X_T}^{-1}(1-\alpha)$ as well. We will conduct our investigations under the condition that $K = k$ (as, due to the estimated model for X_T, the number of residuals exceeding u is known). We will consider two approaches of interval estimation: the first is based on the frequentistic analysis, while the ideas of the second approach derive from the Bayesian analysis.

INTERVAL ESTIMATION OF THE QUANTILE - THE PROPOSED ESTIMATION PROCEDURES

The frequentistic approach

Let $Y_i := (\varepsilon_i - u \mid \varepsilon_i > u)$, where $i = 1, 2, ..., k$ and ε_i, u, k are such as in the previous sections. By the conditions in (A1), (A2), we have that the $Y_i's$ are

independent and have approximately the $Exp(1/\beta(u))$ distribution (see (2)). Hence, the maximum likelihood estimate of the parameter $\beta(u)$ is of the form

$$\hat{\beta} = \sum_{i=1}^{k} Y_i / k. \tag{4}$$

Due to (3), (4), we have the following estimate for the quantile $q_{1-\alpha}^{(\varepsilon)} = F_\varepsilon^{-1}(1-\alpha)$

$$\hat{q}_{1-\alpha}^{(\varepsilon)} = u + \hat{\beta} \ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right). \tag{5}$$

Let us construct the frequentistic confidence interval for $q_{1-\alpha}^{(\varepsilon)} - \hat{q}_{1-\alpha}^{(\varepsilon)}$. In order to obtain it, we need to find v_γ, w_γ, satisfying the following relation

$$P\left(v_\gamma \le q_{1-\alpha}^{(\varepsilon)} - \hat{q}_{1-\alpha}^{(\varepsilon)} \le w_\gamma\right) = 1 - \gamma. \tag{6}$$

By substituting the expressions on the r.h.s. of (3) and (5) for $q_{1-\alpha}^{(\varepsilon)}$ and $\hat{q}_{1-\alpha}^{(\varepsilon)}$ into (6), we obtain $P\left(v_\gamma \le (\beta - \hat{\beta})\ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right) \le w_\gamma\right) = 1 - \gamma$, which (since

$\dfrac{1-F_\varepsilon(u)}{\alpha} > 1$, as $q_{1-\alpha}^{(\varepsilon)} > u$) yields

$$P\left(\frac{-w_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)} \le \hat{\beta} - \beta \le \frac{-v_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)}\right) = 1 - \gamma.$$

By using (4), we have

$$P\left(\left(\frac{-w_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)} + \beta\right)k \le \sum_{i=1}^{k} Y_i \le \left(\frac{-v_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)} + \beta\right)k\right) = 1 - \gamma.$$

Since $\sum_{i=1}^{k} Y_i$ has approximately the gamma $\Gamma(k, 1/\beta)$ distribution (as Y_i are independent and, approximately, $Y_i \sim Exp(1/\beta)$) and a confidence level equals $1 - \gamma$, we may write that:

$$\left(\frac{-w_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)} + \beta\right)k = \Gamma^{-1}\left(\frac{\gamma}{2}, k, \frac{1}{\beta}\right),$$

$$\left(\frac{-v_\gamma}{\ln\left(\dfrac{1-F_\varepsilon(u)}{\alpha}\right)} + \beta \right) k = \Gamma^{-1}\left(1 - \frac{\gamma}{2}, k, \frac{1}{\beta}\right),$$

where $\Gamma^{-1}(\cdot,\cdot,\cdot)$ denote the corresponding quantiles of the gamma distribution. Therefore:

$$v_\gamma = -\left(\frac{\Gamma^{-1}\left(1 - \dfrac{\gamma}{2}, k, \dfrac{1}{\beta}\right)}{k} - \beta \right) \ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right), \qquad (7)$$

$$w_\gamma = -\left(\frac{\Gamma^{-1}\left(\dfrac{\gamma}{2}, k, \dfrac{1}{\beta}\right)}{k} - \beta \right) \ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right). \qquad (8)$$

Obviously, the frequentistic confidence interval for $q_{1-\alpha}^{(\varepsilon)}$ is given by

$$P\left(v_\gamma + \hat{q}_{1-\alpha}^{(\varepsilon)} \le q_{1-\alpha}^{(\varepsilon)} \le w_\gamma + \hat{q}_{1-\alpha}^{(\varepsilon)}\right) = 1 - \gamma. \qquad (9)$$

Thus, due to (9) and (5), (7), (8), we obtain the following left (lower) and right (upper) ends (limits) of the frequentistic confidence interval for $q_{1-\alpha}^{(\varepsilon)} = F_\varepsilon^{-1}(1-\alpha)$ - the $(1-\alpha)$-quantile of the distribution of the model errors:

$l_{fr}(T)$ - the left end of the frequentistic confidence interval for ε_T

$$= u + \hat{\beta}\ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right) - \left(\frac{\Gamma^{-1}\left(1 - \dfrac{\gamma}{2}, k, \dfrac{1}{\beta}\right)}{k} - \beta \right) \ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right), \qquad (10)$$

$r_{fr}(T)$ - the right end of the frequentistic confidence interval for ε_T

$$= u + \hat{\beta}\ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right) - \left(\frac{\Gamma^{-1}\left(\dfrac{\gamma}{2}, k, \dfrac{1}{\beta}\right)}{k} - \beta \right) \ln\left(\frac{1-F_\varepsilon(u)}{\alpha}\right). \qquad (11)$$

As $F_\varepsilon(u)$ is unknown, we may use its empirical distribution

$$\hat{F}_\varepsilon(u) = 1 - k/T, \qquad (12)$$

where (for recollection) k stands for the number of residuals exceeding u.

Thus, by substituting the estimates for β and $F_\varepsilon(u)$ (see (4) and (12), respectively) into the formulas (10), (11), we have:

$$\hat{l}_{fr}(T) = u + \left(2\hat{\beta} - \frac{\Gamma^{-1}\left(1 - \frac{\gamma}{2}, k, \frac{1}{\hat{\beta}}\right)}{k} \right) \ln\left(\frac{k}{T\alpha}\right), \tag{13}$$

$$\hat{r}_{fr}(T) = u + \left(2\hat{\beta} - \frac{\Gamma^{-1}\left(\frac{\gamma}{2}, k, \frac{1}{\hat{\beta}}\right)}{k} \right) \ln\left(\frac{k}{T\alpha}\right). \tag{14}$$

The relations above determine the ends (limits) of the approximated frequentistic confidence interval for $q_{1-\alpha}^{(\varepsilon)} = F_\varepsilon^{-1}(1-\alpha)$. Consequently, the approximated frequentistic confidence interval for $q_{1-\alpha} = F_{X_T}^{-1}(1-\alpha)$ is given by:

$$\hat{L}_{fr}(T) = \hat{a} + \hat{b}T + \hat{l}_{fr}(T), \qquad \hat{R}_{fr}(T) = \hat{a} + \hat{b}T + \hat{r}_{fr}(T), \tag{15}$$

where $\hat{a}$, $\hat{b}$ stand for the parameter estimates of the model $X_t = a + bt + \varepsilon_t$.

The Bayesian approach

Assume that the prior distribution of the parameter β is the inverse gamma distribution $IG(c, d)$, where c and d are some constants, which may be chosen on subjective grounds by using our knowledge before any data set is available. It follows from (A1) and (A2) that the r.v.'s $Y_i = (\varepsilon_i - u \mid \varepsilon_i > u)$; $i = 1, 2, \ldots, k$, where $k = \sum_{t=1}^{T} I(\varepsilon_t > u)$, are independent and (approximately) distributed according to the exponential distribution $Exp(1/\beta)$ (see the approximation in (2)). Let (y_i) denote the realizations of (Y_i). Then, the posterior distribution of $\beta \mid y_1, \ldots, y_k$ is the inverse gamma distribution $IG(c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1})$. Our goal now is to construct the Bayesian confidence interval for the quantile $q_{1-\alpha}^{(\varepsilon)} = F_\varepsilon^{-1}(1-\alpha)$. It means that, for a fixed confidence level $1 - \gamma$, we need to find $l_{Ba} = l_{Ba}(T, c, d, k)$, $r_{Ba} = r_{Ba}(T, c, d, k)$, such that the following condition holds: $P(l_{Ba} \leq q_{1-\alpha}^{(\varepsilon)} \leq r_{Ba} \mid y_1, \ldots, y_k) = 1 - \gamma$. This and the approximation in (3)

imply $P(l_{Ba} \leq u + \beta \ln\left(\dfrac{1 - F_\varepsilon(u)}{\alpha}\right) \leq r_{Ba} \mid y_1, ..., y_k) = 1 - \gamma$. Since $\dfrac{1 - F_\varepsilon(u)}{\alpha} > 1$

(as $q_{1-\alpha}^{(\varepsilon)} > u$), the last relation is equivalent to the following one:

$$P\left(\frac{l_{Ba} - u}{\ln\left(\dfrac{1 - F_\varepsilon(u)}{\alpha}\right)} \leq \beta \leq \frac{r_{Ba} - u}{\ln\left(\dfrac{1 - F_\varepsilon(u)}{\alpha}\right)} \,\middle|\, y_1, ..., y_k \right) = 1 - \gamma.$$ Due to the fact that

$\beta \mid y_1, ..., y_k$ has the $IG\left(c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1}\right)$ distribution, we get:

$$\frac{l_{Ba} - u}{\ln\left(\dfrac{1 - F_\varepsilon(u)}{\alpha}\right)} = IG^{-1}\left(\frac{\gamma}{2}, c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1} \right),$$

$$\frac{r_{Ba} - u}{\ln\left(\dfrac{1 - F_\varepsilon(u)}{\alpha}\right)} = IG^{-1}\left(1 - \frac{\gamma}{2}, c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1} \right),$$

where $IG^{-1}(\cdot, \cdot, \cdot)$ denote the corresponding quantiles of the inverse gamma distribution. Hence, the Bayesian confidence interval for the quantile $q_{1-\alpha}$ is determined by:

$$l_{Ba} = u + IG^{-1}\left(\frac{\gamma}{2}, c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1} \right) \ln\left(\frac{1 - F_\varepsilon(u)}{\alpha}\right),$$

$$r_{Ba} = u + IG^{-1}\left(1 - \frac{\gamma}{2}, c + k, \left(d + \sum_{i=1}^{k} y_i\right)^{-1} \right) \ln\left(\frac{1 - F_\varepsilon(u)}{\alpha}\right).$$

As $IG^{-1}(1 - \alpha, c, d) = 1/\Gamma^{-1}(\alpha, c, 1/d)$, we may rewrite l_{Ba} and r_{Ba} by means of the quantiles of the gamma distribution as follows:

$$l_{Ba} = u + \ln\left(\frac{1 - F_\varepsilon(u)}{\alpha}\right) \Big/ \Gamma^{-1}\left(1 - \frac{\gamma}{2}, c + k, d + \sum_{i=1}^{k} y_i \right),$$

$$r_{Ba} = u + \ln\left(\frac{1 - F_\varepsilon(u)}{\alpha}\right) \Big/ \Gamma^{-1}\left(\frac{\gamma}{2}, c + k, d + \sum_{i=1}^{k} y_i \right).$$

Let us put: $c = 1/\beta_0 + 1$, $d = 1$, where $\beta_0 = \sum_{i=1}^{k} y_i / k$ is the initial estimate of β. Then, since $\beta \sim IG(c, d)$, we have $E\beta = d/(c - 1) = \beta_0$.

Furthermore, as $F_\varepsilon(u)$ is unknown, we may replace it by its natural estimate in (12). Therefore:

$$\hat{l}_{Ba} = \hat{l}_{Ba}(T) = u + \ln\left(\frac{k}{T\alpha}\right)/\Gamma^{-1}\left(1 - \frac{\gamma}{2}, \frac{k}{\sum\limits_{i=1}^{k} y_i} + 1 + k, 1 + \sum\limits_{i=1}^{k} y_i\right), \qquad (16)$$

$$\hat{r}_{Ba} = \hat{r}_{Ba}(T) = u + \ln\left(\frac{k}{T\alpha}\right)/\Gamma^{-1}\left(\frac{\gamma}{2}, \frac{k}{\sum\limits_{i=1}^{k} y_i} + 1 + k, 1 + \sum\limits_{i=1}^{k} y_i\right). \qquad (17)$$

The relations above determine the ends (limits) of the approximated Bayesian confidence interval for $q_{1-\alpha}^{(\varepsilon)} = F_\varepsilon^{-1}(1-\alpha)$. Consequently, the approximated Bayesian confidence interval for $q_{1-\alpha} = F_{X_T}^{-1}(1-\alpha)$ is given by:

$$\hat{L}_{Ba}(T) = \hat{a} + \hat{b}T + \hat{l}_{Ba}(T), \qquad \hat{R}_{Ba}(T) = \hat{a} + \hat{b}T + \hat{r}_{Ba}(T), \qquad (18)$$

where $\hat{a}$, $\hat{b}$ denote the parameter estimates of the model $X_t = a + bt + \varepsilon_t$.

In the following section, we will present the computed realizations of the confidence intervals for the quantiles of the distribution of the 3-day maxima of sea levels, obtained according to the derived formulas for confidence intervals.

THE CONFIDENCE INTERVALS FOR THE QUANTILES OF THE MAXIMA OF SEA LEVELS - SOME COMPUTATION RESULTS

As we have already mentioned, our data set consists of the 3-day maxima of the North Sea levels along the Dutch coast at Hook of Holland, collected in the period 1945-1995. It is graphically presented on Figure 1. The sample size equals $T = 5963$ observations and some autocorrelations, as well as the increasing linear trend can be seen in the sample of the 3-day maxima X_t. By using the least squares method, we obtained the estimation of the model $X_t = a + bt + \varepsilon_t$. It had the form $\hat{X}_t = 100.1 + 0.006507 \cdot t$ (both the parameters were statistically significant; p-value $= 2e\text{-}16$).

Let: (x_t) denote our empirical data, $\hat{x}_t = 100.1 + 0.006507 \cdot t$, $\hat{\varepsilon}_t = x_t - \hat{x}_t$; $t = 1, 2, \ldots, 5963$. We carried out the plot of the sample autocorrelation functions for the residuals $\hat{\varepsilon}_t$ (see Figure 1.), which denied the independence of the model errors. On the other hand, there was no trend in the sequence of residuals

$\left(\hat{\varepsilon}_t\right)$ (see also Figure 1.), which confirmed the stationarity of the error terms of the model. We also checked that the tail of the d.f. of the sequence $\left(\hat{\varepsilon}_t\right)$ declined exponentially (at least approximately). Thus, as we might assume that (approximately) the tail of the d.f. of the error term ε declined exponentially (i.e., $\overline{F}_\varepsilon(x) = P(\varepsilon > x) \approx \exp(-x/\beta)$ for some $\beta > 0$, if $x \to \infty$ - see the condition in (A1)), we obtained that, for each i and sufficiently large threshold u, $Y_i = \left(\varepsilon_i - u \mid \varepsilon_i > u\right)$ had approximately the exponential distribution $Exp(1/\beta(u))$ (see (2)). Next, we determined several sufficiently large thresholds u. For each threshold, we chose those of the values of residuals $\hat{\varepsilon}_t$, which exceeded u and constructed the sample $y_i = \left(\hat{\varepsilon}_i - u \mid \hat{\varepsilon}_i > u\right);$ $i = 1,2,...,k$, where $k = \sum_{t=1}^{5963} I(\hat{\varepsilon}_t > u)$. Next, we carried out the plot of the sample autocorrelation functions for y_i and concluded that it might be assumed that (Y_i) was a sequence of independent r.v.'s (see the condition in (A2)). Moreover, by using the Kolmogorov test, we also verified the accordance of $Y_i = \left(\varepsilon_i - u \mid \varepsilon_i > u\right)$ with the exponential distribution. Thus, we were in a position to apply the derived formulas for the approximated confidence intervals.

Below, we present the results of our computations. We considered the following thresholds u: 30, 40, 45, 50 and the confidence level $1 - \gamma = 0.95$. It turned out that the shortest confidence interval for $F_\varepsilon^{-1}(1-\alpha) = q_{1-\alpha}^{(\varepsilon)}$ was reached for $u = 30$. The realizations of the frequentistic (fr) and Bayesian (Ba) confidence intervals for $F_\varepsilon^{-1}(1-\alpha) = q_{1-\alpha}^{(\varepsilon)}$, calculated according to (13), (14) and (16), (17), in the case where $u = 30$, are given in the table below:

Table 1. The frequentistic and Bayesian confidence intervals for $q_{1-\alpha}^{(\varepsilon)}$; $u = 30$

	$1 - \alpha = 0.95$	$1 - \alpha = 0.99$	$1 - \alpha = 0.999$	$1 - \alpha = 0.9999$
fr	$(54.8;58.4)$	$(93.9;103.3)$	$(149.9;167.4)$	$(205.8;231.6)$
Ba	$(54.9;58.5)$	$(94.1;103.5)$	$(150.3;167.9)$	$(206.5;232.2)$

Source: own calculations

Thus, due to (15), (18), by taking into account the computed correction of trend, which equaled $100.1 + 0.006507 \cdot 5963 = 138.9012$ cm, we obtained the following realizations of the confidence intervals for $F_{X_T}^{-1}(1-\alpha) = q_{1-\alpha}$, if $u = 30$:

Table 2. The frequentistic and Bayesian confidence intervals for $q_{1-\alpha}$; $u = 30$

	$1-\alpha = 0.95$	$1-\alpha = 0.99$	$1-\alpha = 0.999$	$1-\alpha = 0.9999$
fr	$(193.7; 197.3)$	$(232.8; 242.2)$	$(288.8; 306.3)$	$(344.7; 370.5)$
Ba	$(193.8; 197.4)$	$(233.0; 242.4)$	$(289.2; 306.8)$	$(345.4; 371.1)$

Source: own calculations

THE CORRECTNESS AND ACCURACY OF THE OBTAINED ESTIMATION PROCEDURES – A SIMULATION STUDY

Since the approximation in (2) is valid when F_ε is of the thin-tailed type d.f., we conducted our simulations assuming that the data come from one of the following distributions (the values in parentheses denote the threshold values, which had been chosen in such a manner that, for the given quantile rank $1-\alpha$, u satisfied $1 - F_\varepsilon(u) > \alpha$, i.e., $\ln\dfrac{1 - F_\varepsilon(u)}{\alpha} > 0$): i) the exponential distribution $Exp(1/100)$ ($u = 200$), ii) the gamma distribution $\Gamma(30, 1/4)$ ($u = 150$), iii) the Gumbel distribution $Gumbel(100, 50)$ ($u = 220$).

For the chosen d.f. F_ε and the threshold u, we carried out the simulations according to the following scheme: *step 1)* we simulated a sample of size 10000 from F_ε, *step 2)* for the chosen threshold u and the obtained sample $\varepsilon_1, ..., \varepsilon_{10000}$, we created the sample $\varepsilon_1 - u, ..., \varepsilon_{10000} - u$, *step 3)* we chose those of $\varepsilon_t - u$, which satisfied $\varepsilon_t - u > 0$, and denoted the obtained sample by $y_1, ..., y_k$, *step 4)* we calculated an average of $y_1, ..., y_k$ and this way, we obtained the value of the estimate of the parameter β (see (4)), *step 5)* by using the Kolmogorov test, we checked the accordance of the sample $y_1, ..., y_k$ with the exponential distribution, *step 6)* we generated 10000 samples of size 10000 from F_ε; in this way, we obtained (by proceeding as in the steps 2)-4)) 10000 estimates of the parameter β, *step 7)* we calculated an average of 10000 estimates of β obtained in the step 6); we took this average for a real value of the parameter β, *step 8)* by using the value of β (calculated in the step 7)) and the formula in (3), we obtained the approximated quantiles of the ranks 0.95, 0.99, 0.999, 0.9999, *step 9)* we repeated 100 times the simulations from the steps 1)-4) and, by applying (13), (14), we obtained the realizations of the 95% frequentistic confidence intervals for the quantiles of appropriate ranks, *step 10)* we repeated 100 times the simulations

from the steps 1)-4) and, by applying (16), (17), we obtained the realizations of the 95% Bayesian confidence intervals for the quantiles of appropriate ranks.

The results of our simulations in the case of the frequentistic approach are:

For the case in i):

The exact quantiles from the $Exp(1/100)$ distribution are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Exact quantile value	299.6	460.5	690.8	921.0

The value of β, obtained by the MC method (see the step 7)): $\beta = 100.03$

The approximated quantiles (see the step 8)) are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Approximated quantile	299.6	460.6	690.9	921.3

The estimate of β, obtained according to the step 3): $\hat{\beta} = 99.95$

The result of the Kolmogorov test on the accordance of the sample $y_1,...,y_k$ with the $Exp(1/100.03)$ distribution: p-value $= 0.9326$; it means that we may accept the approximation by the exponential distribution.

For the case in ii):

The exact quantiles from the $\Gamma(30,1/4)$ distribution are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Exact quantile value	158.2	176.8	199.2	219.0

The value of β, obtained by the MC method (see the step 7)): $\beta = 12.38$

The approximated quantiles (see the step 8)) are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Approximated quantile	157.5	177.5	206.0	234.5

The estimate of β, obtained according to the step 3): $\hat{\beta} = 11.94$

The result of the Kolmogorov test on the accordance of the sample $y_1,...,y_k$ with the $Exp(1/12.38)$ distribution: p-value $= 0.4749$; it means that we may accept the approximation by the exponential distribution.

For the case in iii):

The exact quantiles from the $Gumbel(100,50)$ distribution are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Exact quantile value	248.5	330.0	445.4	560.5

The value of β, obtained by the MC method (see the step 7)): $\beta = 51.15$

The approximated quantiles (see the step 8)) are as follows:

Quantile rank	0.95	0.99	0.999	0.9999
Approximated quantile	248.2	330.5	448.3	566.0

The estimate of β, obtained according to the step 3): $\hat{\beta} = 52.41$

The result of the Kolmogorov test on the accordance of the sample $y_1,...,y_k$ with the $Exp(1/51.15)$ distribution: p-value $= 0.3043$; it means that we may accept the approximation by the exponential distribution.

Below, we give the graphical presentation of the realizations of the frequentistic and Bayesian confidence intervals for the quantiles of appropriate ranks. For brevity, we only show the results for the case in i):

Figure 2. The realizations of the $(1-\gamma)\cdot100\% = 95\%$ frequentistic confidence intervals for the quantiles of appropriate ranks in the case i)
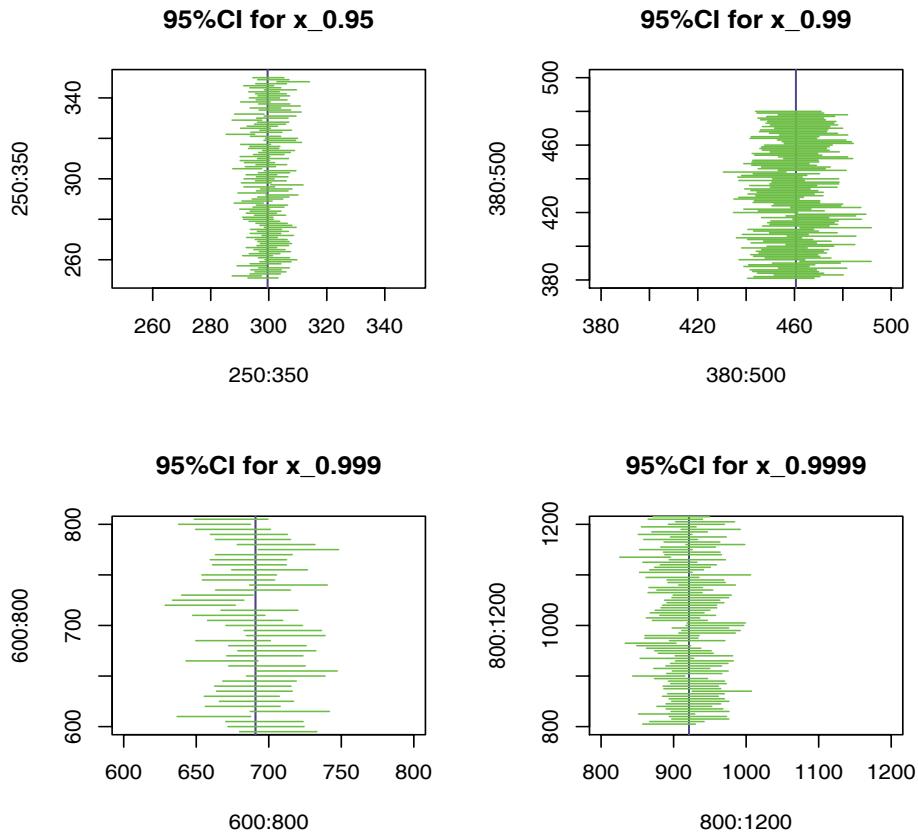
Source: own preparation

Figure 3. The $(1 - \gamma) \cdot 100\% = 95\%$ Bayesian confidence intervals for the quantiles $x_{0.95}$, $x_{0.99}$, $x_{0.999}$, $x_{0.9999}$ in the case i)

95%CI for x_0.95

95%CI for x_0.99

95%CI for x_0.999

95%CI for x_0.9999
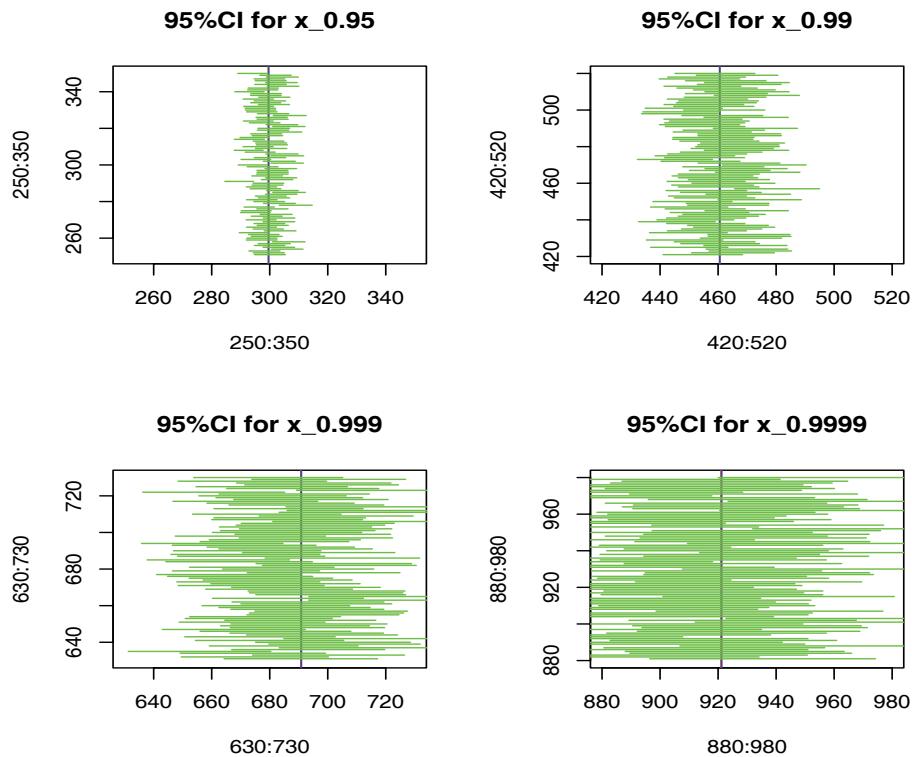
Source: own preparation

FINAL CONCLUSIONS

Conclusions concerning the obtained results for the frequentistic confidence intervals are as follows: a) the received confidence intervals estimate the corresponding quantiles reasonably well, b) in the case where the approximation by the exponential distribution is appropriate, the estimation errors are small, c) the following procedure can be proposed: choose (by applying the Kolmogorov test) the threshold u in such a manner that the sample $y_1,...,y_k$, where $y_i = \varepsilon_i - u | \varepsilon_i > u$ is i.i.d. and comes from the exponential distribution, and compute the approximated quantile and the realizations of the approximated confidence

intervals for the quantile by means of the formulas derived in Section "Interval estimation of the quantile - the proposed estimation procedures".

REFERENCES

Katz R. W., Parlange M. B. and Naveau P. (2002) Statistics of extremes in hydrology, Advances in water resources, 25, pp. 1287-1304.

Knox J. C and Kundzewicz Z. W. (1997) Extreme hydrological events, palaeo-information and climate change, Hydrological Sciences-Journal-des Sciences Hydrologiques, 42, pp. 765-779.

Shiau J. T. (2003) Return period of bivariate distributed extreme hydrological events, Stochastic environmental research and risk assessment, 17, pp. 42-57.

Van Gelder P. H. A. J. M. (1996) A Bayesian analysis of extreme water levels along the Dutch coast using flood historical data, Stochastic Hydraulics, 7, pp. 243-251.

Van Gelder P. H. A. J. M., Roos, A. and Tonneijck, M. R. (1995) On the probabilistic design of dikes in the Netherlands, Applications of Statistics and Probability, 3, pp. 1505-1508.

FACTORS AFFECTING INWARD FOREIGN DIRECT INVESTMENT FLOWS INTO THE UNITED STATES: EVIDENCE FROM STATE-LEVEL DATA

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Abstract: This paper investigates factors affecting the inward foreign direct investment (FDI) flows among fifty states of the United States. The analysis uses annual data for the period 1997 until 2007. The study identifies several state-specific determinants of FDI and investigates the changes in their importance during the period. Our results show that among the major determinants; real per capita income, real per capita expenditure on education, FDI related employment, real research and development expenditure, and capital expenditure have a significant positive impact on FDI inflows. There is also evidence that the share of scientists and engineers in the workforce exerts a small positive impact on inward FDI flows. In addition, per capita state taxes, unit labor cost, manufacturing density, unionization, and unemployment rate exert a negative impact on FDI inflows.

INTRODUCTION

During the past three decades, foreign direct investment (FDI) undertaken by transnational corporations has become one of the leading factors promoting the process of globalization. Foreign direct investment in the United States in particular has grown significantly during this period. For example, according to the United Nations Conference on Trade and Development (UNCTAD)'s *World Investment Report 2010*, the stock of FDI in the U.S. grew from $83.0 billion in 1980 to $539.6 billion in 1990, to $2,783.2 billion in 2000 and to $3,120.6 billion in 2009 (see Table 1). Though there has been a significant increase in the FDI to

the developing countries in recent years, the majority of these inflows still goes to developed countries, with developed countries accounting for 50.8% of FDI in 2009. Of these total worldwide FDI inflows, the U.S. received 11.7% in 2009. The FDI inflows to the U.S. increased from $48.4 billion in 1990 to $324.6 billion in 2008 but dropped to $129.9 billion in 2009 (see Table 2).

While the FDI inflows to the U.S. has grown significantly over the past two decades, the largest part of these flows went to four states, namely, Texas, California, New York, and Illinois (see Table 3). These four states have been the top recipient states of FDI since 1990. A significant research effort has been directed at establishing the determinants of foreign direct investment (FDI). However, only a very limited number of studies have focused on state-specific locational determinants. Moreover, the empirical literature has been limited in several respects, with most work focused exclusively on host country tax regimes. This paper investigates locational determinants of the inward foreign direct investment (FDI) among fifty states of the United States. The analysis uses annual data for the period 1997 until 2007.

The paper is structured as follows: The next section presents a survey of the literature. Section Model Specification presents the specification of the econometric model. Section Data Sources and Variables discusses the variables and data sources. The empirical results are presented and discussed in Section Empirical Results and finally, Section Summary and Conclusions summarizes the main results and concludes with some policy implications.

Table 1(a). Inward Foreign Direct Investment Stock, 1980-2009
(Billions of Current US Dollars)

Economy	1980	1990	2000	2009
Developed economies	401.6	1,555.6	5,653.2	12,352.5
of which: United States	83.0	539.6	2,783.2	3,120.6
Developing economies	298.6	524.5	1,728.5	4,893.5
Developing economies: Africa	41.1	60.7	154.2	514.8
Developing economies: America	41.8	111.4	502.1	1,472.7
Developing economies: Asia	214.2	349.6	1,067.7	2,893.8
Developing economies: Oceania	1.5	2.8	4.4	12.2
Developed economies: America	137.2	652.4	2,996.2	3,648.6
Developed economies: Asia	6.4	14.3	72.9	271.4
Developed economies: Europe	230.8	807.3	2,440.3	8,037.8
Developed economies: Oceania	27.1	81.6	143.8	394.7
World	**700.3**	**2,081.8**	**7,442.5**	**17,743.4**

Table 1(b). Share of Inward Foreign Direct Investment Stock, 1980-2009
(Percent)

Economy	1980	1990	2000	2009
Developed economies	57.4	74.7	76.0	69.6
of which: United States	11.9	25.9	37.4	17.6
Developing economies	42.6	25.2	23.2	27.6
Developing economies: Africa	5.9	2.9	2.1	2.9
Developing economies: America	6.0	5.4	6.7	8.3
Developing economies: Asia	30.6	16.8	14.3	16.3
Developing economies: Oceania	0.2	0.1	0.1	0.1
Developed economies: America	19.6	31.3	40.3	20.6
Developed economies: Asia	0.9	0.7	1.0	1.5
Developed economies: Europe	33.0	38.8	32.8	45.3
Developed economies: Oceania	3.9	3.9	1.9	2.2
World	**100.0**	**100.0**	**100.0**	**100.0**

Source: The United Nations Conference on Trade and Development (UNCTAD), *World Investment Report 2010*.

LITERATURE REVIEW

This section presents a brief overview of some related work. Although there has been considerable research concerning determinants of foreign direct investment, we only present findings of studies that analyze the locational determinants of foreign investment in the U.S.

Axarloglou and Pournarakis (2007) investigate the impact of FDI inflows on the local economies of the U.S. states that receive most of the FDI inflows in the country (Texas, California, New York, and Illinois). It appears that FDI inflows in manufacturing have rather weak effects on local employment and wages in most of the states in the sample. However, these results are primarily due to the industry composition of the FDI. FDI inflows in Printing and Publishing, Transportation Equipment and Instruments have positive effects on local employment and wages, while FDI inflows in Leather and Stone/Clay/Glass have detrimental effects on local labor markets in most of the states in the sample. These findings indicate the importance of industry characteristics in evaluating the effects of FDI inflows on local communities. Also, they emphasize the need for U.S. states to selectively target and attract FDI inflows in specific industries.

Table 2(a). Inward Foreign Direct Investment Inflows, 1980-2009
(Billions of Current US Dollars)

Economy	1980	1990	2000	2009
Developed economies	46.6	172.5	1,138.0	565.9
of which: United States	16.9	48.4	314.0	129.9
Developing economies	7.5	35.1	256.5	478.3
Developing economies: Africa	0.4	2.8	9.8	58.6
Developing economies: America	6.4	8.9	97.7	116.6
Developing economies: Asia	0.5	22.6	148.7	301.4
Developing economies: Oceania	0.1	0.7	0.2	1.9
Developed economies: America	22.7	56.0	380.9	148.8
Developed economies: Asia	0.3	1.9	15.3	15.8
Developed economies: Europe	21.4	104.4	724.9	378.4
Developed economies: Oceania	2.2	10.2	17.0	22.9
World	**54.1**	**207.7**	**1,401.5**	**1,114.2**

Table 2(b). Share of Inward Foreign Direct Investment Inflows, 1980-2009
(Percent)

Economy	1980	1990	2000	2009
Developed economies	86.1	83.1	81.2	50.8
of which: United States	31.3	23.3	22.4	11.7
Developing economies	13.8	16.9	18.3	42.9
Developing economies: Africa	0.7	1.4	0.7	5.3
Developing economies: America	11.9	4.3	7.0	10.5
Developing economies: Asia	1.0	10.9	10.6	27.0
Developing economies: Oceania	0.2	0.3	0.0	0.2
Developed economies: America	42.0	27.0	27.2	13.4
Developed economies: Asia	0.5	0.9	1.1	1.4
Developed economies: Europe	39.5	50.3	51.7	34.0
Developed economies: Oceania	4.1	4.9	1.2	2.1
World	**100.0**	**100.0**	**100.0**	**100.0**

Source: The United Nations Conference on Trade and Development (UNCTAD), *World Investment Report 2010*.

Table 3. Top 10 States with Largest Stock of Foreign Direct Investment, 1990-2007
(Millions of Current US Dollars)

1990		2000		2007	
State	FDI Stock	State	FDI Stock	State	FDI Stock
California	75,768	California	121,040	Texas	128,424
Texas	57,079	Texas	110,032	California	108,572
New York	36,424	New York	68,522	New York	80,474
Illinois	23,420	Illinois	48,425	Illinois	48,626
Ohio	20,549	Michigan	39,238	Ohio	43,438
Alaska	19,435	Florida	38,755	Pennsylvania	39,824
Florida	18,659	Ohio	37,530	New Jersey	38,425
New Jersey	18,608	New Jersey	35,115	Indiana	38,145
Louisiana	17,432	Pennsylvania	34,106	Florida	35,052
Georgia	16,729	Louisiana	31,160	Alaska	34,473

Source: U.S. Department of Commerce, *Bureau of Economic Analysis*.

Wijeweera, Dollery, and Clark (2007) analyze the relationship between corporate tax rates and foreign direct investment in the United States. The study uses a panel of nine investing tax exemptions and tax credits over the period 1982-2000 to find answers to two questions, namely, are corporate income tax rates an important determinant of FDI in the US? Also, do investors from tax credit countries differ significantly in their tax response relative to those from tax exemption countries?

Axarloglou (2005) evaluates the relative impact of industry and state specific economic factors on inward FDI in several U.S. states that compete for the same inward FDI. The study finds evidence that relative labor productivity, relative spending on education, and relative crime rate are important in inter-state competition for the same inward FDI. The findings also suggest that relative tax incentives become important in attracting FDI inflows when the contest in attracting inward FDI relates to two states.

In another study, Axarloglou (2004) evaluates the impact of industry and state specific economic conditions on inward FDI in several U.S. states. The study uses annual data for the 1974-1991 period. The results suggest that FDI inflows in the U.S. are strongly influenced by industry and state-specific labor productivity and by state spending on education. The findings also suggest that the quality of the local labor force, along with the efforts to improve this quality, is pivotal in attracting FDI inflows.

Chung and Alcácer (2002) examine whether and when state technical capabilities attract foreign investment in manufacturing from 1987-1993. The study finds that on average state R&D intensity does not attract foreign direct investment. Most investing firms are in lower-tech industries and locate in low R&D intensity states, suggesting little interest in state technical capabilities. In contrast, the study finds that firms in research-intensive industries are more likely to locate in states with high R&D intensity. Foreign firms in the pharmaceutical industry value state R&D intensity the most, at a level twice that of firms in the semiconductor industry, and four times that of electronics firms. Interestingly, not only firms from technically lagging nations, but also some firms from technically leading countries are attracted to R&D intensive states.

Keller and Levinson (2002) estimate the effect of changing environmental standards on patterns of international investment. The study employs an 18-year panel of relative abatement costs covering the period from 1977 to 1994 and controls for unobserved state characteristics. They find robust evidence that abatement costs have had moderate deterrent effects on foreign direct investment.

Hines (1996) compares the distribution between U.S. states of investment from countries that grant foreign tax credits with investment from all other countries. The ability to apply foreign tax credits against home-country tax liabilities reduces an investor's incentive to avoid high-tax foreign locations. The study uses data for 1987 and finds evidence that state taxes significantly influence the pattern of foreign direct investment in the United States.

Friedman, et al. (1996) examine the aggregation bias in Coughlin, Terza, and Arromdee's (1991) study. They find evidence that marked differences exist between the locational preferences of those investing in new manufacturing plants and those investing in mergers and acquisitions.

Hennart and Park (1994) examine the impact of location and governance factors for four types of strategic interactions, on a Japanese firm's propensity to manufacture in the U.S. The results support the view that foreign direct investment is explained by location, governance, and strategic variables. Economies of scale and trade barriers encourage Japanese FDI in the U.S. The larger a Japanese firm's R&D expenditures, the greater the probability it will manufacture in the U.S., but this is not the case for advertising expenditures. Some strategic factors are also important: Japanese firms with medium domestic market shares have the highest propensity to invest in the U.S. There is evidence of follow-the-leader behavior between firms of rival enterprise groups, but none of 'exchange-of-threat' between American and Japanese firms. Japanese investors are also attracted by concentrated and high-growth U.S. industries.

Coughlin, Terza, and Arromdee (1991) use a conditional logit model of the location decision of foreign firms investing in manufacturing facilities in the United States using annual data for the 1981-1983 period. They find evidence that states with higher per capita incomes, higher densities of manufacturing activity, higher unemployment rates, higher unionization rates, more extensive

transportation infrastructures, and larger promotional expenditures attract relatively more foreign direct investment. In addition, higher wages and higher taxes deter foreign direct investment.

This study uses annual data on state-level foreign direct investment covering all 50 states over the 11-year period from 1997 until 2007. The study tests the importance of several state-specific determinants of foreign direct investment.

MODEL SPECIFICATION

Drawing on the existing empirical literature in this area, we specify the following model:

$$FDI_{it} = \beta_0 + \beta_1 PCI_{it} + \beta_2 TAX_{it} + \beta_3 EDU_{it} + \beta_4 SE_{it} + \beta_5 FDIEMP_{it} + \beta_6 RD_{it} + \beta_7 CAP_{it}$$
$$+ \beta_8 LCOST_{it} + \beta_9 MANDEN_{it} + \beta_{10} UNION_{it} + \beta_{11} UNEMP_{it} + u_t \tag{1}$$

where FDI_{it} is the real foreign direct investment (FDI) inflows in state i in year t ($i = 1, 2, ..., 50$ and $t = 1, 2, ..., 11$); PCI_{it} is the per capita real disposable income of state i in year t; TAX_{it} is the per capita state taxes of state i in year t; EDU_{it} is the real per capita expenditure on education in state i in year t; SE_{it} is an indicator of labor quality as measured by the share of scientists and engineers in the workforce in state i in year t; $FDIEMP_{it}$ is the FDI related employment in state i in year t; RD_{it} is the real research and development (R&D) expenditure in state i in year t; CAP_{it} is the real capital expenditure in state i in year t; $LCOST_{it}$ is the unit labor cost in state i in year t; $MANDEN_{it}$ is the manufacturing density in state i in year t; $UNION_{it}$ is the share of the workforce that is unionized state i in year t; and $UNEMP_{it}$ is the unemployment rate in state i in year t.

The first variable, real per capita income is a measure of market demand in a state and is expected to be positively related to foreign direct investment. Therefore, *a priori*, we would expect that $\beta_1 > 0$. The real per capita state taxes usually deter FDI flows and, therefore, is expected to be negatively related to foreign direct investment; thus, we would expect that $\beta_2 < 0$. The third variable, the real per capita expenditure on education is expected to have a positive effect on foreign direct investment, i.e., $\beta_3 > 0$.

The next variable, the share of scientists and engineers in the workforce is expected to have a positive effect on foreign direct investment ($\beta_4 > 0$). Our fifth variable, the FDI related employment as a share of state total employment is expected to have a positive effect on foreign direct investment ($\beta_5 > 0$). Our sixth variable, the real research and development expenditure is expected to have a positive effect on foreign direct investment ($\beta_6 > 0$). Our seventh variable, the real capital expenditure is expected to have a positive effect on foreign direct

investment ($\beta_7 > 0$). Our eighth variable, the unit labor cost is expected to have a negative effect on foreign direct investment ($\beta_8 < 0$).

States with higher densities of manufacturing activity are expected to attract more foreign direct investment because the foreign investors might be serving existing manufacturers. As Coughlin, Terza, and Arromdee (1991) and Head, Ries and Swenson (1995, 1999) point out, manufacturing density could also be used as a proxy for agglomeration economies. The manufacturing density is expected to be positively related to foreign direct investment. Therefore, we expect β_9 to be greater than 0. The next variable, unionizatiotn of the workforce is considered to be a deterrent and therefore expected to be related negatively to foreign direct investment. Thus, we would expect that $\beta_{10} < 0$. The effect of unemployment on foreign direct investment could either be positive or negative. On one hand, unemployment rate reflects a pool of potential workers, thus higher unemployment rates across the states will likely be related positively to foreign direct investment. On the other hand, as Coughlin, Terza, and Arromdee (1991) argue higher unemployment rates could increase the amount that a firm must pay in unemployment insurance premiums. This would deter foreign firms with low labor turnover from investing in a state because they would be required to subsidize the unemployed workers who become unemployed by other firms. Thus, the expected sign of β_{11} could either be positive or negative.

DATA SOURCES AND VARIABLES

In order to test the implications of the model, we collect a panel of aggregate data on foreign direct investment in all U.S. states, excluding the District of Columbia. The data set includes 50 states for which foreign direct investment and all other relevant variables are reported over the 1997–2007 period. The real stock of FDI is measured in this study as the nominal stock of FDI deflated by the GDP deflator in constant (2000) U.S. dollars. The data on nominal stock of FDI are from the U.S. Department of Commerce, *Bureau of Economic Analysis* (BEA). The GDP deflators for states are derived by dividing the nominal gross state product by the real gross state product (base year = 100), both of which are obtained from the *Bureau of Economic Analysis*. The real per capita disposable income is measured as the nominal per capita disposable income deflated by the GDP deflator in constant (2000) U.S. dollars.

The real per capita taxes is measured by dividing the real state tax revenue by the state population. The nominal tax revenue for states are from various issues of the *Annual Survey of State Government Finances* published by the U.S. Department of Commerce. The nominal tax revenue is deflated by the GDP deflator to derive the real state tax revenue. The data on state population are from the *U.S. Census Bureau*. The real per capita expenditure on education is measured by dividing the real state education expenditure by the state population.

The nominal education expenditure for states are from various issues of the *Annual Survey of State Government Finances* published by the U.S. Department of Commerce. The nominal education expenditure is deflated by the GDP deflator to derive the real state education expenditure.

The share of scientists and engineers in the workforce, a proxy for labor quality, is collected from the National Science Foundation, Division of Science Resources Statistics, *Science and Engineering Indicators 2010*. The FDI related employment variable is measured as the ratio of FDI related employment to total state employment. The data on FDI related employment are from the *Bureau of Economic Analysis* while the data on state employment are from the U.S. Department of Labor, *Bureau of Labor Statistics*. The information on real research and development expenditure is from the National Science Foundation, Division of Science Resources Statistics, *Science and Engineering Indicators 2010*. Data on real capital expenditure at the state level are not readily available. Therefore, the capital expenditure on manufacturing is used as a proxy. The information on capital expenditure on manufacturing is from the U.S. Census Bureau, *Annual Survey of Manufactures: Geographic Area Statistics* series.

The unit cost variable is measured following the procedure used by Axarloglou (2004). The unit labor cost is defined as:

$$LCOST_{it} = \frac{w_{it}}{APL_{it}} \tag{2}$$

where w_{it} is the average wage rate in state i in year t and APL_{it} is the average product of labor in state i in year t. The average product of labor is calculated as:

$$APL_{it} = \frac{RGSP_{it}}{EMP_{it}} \tag{3}$$

where $RGSP_{it}$ is the real gross state product of state i in year t and EMP_{it} is the total employment in state i in year t. The data on the average wage and total state employment are from the U.S. Department of Labor, *Bureau of Labor Statistics*. Following Coughlin, Terza, and Arromdee (1991), the manufacturing density variable is measured as the manufacturing employment per square mile of state land excluding federal land. The data on manufacturing employment are from the U.S. Department of Labor, *Bureau of Labor Statistics*. The information on union membership is collected from http://www.unionstats.com/ maintained by Barry Hirsch (Georgia State University) and David Macpherson (Trinity University). The data on state unemployment rate are collected from the U.S. Department of Labor, *Bureau of Labor Statistics*.

EMPIRICAL RESULTS

The results of the empirical analysis are presented in Table 4. In addition to the eleven independent variables included in Equation (1), we experimented with several other variables including the growth rate of real gross state product, highway mileage, land area, number of airports, railway mileage, labor productivity, average hourly wage rate, real per capita exports, and right-to-work regulation. However, they were dropped from the model to minimize the problems of multicolinearity and incorrect signs. All the variables presented in Table 4 are expressed in logarithm and the coefficient of each variable can be interpreted as elasticity.

Table 4. Determinants of FDI in the United States Panel Least Squares Estimates
        (Dependent variable: Real FDI Inflows)

Variable	Coefficient	t-statistic
Constant	33,2684***	3,25
Real Per Capita Income	0,8839	0,92
Real Per Capita Taxes	- 3,3844**	-2,41
Real Education Expenditure	0,5549*	1,80
Scientists and Engineers	0,0558	0,29
FDI Related Employment	2,2268***	8,49
Research and Development	0,2373***	4,31
Real Capital Expenditure	0,5568***	7,68
Unit Labor Cost	- 2,5333	-1,00
Manufacturing Density	- 0,1328***	-3,53
Unionization	- 0,7159*	-1,83
Unemployment	- 3,5858***	-13,60
Adjusted R^2	0,3669	
Number of Observations	376	

Source: own calculations

Note: ***, **, and * indicate the statistical significant at 1%, 5%, and 10% level, respectively.

Real per capita disposable income variable has the expected positive sign but it is not statistically significant. This result is similar to the findings of Coughlin, Terza, and Arromdee (1991) and Axarloglou (2004). The real per capita taxes also has the expected negative sign and it is statistically significant at the 5% level of significance. This finding is consistent with the findings of previous studies.

The results suggest that the real inflow of FDI in the U.S. is influenced by the state spending on education. The coefficient of this variable is positive and statistically significant at the 10% level of significance. This result is consistent with the findings of the study by Axarloglou (2004). The share of scientists and

engineers in the workforce has the expected positive sign but it is not statistically significant.

The FDI related employment variable has a positive statistically significant effect on the real inflow of FDI. This variable is statistically significant at the 1% level of significance. This could be due to the fact that the states with high level of FDI inflows also have larger FDI related employment. The state's expenditure on research and development is also found to have a positive effect on the real stock of FDI. This variable is statistically significant at the 1% level of significance. The real capital expenditure variable also has the expected positive sign and it is statistically significant at the 1% level of significance. This could be due to the fact that of the capital expenditure on manufacturing a larger part of FDI flows are in the manufacturing sector.

The unit labor cost variable has the expected negative sign. However, this variable is not statistically significant. Manufacturing density variable has an unexpected negative sign but it is statistically significant at the 1% level of significance, what can be explain by the fact that recent FDI flows relate to a green field investment located in the newly explored regions of the U.S.

The unionization variable has the expected negative sign and it is statistically significant at the 10% level of significance. This result is not consistent with the findings of Coughlin, Terza, and Arromdee (1990, 1991), Beeson and Husted (1989) and Bartik (1985). Finally, the results show that the unemployment rate is a negative, statistically significant determinant of foreign direct investment. This result is not consistent with our prior expectations. We can conclude that, the unemployment rate is a signal of the availability of labor that affects investors.

SUMMARY AND CONCLUSIONS

This paper investigates locational determinants of inward foreign direct investment (FDI) flows among the fifty states of the United States. In order to test the implications of the model, we use a panel of aggregate data on foreign direct investment on all U.S. states, excluding the District of Columbia. The data set includes 50 states for which foreign direct investment and all other relevant variables are reported over the 1997–2007 period.

Findings of our results show that real per capita disposable income has the expected positive sign but it is not statistically significant. The real per capita taxes also has the expected negative sign it is statistically significant at the 5% level of significance. These findings are consistent with the findings of previous studies.

The study shows that the real inflows of FDI in the U.S. is influenced by the state spending on education. The coefficient of this variable is positive and statistically significant at the 10% level of significance. As expected, the share of scientists and engineers in the workforce has the expected positive sign. However, it is not statistically significant.

The FDI related employment variable has a positive and highly statistically significant effect on the real inflow of FDI. This may be due to the fact that the states with high level of FDI inflows also have larger FDI related employment. The state's expenditure on research and development is also found to have a positive and significant effect on the real stock of FDI. This variable is statistically significant at the 1% level of significance. The real capital expenditure variable also has the expected positive sign, statistically significant at the 1% level of significance. This is due to the fact that of the capital expenditure on manufacturing a larger part of FDI flows are in the manufacturing sector.

Among other findings, unit labor cost has the expected negative sign; manufacturing density has an unexpected negative sign but it is statistically significant at the 1% level of significance; unionization variable also has the expected negative sign, statistically significant at the 10% level of significance; and the unemployment rate is a negative, statistically significant determinant of foreign direct investment. Some of these findings are consistent with findings of previous studies.

Given that the current results suggest that state government taxation negatively affect foreign direct investment inflows, state governments may consider providing more fiscal incentives to foreign investors in order to attract more foreign direct investment to their states. Another way for states to attract more investment is to spend more on education, improvements in labor quality, research and development activities and capital expenditure. This could, however, be a long-term goal. While the present study use aggregate data, another avenue of future research is to investigate the possibility that the location determinants vary across both countries and industries.

REFERENCES

Axarloglou, K. and Pournarakis, M. (2007) Do all foreign direct investment inflows benefit the local economy? The World Economy 30(3), 424-432.

Axarloglou, K. (2005). What attracts foreign direct investment inflows in the United States. The International Trade Journal 19(3): 285-308.

Axarloglou, K. (2004) Local labor market conditions and foreign direct investment flows in the U.S. Atlantic Economic Journal 32(l):62-66.

Bartik, T. J. (1985) Business location decisions in the United States: Estimates of the effects of unionization, taxes, and other characteristics of States. Journal of Business and Economics Statistics 3:14-22.

Beeson, P. E. and Husted, S. (1989) Patterns and determinants of productive efficiency in state manufacturing. Journal of Regional Science 29(1): 15-28.

Chung, W. and Alcácer, J. (2002) Knowledge seeking and location choice of foreign direct investment in the United States. Management Science 48(12): 1534-1554.

Coughlin, C. C, Terza, J. V., and Arromdee, V. (1990) State government effects on the location of foreign direct investment. Regional Science Perspectives 20:194-207.

Coughlin, C. C, Terza, J. V., and Arromdee, V. (1991) States characteristics and the location of foreign direct investment within the United States. Review of Economics and Statistics 73:675-683.

Friedman, J., Fung, H. G., Gerlowski, D. A., and Silberman, J. (1996) A Note on state characteristics and the location of foreign direct investment within the United States. Review of Economics and Statistics 78(2):367-368.

Head, C. K., Ries, J. C, and Swenson, D. L. (1995) Agglomeration benefits and location choice: Evidence from Japanese manufacturing Investments in the United States. Journal of International Economics 38:223-247.

Head, C. K, Ries, J. C, and Swenson, D. L. (1999) Attracting foreign manufacturing: investment promotion and agglomeration. Regional Science and Urban Economics 29: 197-218.

Hennart, J-F. and Park, Y-R. (1994) location, governance, and strategic determinants of Japanese manufacturing investment in the United States. Strategic Management Journal. 15(6): 419-436.

Hines, Jr. J. R. (1996) Altered States: Taxes and the location of foreign direct investment in America. The American Economic Review. 86(5): 1076-1094.

Hirsch, B. and Macpherson, D. (2010). http://www.unionstats.com/

Keller, W. and Levinson, A. (2002). Pollution abatement costs and foreign direct investment inflows to U.S. States. The Review of Economics and Statistics. 84(4): 691-703.

National Science Foundation, Division of Science Resources Statistics, Science and Engineering Indicators (2010).

United Nations Conference on Trade and Development (UNCTAD), World Investment Report 2010.

U.S. Census Bureau, Annual Survey of Manufactures: Geographic Area Statistics (various issues).

U.S. Department of Commerce, Bureau of Economic Analysis (BEA).

U.S. Department of Commerce, Annual Survey of State Government Finances (various issues).

U.S. Department of Labor, Bureau of Labor Statistics.

Wijeweera, A., Dollery, B. and Clark, D. (2007) Corporate tax rates and foreign direct investment in the United States. Applied Economics 39(1): 109-117.

STOCHASTIC FRONTIER ANALYSIS
OF REGIONAL COMPETITIVENESS[1]

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Abstract: The regional competitiveness is the source of national competitiveness and the efficiency measuring and relative regional efficiency comparison are crucial questions for analysts as well as for economic policy creators. Regional competitiveness becomes a subject of evaluation due to increasing significance of regions in concept of European Union. This paper deals with the application of parametric benchmarking method – Stochastic Frontier Analysis (SFA) for measuring technical efficiency of NUTS2 regions of V4 countries within the time period of 8 years.

Key words: Stochastic Frontier Analysis, Technical Efficiency, NUTS2 regions, Competitiveness

INTRODUCTION

During the last few years economic policy making and research have shown increasing interest for regional competitiveness evaluation. The increasing significance of regional competitiveness evaluation deserves more attention especially because of the economic efficiency of regions representing the basis of economic success for micro-economic level and also the competitiveness of the country. Competitiveness is a complex economic phenomenon, with many definitions and quantification methods upon which the specialists have not yet

reached full consensus, but the need of competitiveness gaining is frequently discussed both in the economic literature and in the everyday practice. At the same time, the increasing importance of competitiveness issues may be explained by the deeper economic integration and increased globalization, which require a constant increase in the competitive power of every economic entity belonging to a certain country, as well as in the competitive power of the country itself. There is no general consensus about what regional competitiveness means and there are many different its definition. For example the European Commission interprets the term the following way: "Competitiveness is the ability to produce goods and services which meet the test of international markets, while at the same time maintaining high and sustainable levels of income or, more generally, the ability of companies, industries, regions, nations and supra-national regions to generate, while being exposed to international competition, relatively high income and employment levels" [European Commission 1999].

METHODOLOGY FOR ANALYZING REGIONAL COMPETITIVENESS

Approaches to evaluation of regional competitiveness went through many debates because of non existence of unique methodology for measuring and evaluation competitiveness. Even the definition of „region" is problematic because the regional competitiveness is not a simple sum of competitiveness of the firms located in a given region. For this reason the regional competitiveness evaluation is determined by the selection of the regional unit. Growing interest in Europe for regional competitiveness may be explained by the strength of sub-national territorial EU units and the NUTS (Nomenclature of Units for Territorial Statistics) nomenclature can be used for region classification. The non existence of the unique method for regional competitiveness evaluation caused that there are various evaluation methodologies and approaches. One of these regional evaluation methods is the method according to Viturka [Nevima et al. 2009] and his method is oriented to a long term horizont. For short term horizont regional competitiveness evaluation could be used the group of specific economic indicators of efficiency [Nevima et al. 2009]. The basic idea is to identify the internal sources of regional competitiveness in detail. There are proposed following specific indicators: coefficient of employment, coefficient of efficiency of disposability, coefficient of efficiency in development, coefficient of efficiency of investment construction, coefficient of efficiency of revenues and coefficient of efficiency of building works. Each specific coefficient compares a concrete level of the value in the region with respect to its total level in the country. These specific economic indicators are the basis for further analysis and through the techniques of multicriteria decision making methods should be obtained the comparison and final ranking of the regions. From multicriteria decision making methods could be

exploit Ivanovic deviation (for more details see [Nevima et al. 2009]), variety of multicriteria alternative evaluation methods (e.g. TOPSIS, PROMETHEE, ELECRE) or models of Data Envelopment Analysis – DEA[2]. DEA models are able to identify the best performers (regions) and separate them from their inefficient counterparts and in addition to identify the sources of inefficiency of units (regions). DEA models are based on benchmarking, that is, measuring a unit's (region's) efficiency compared with a reference performance (so-called efficient frontier). Inefficiency of unit's can result from technological deficiencies or non-optimal allocation of resources into production. Both technical and allocative inefficiencies are included in cost inefficiency, which is by definition, the deviation from minimum costs to produce a given level of output with given input prices. The efficient frontier (cost or production) is unknown and must be empirically estimated from the real data set by parametric and nonparametric techniques (see [Coelli et al. 2005]). Due to the impossibility of the separation of the inefficiency effect from the statistical noise which is the shortcoming of DEA we decided to exploit parametric benchmarking method, namely Stochastic Frontier Analysis (SFA) for regional competitiveness evaluation. This methodology is based on econometric theory and pre-specified functional form is estimated and inefficiency is modeled as an additional stochastic term. In our analysis we tried to use SFA for measuring technical efficiency of NUTS2 regions of V4 countries within the time period of 8 years. The analysis is based on production function principle in macroeconomic context, evaluated regions are treated as producers of output given some inputs. In our analysis are applied various versions of stochastic frontier production function models only for panel data due to the fact that panel data provide information on the same units over several periods that is not possible with cross section data. We estimated levels of technical efficiency for each NUTS2 region and the differences in estimated scores, parameters and ranking of regions are compared across different panel data models.

STOCHASTIC FRONTIER ANALYSIS – PARAMETRIC BENCHMARKING TECHNIQUE

One of the simplest structures we can impose on the inefficiency effect is

$$u_{it} = u_i \quad i = 1,...,N \quad t = 1,...,T \tag{1}$$

where u_i is treated as either a fixed parameter or random variable. These models are known as the fixed effects model and random effects model respectively. Supposing that technical efficiency is time invariant, a Cobb-Douglas production frontier with time invariant technical efficiency can be written as:

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[2] DEA is non-parametric benchmarking method and originates from operations research and uses linear programming to calculate an efficient deterministic frontier against which units are compared.

$$\ln y_{it} = \beta_0 + \sum_n \beta_n \ln x_{nit} + v_{it} - u_i \quad i = 1,\dots,N \quad t = 1,\dots,T \qquad (2)$$

where $\ln y_{it}$ is logarithm of output quantities, $\ln x_{nit}$ is logarithm of inputs quantities, β is unknown vector of parameters to be estimated, v_{it} is symmetric random variable, u_i is time invariant technical inefficiency term of compound error term $\varepsilon_{it} = v_{it} - u_i$.

In this specification the error term is composed of two uncorrelated parts. The first part u_i is capturing the effect of technical inefficiency and the second part v_{it} is reflecting effect of statistical noise. This random effect model can be estimated using Maximum Likelihood Estimation (MLE) method or Method of Moments. Using Maximum Likelihood Estimation method requires make distribution assumptions for stochastic terms. Usually we assume that v_{it} are random variables to be normally distributed ($v_{it} \sim$ iid N(0, σ_v^2)) and u_i are non negative time-invariant random variables to be half normal distributed ($u_i \sim$ iid $N^+(0, \sigma_u^2$)) or truncated normal distribution ($u_i \sim$ iid $N^+(\mu, \sigma_u^2)$)) can be also considered. The next step is to obtain estimates of the technical efficiency of each unit. The problem is to extract the information that ε_i contains on u_i (we have estimates of $\varepsilon_{it} = u_i + v_{it}$, which obviously contain information on u_i). A solution to the problem is obtained from the conditional distribution of u_i given ε_i, which contains whatever information ε_i contains concerning u_i. This procedure is known as JLMS decomposition (for more details see [Jondrow et al. 1982]). For separation the inefficiency effect from the statistical noise can be also used an alternative minimum squared error predictor estimator (for more details see [Kumbhakar et al. 2000]). Once the point estimates of u_i are obtained, estimates of the technical efficiency of each unit can be obtained by substituting them into equation (3). If the production frontier is specified as being stochastic, the appropriate measure of individual technical efficiency becomes:

$$TE_i = \frac{y_{it}}{f(x_{it}, \beta) \exp\{v_{it}\}} = \exp\{-u_i\} \quad i = 1,\dots,N \quad t = 1,\dots,T \qquad (3)$$

which defines technical efficiency as the ratio of observed outputs quantities to the maximum outputs quantities attainable in an environment characterized by $\exp\{v_{it}\}$.

If we allow efficiency changes in time, inefficiency component will consist of two parts, namely cross-section component (u_i) and time component (β_t):

$$u_{it} = u_i + \beta_t \quad i = 1,\dots,N \quad t = 1,\dots,T \qquad (4)$$

The time invariant production efficiency model given by equation (2), we reformulate as follows:

$$\ln y_{it} = \beta_{0t} + \sum_n \beta_n \ln x_{nit} + v_{it} - u_{it} \quad i = 1,...,N \quad t = 1,...,T \tag{5}$$

where β_{0t} is the production frontier intercept common to all units in period t, $\ln y_{it}$ is logarithm of output quantities, $\ln x_{nit}$ is logarithm of inputs quantities, β is unknown vector of parameters to be estimated, v_{it} is symmetric random variable, u_{it} is time variant technical inefficiency term of compound error term $\varepsilon_{it} = v_{it} - u_{it}$. There are proposed various approaches to estimated time varying production frontier model given by equation (5) (for more details see [Kumbhakar et al. 2000]).

Battese and Coelli (see [Coelli et al. 2005] or [Kumbhakar et al. 2000]) presented a model where they model the inefficiency component in (4) according to following exponential time function:

$$u_{it} = \exp\{-\eta(t - T)\}u_i \quad i = 1,...,N \quad t = 1,...,T \tag{6}$$

where η is unknown parameter to be estimated. The function value is determined by value of parameter η and number of observations. The function is decreasing for $\eta > 0$, increasing for $\eta < 0$ or constant for $\eta = 0$, i.e. if $\eta > 0$ technical inefficiency will have decreasing effects through time (positive effect in technical efficiency over time) and $\eta < 0$ inefficiency will be always increasing through time. This function does not allow a change in the rank ordering of unites over time, the unit that is ranked n-th at the first period is always ranked n-th. This fact is main shortcoming of this formulation. On the other hand this model requires additional estimation only of one parameter η. This model can be estimated by using the method of maximum likelihood. The likelihood function of this model is a generalization of the likelihood function for the conventional model (for more details see e.g. [Kumbhakar et al. 2000]). Estimates of the technical efficiency of each unit at time t can be obtained by substituting estimates of u_{it} into equation (7):

$$TE_{it} = \exp\{-u_{it}\} \quad i = 1,...,N \quad t = 1,...,T \tag{7}$$

MODEL SPECIFICATION AND DATA

Above mentioned SFA models have been used for analyzing the process of regional competitiveness differentiation of V4 (Visegrad four countries) countries regions. The territorial unit of our analysis will be the sub-national territorial unit of public administration (NUTS2). Slovakia has 4 NUTS2 regions, Czech Republic has 8 NUTS2 regions, Hungary has 7 NUTS2 regions and Poland is divided into 16 NUTS2 regions. Our balanced panel data set of 35 NUTS2 regions observed over a period from 2001 to 2008 includes 280 observations in total. All data are based on information from regional statistics of OECD and Eurostat. The evaluated regions are treated as producers of output given some

inputs. The output and input selection is a crucial step in our analysis and must be done with respect to the competitiveness definition. We shall compare the competitiveness of regions through the estimated levels of technical efficiency as the efficiency we perceive as the "mirror" of the competitiveness. Overall performance of the regional economy affects employment in various sectors, therefore, we selected as the first input Employment Rate - *ER* (annually in %). Efficiency in our model should demonstrate the ability of the regions to transform its capital for its further development. For this reason as the second input was chosen Gross Fixed Capital Formation – *GFCF* (in % of GDP). This indicator includes investment activity of domestic companies and fixed assets of foreign companies and is largely influenced by the inflow of foreign investment. The third included input is the Net Disposable Income of Households - *NI* (per capita). In terms of competitiveness the disposable income plays an important role because it directly reflects the purchasing power of the region. Full picture of efficiency in regional competitiveness investigating might provide also input Income of Corporation. Our analysis could not be improved by adding this additional input data due to unavailability of this indicator in the structure of NUTS2 in regional OECD and Eurostat statistics. Previously mentioned inputs are used to produce one output, output is measured by Gross Domestic Product – *GDP* (in purchasing power parity standards per capita) the most important macroeconomic indicator.

The first part of analysis was based on the assumption of time invariant technical efficiency. We applied SFA panel data models with time invariant technical efficiency assumption (Model1 and Model2) and the analysis was based on the estimation of the model given by equation (8):

$$\ln\left(GDP_{it}\right) = \beta_0 + \beta_1 \ln\left(ER_{it}\right) + \beta_2 \left(\ln GFCF_{it}\right) + \beta_3 \left(\ln NI_{it}\right) + v_{it} - u_i \tag{8}$$
$$i = 1,...,N \quad t = 1,...,T$$

In order to estimate the model with time varying technical efficiency (Model3) we formulated model given by equation (9):

$$\ln\left(GDP_{it}\right) = \beta_0 + \beta_1 \left(\ln ER_{it}\right) + \beta_2 \ln\left(GFCF_{it}\right) + \beta_3 \left(\ln NI_{it}\right) + v_{it} - u_{it} \tag{9}$$
$$i = 1,...,N \quad t = 1,...,T$$

where

$$u_{it} = \exp\left\{-\eta(t-T)\right\}u_i$$

In all models v_{it} is reflecting effect of statistical noise and u_i or u_{it} are random variables reflecting time invariant or time varying technical inefficiency respectively. Remaining variables have been defined before. The traditional Cobb-Douglas production function has been considered and MLE method has been used in all three models. The MLE method requires making the distributional assumptions for stochastic terms. We made following distributional assumptions:

Model1*: $v_{it} \sim iid\ N(0,\ \sigma_v^2)$, $u_i \sim iid\ N^+(0,\sigma_u^2)$,*

Model2*: $v_{it} \sim iid\ N(0,\ \sigma_v^2)$, $u_i \sim iid\ N^+(\boldsymbol{\mu},\sigma_u^2)$,*

Model3*: $v_{it} \sim iid\ N(0,\ \sigma_v^2)$, $u_{it} \sim iid\ N^+(\boldsymbol{\mu}, \sigma_u^2)$.

In all models for separation the inefficiency effect from the statistical noise was used Battese and Coelli point estimator (see [Coelli et al. 2005]). The individual technical efficiency estimates were obtained by substituting the inefficiency effects to the equations (3) and (7). The final estimates of the parameters all frontier models are listed in Table 1. Table 2 provides efficiency scores according to all models.

Table 1. Parameters of the Production Function

	Model1		Model2		Model3	
	Estimate	Stand. error	Estimate	Stand. error	Estimate	Stand. error
β_0	1,6009*	0,1694	-2,7287*	0,4302	-4,2246*	0,9951
β_1	0,1795*	0,0426	0,8061*	0,0957	0,4728	0,9104
β_2	-0,0567*	0,0171	-0,1654*	0,0403	-0,0611	0,4835
β_3	0,8249*	0,0099	1,0046*	0,0343	1,3508*	0,9587
σ^2	0,0771*	0,0089	0,0208*	0,0042	0,0377	0,1070
γ	0,9890*	0,0016	0,8951*	0,0203	0,8502	0,9376
μ			0,2728*	0,0812	0,1151	0,3277
η					-0,0961	0,1808
logLF	399,1864		305,3365		298,4673	

Source: own calculations (Frontier 4.1)
* significant at $\alpha = 0,05$

CONCLUSION

Inefficiency effects were estimated by using three models SFA. As for Model1 and Model2 we can see (Table1) that all the parameters are significant and the parameters are mildly different from one model to another. All estimated parameters have expected positive signs besides surprisingly negative sign of parameter β_2 corresponding to variable Gross Fixed Capital Formation. Model1 and Model2 are both based on the assumption time invariant technical efficiency but were estimated under two different distributional assumptions for inefficiency term, the first is the half normal distribution (Model1) and the second is truncated normal distributional assumption (Model2). Therefore additional estimated parameter μ is listed for Model2 in Table 1. Although the mean of truncated normal distribution μ is found to be significant, half normal specification is preferred for the distribution of u following results of LR test. If the model has been estimated by the method of maximum likelihood, hypothesis concerning individual coefficients or more than one coefficient can be tested using LR test. Our null hypotheses of special interest was set as $H_0 : \mu = \eta = 0$, which implies time

invariant half normal inefficiency effects and this hypotheses was confirmed by this LR test.

Table 2. Efficiency scores – Model1, Model2

Name of region	Code of region	Model1	Model2
Bratislavský kraj	SK01	1,0000	0,9123
Západné Slovensko	SK02	0,7427	0,6904
Stredné Slovensko	SK03	0,6688	0,6612
Východné Slovensko	SK04	0,6499	0,6764
Average Efficiency		0,7653	0,7351
Praha	CZ01	1,0000	0,9948
Střední Čechy	CZ02	0,7781	0,6464
Jihozápad	CZ03	0,8106	0,6871
Severozápad	CZ04	0,7735	0,7009
Severovýchod	CZ05	0,7762	0,6597
Jihovýchod	CZ06	0,8163	0,7062
Střední Morava	CZ07	0,7412	0,6565
Moravskoslezko	CZ08	0,7869	0,7406
Average Efficiency		0,8104	0,7240
Kosep-Magyarorszag	HU10	0,9758	0,8077
Kosep-Dunantul	HU21	0,7688	0,7047
Nyugat-Dunantul	HU22	0,8469	0,7600
Del-Dunantul	HU23	0,6769	0,6901
Eszak-Magyarorszag	HU31	0,6479	0,6921
Eszak-Alfold	HU32	0,6839	0,7322
Del-Alfold	HU33	0,6619	0,6537
Average Efficiency		0,7517	0,7201

| Name of region	Code of region	Model1	Model2
Lódzkie	PL11	0,6266	0,5489
Mazowieckie	PL12	0,9048	0,8313
Malopolskie	PL21	0,6546	0,6296
Slaskie	PL22	0,7104	0,7312
Lubelskie	PL31	0,5639	0,5132
Podkarpackie	PL32	0,6122	0,6283
Swietokrzyskie	PL33	0,6007	0,5682
Podlaskie	PL34	0,6088	0,6007
Wielkopolskie	PL41	0,7206	0,6877
Zachodnio-pomorskie	PL42	0,6773	0,7184
Lubelskie	PL43	0,6741	0,6712
Dolnoslaskie	PL51	0,7339	0,7675
Opolskie	PL52	0,6948	0,7601
Kujawsko-Pomorskie	PL61	0,6602	0,6437
Warminsko-Mazurskie	PL62	0,6303	0,6637
Pomorskie	PL63	0,7428	0,7985
Average Efficiency		0,6760	0,6726

Source: own calculations (Frontier 4.1)

The time invariant technical efficiency assumption was relaxing in Model3 following model defined in equation (5) and (6). This model is preferred in applied work due to its simplicity and flexibility. Modeling the variation in efficiency across time requires estimation of only one extra parameter, namely parameter η. In this model, the inefficiency term is modeled as a truncated normal random variable multiplied by a specific function of time. However, this simplicity comes at a cost. This specification does not allow for efficiency to increase or decrease at a decreasing rate. The time invariant technical assumption relaxation has led to non signification almost all estimated parameters. Moreover negative sign of insignificant parameter η means that inefficiency will be always increasing through time or technical efficiency of regions will have decreasing effects through time.

According to results of LR test applied before we decided to prefer Model1 to Model2 and Model3. In our data set is not necessary apply time varying efficiency model and half normal distribution assumption for inefficiency term is preferred.

Figure 1 provides efficiency estimates and ordering of V4 NUTS2 regions according to Model1. The scores can move between 0 and 1, where the highest value implies a perfectly efficient region. The highest competitive regions (see Table2 or Graph1) are regions CZ01 – Praha and SK01 – Bratislavský kraj (perfectly technically efficient regions, the efficiency scores equals 1). Next positions belong to regions HU10 – Kosep-Magyarorszag (0,9758) and PL12 – Mazowieckie (0,9048). We can notice that these "best regions" are capital cities regions and usually have better economic positions than other regions of country with weaker economic positions. As low competitive regions according to our analysis were evaluated polish regions and the lowest competitive region was region PL31 – Lubelskie and its technical efficiency level was only 56,39 % . The highest average efficiency achieved Czech regions (81,04 %) and the lowest average efficiency had Polish regions (67,60 %).

The individual efficiency estimates are relatively stable across Model1 and Model2 even despite of different distribution assumption about inefficiency component. In consequence of relaxation time invariant efficiency assumption we obtained efficiency scores for each observed year in Model3. In spite of this advantage and for reason of non signification of η and following results of LR test mentioned before we prefer Model1 in our data set (by this reason and insufficient space we present efficiency scores results only according to Model1). Due to the absence of the methodological approach to regional competiveness mainstream we presented SFA methodology as a contribution to the discussion about quantitative measurement of competitiveness at the regional level.

Figure 1. Efficiency scores for NUTS 2 regions – Model1
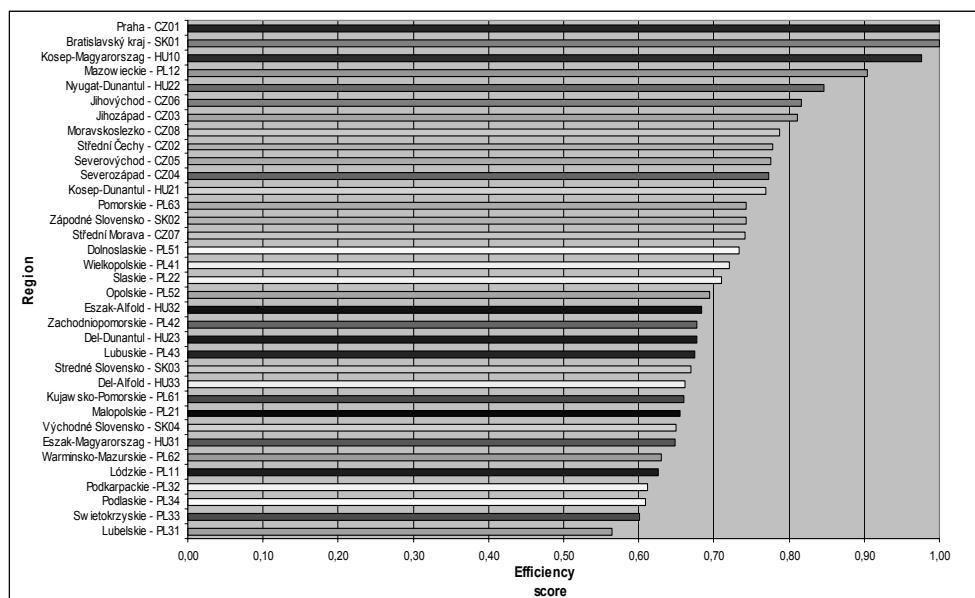
Source: own calculations

REFERENCES

Battese, G. E., Coelli, T. J. (1992) Frontier production functions, technical efficiency and panel data: with application to Paddy farmers in India, Journal of Productivity Analysis, Volume 3, pp. 153 – 169, ISSN 0895-562X.

Coelli, T. J., Rao Prasada, D., Battese, G. (2005) An introduction to efficiency and productivity analysis, Springer, ISBN 0387242661.

Enyedi, G. (2009): Competitiveness of the Hungarian regions, Hungarian Geographical Bulletin. Volume 58, No 1, pp. 33–48.

European Commission 1999 (1999) Sixth periodic report on the social and economic situation of regions in the EU, European Commission, Brussels.

Jondrow, J., Lovell, C. A. K., Materov, I. S, Schmidt, P. (1982) On the estimation of technical inefficiency in the stochastic frontier production function model, Journal of Econometrics, Volume 19, pp. 233 – 238, ISSN 0304-4076.

Kumbhakar, S. C., Lovell, C. A. K. (2000) Stochastic frontier analysis. Cambridge University Press, ISBN 05216666635.

Nevima, J., Ramík, J. (2009) Application of multicriteria decision making for evaluation of regional competitiveness, Mathematical Methods in Economics, 27th International Conference, Praha, ISBN 978-80-213-1963-9.

Ramík, J. (2010) Multictriteria approaches to competitiveness, Quantitative Methods in Economics Multiple Criteria Decision Making XV – International Scientific Conference, Smolenice, ISBN 978-80-8078-364-8.

IDENTIFICATION OF WEB PLATFORMS USAGE PATTERNS
WITH DYNAMIC TIME SERIES ANALYSIS METHODS

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Summary: The paper proposes a new approach to modelling online social systems users' behaviours based on dynamic time wrap algorithm integrated with online system's databases. The proposed method can be applied in the field of community platforms, virtual worlds and massively multiplayer online systems to capture quantitative characteristic of usage patterns.

Key words: social platforms, time series analysis, web users' behaviour

INTRODUCTION

Analyses of behaviour of Internet systems users plays a crucial role in decision-making processes and management of online platforms. This also relates to users of social services, in which analyses enable users to get to know the behaviour patterns and determine users' needs in a more efficient way. The research can be done in many scopes, and there are different approaches applied concerning data processing that are based on web mining algorithms or on social network analysis. In services focused on communities, the significant area that is a subject to exploration is the activeness of users and identification of behaviour patterns. The article herein presents new approach based on dynamic time wrap algorithm integrated with online system database that identifies two dimensions of temporal data analysis based on real and virtual time that enables the segmentation of users focused on time conditioning and trends identification. Research conducted in the real environment indicates numerous areas of applications and gives the basis for development of new methods and theoretical models. The proposed method allows the comparison of sequences in relation to the ideal pattern that represents the maximum possible number of logons or reference to the behaviour of other users.

MOTIVATION AND RELATED WORK

The development of social services, online games systems, and virtual worlds creates the demand of implementation of new methods for processing of data focused on those environments. The analyses aim to recognize different occurrences and trends and to acquire knowledge that can have a significant role in the development of online services. In this trend there is increased interest in, among others, methods of social network analysis, which applications in relation to Internet web systems is numerously present in literature concerning detection of social leaders and communities structures [Newman 2006][Newman, Girvan 2006], diffusion and marketing message [Marciniak, Budnarowska 2009][Acar, Polonsky 2008], recommendation message [Jung 2005][Gursel, Sen 2009][Che, Hu 2010] or dynamics of network structure development [Marsili, Slanina 2010], as well as in other fields. There are also developed solutions deriving from methods of knowledge discovery in databases, which with reference to Internet systems acquire a specialized form of Web mining algorithms, which are widely discussed in literature. They can be applied, among others, at the basis of agent systems [Gomes, Canuto 2006][Chau et al. 2003], in recommending systems [Kazienko 2009], electronic marketing [Chun-Ling et al. 2010], and in design of intelligent interfaces and adaptive portals. In most kind of appliances, the data has a temporal form, and time conditioning is taken into consideration. According to A.S. Dick and K. Basu, identification of consumers and increasing their loyalty constitutes a significant element of marketing actions [Dick, Basu 1994]. What is also important in creating loyalty towards advanced online services is the habit and limitations of introduction threshold where getting to know a new system is time-consuming. According to H. Tsai and H. Huang [Tsai, Huang 2007], it can discourage searching for alternatives. F. Wangenheim and T. Bayona state that behavioural aspects of loyalty translate into the purchase of a service and recognition of the superiority of an individual service over other subjects on the market [Wangenheim, Bayon 2004]. The loyalty of Internet system users constitutes a subject of separate research. H.P. Lu and S. Wang identify a dependency between users' satisfaction and loyalty in connection to Internet addiction [Lu, Wang 2008]. The development of technologies based on communities increases users' interactions and can be an element of strategies focused on loyalty building, e.g. in a form of weblogs in the scope of corporative systems, which was indicated by S.C. Herring and others [Herring et al. 2005]. Interpersonal engagement is an additional factor of loyalty building; bonds occur not only on the line of users - technology supplier, but also in the scope of technological platform, which was proved by C. Wagner and N. Bolloju [Wagner, Bolloju 2005]. Revealing hidden information and its time characteristic is a crucial element of data processing. Its specification both in interpretation and representation aspect requires the usage of new approaches that provide better

possibilities of occurrence modelling and identification of its characteristics. Temporal data mining is one of the developing fields of methods for knowledge extraction from data bases focused on time characteristics. C.M. Antunes and A.L.Oliveira present a review of approaches and methods used in this field [Antunes, Oliveira 2001]. The methods of time representation and scope of appliance in different fields were identified. The review of methods presented in the study relates to temporary data bases and indicate development direction of methods focused on acquiring knowledge about hidden patterns and analysis of changeability of sets with temporal characteristics [Roddick, Myra 2002]. The unresolved problems of data exploration were also presented. The study [Hu et al. 2007] develops the aspect of Web data specification and the need to search for dedicated solutions that include local solutions, which in case of searching for generalized dependencies for all data set are not detected. The presented methods are focused on acquiring temporal, indirect, frequent patterns and their temporal, extended patterns in the area of identifying Web users with distinct interests.

As B. Hernandez-Ortega, J. Jimenez and others proved, there are significant differences in behaviour of potential, new or permanent consumers [Hernández-Ortega et al. 2008]. The identification of their behaviour enables adequate segmentation and offers targeting or adaptation of functionality to a given group of recipients. S.M. Beitzel and others presented the analysis focused on the temporal aspects and arrangement in time of data from logs from servers connected with search engines, and they proved that there is a possibility of temporal analyses in this environment [Beitzel et al. 2004]. Changeability of consumers' behaviour in time and evolution in time connected with different factors, as well as developing new technologies in time, is emphasized in the works of V. Venkatesh and M.G. Morris [Venkatesh et al. 2003], E. Karahanna and others [Karahanna 1999]. The concept of analysis of time series focused on analysis of Web search system logs was presented by Y. Zhang, B.J. Jansen and others [Zhang et al. 2009]. The analysis of time series enabled the detection of dynamic occurrences and consumers' behaviour patterns. The presented studies focus on the aspect of representation and temporal data analyses. Nevertheless, they do not include the specification of social services functioning and functioning of users in virtual reality, for which different structuring of time lapse can be determined than in real systems, and with a use of dimension analysis, the evaluation of loyalty level or characteristic of using individual service or platform can be done.

CONCEPTUAL FRAMEWORK

The article herein proposes a new approach towards data analysis and characteristics of web platforms usage, which relate to the users' activeness and engagement. Internet systems focused on communities very often function in a form of asynchronous virtual worlds where users create their identities and

general activeness for their virtual beings. In systems of that type, the communication is done asynchronously, contrary to virtual worlds like Second Life, where data transmission is done in direct mode. Both in the first and the second case, the user's virtual representation functions in other reality in a sense of place (cyber space), so it can be assumed that the virtual representation functions also in another dimension of time. Lapse of time in a virtual system herein referred to as virtual time can be considered regardless of real time. For mapping and parameters' determination purposes, the two-dimensional model of time representation in bi-temporal scope was assumed, where occurrences are registered in a scope of real and virtual time. There is an activated snapshot of users' states and activities with assigned virtual world timestamp. The significant matter is to determine granulation of virtual time and its relation to real time. In the easiest presentation the unit of one virtual day lapse can directly reflect a unit of real time and can be identical with a real day. There can be different time separations introduced that will lead to an increase of analysis accuracy by decomposition of time period into sub periods and determination of time granulation. For the use of describing such dependency, the Virtual Time Factor was defined, which relates to relation of virtual and real time, according to the following formula:

$$VTF = \frac{V * I}{d}, \tag{1}$$

where d determines number of real time days, V determines number of virtual time days and I is a number of time intervals taken into consideration for monitoring of users' activeness. In example, if we assume that virtual day is equal to real day VTF=1. If during single real day, we can identify three virtual days, in example 3 logins per day can happen in the morning, afternoon and evening then VTF=3. Time lapse in virtual world has a different scope, and each virtual day for an individual user is initiated at the moment of logging into a system. Such approach enables monitoring the lapse of both types of time for each user and determining a level of activeness. For the purpose of data analysis consistently with assumed time granulation at the t moment for every *i-th* user the S_t snapshot is generated in parameters that are monitored in an individual time period. A set of snapshot parameters depends on parameters and abilities of a system and aims to present the dynamics of system usage and changeability of user's determined features. The m determination was assumed of $P_{s,t}^i$ parameters for every *i-th* user , that $i = 1,...,n$ for every type of s parameter that, $s= 1,...,m$ and for every t time moment the aggregated parameter values are determined in relation to previous periods, as well as dynamics of change $D_{1,t}^i$ in relation to previous period, according to formula:

$$S_t = \{\{ P_{1,t}^i = \sum_{k=t-1}^{t_0} P_{1,k}^i \quad D_{1,t}^i = P_{1,t}^i - \sum_{k=t-1}^{t_0} P_{1,k}^i \}, ..., \{ P_{n,t}^i = \sum_{k=t-1}^{t_0} P_{n,k}^i \quad D_{n,t}^i = P_{n,t}^i - \sum_{k=t-1}^{t_0} P_{n,k}^i \}\} \quad (2)$$

Analysis of individual parameters' changeability enables the determination of characteristics of audiences. Parameters can include social characteristics, user's activeness, a number of loggings and use of particular system functions. On the basis of collected snapshots from a user's account and a presence in a service, the relation of real time to virtual time is determined. Real time has a constant lapse with division into time intervals determining its granulation. For each user there can be determined characteristics, as well as dependency of virtual time on real time, and there is a possibility to model in such a way a dynamics of service usage. Time dependency between real and virtual time is a measure of user's engagement and dynamics of service usage. For the purpose of sequence analysis, the methods focused on measurement of similarity between sequence A and B can be introduced in a form of S similarity (A,B), which can be determined with a use of available methods. To evaluate similarity, time warping distance method was applied. For two series $X(x_1, x_2, ..., x_n)$ i $Y(y_1, y_2, ..., y_m)$ with lengths respectively n and m and M matrix is defined as interpoint relation between series X and Y where M_{ij} element indicates the distance $d(x_i, y_j)$ between x_i and y_j. Dependency among a series is determined by time warping path. The algorithm determines warping path with the lowest cost between two series in accordance with the formula [Rabiner, Juang 1993]:

$$DTW(X,Y) = \min_W \{ d_k, w = \langle w_1, w_2, ..., w_k \rangle \} \quad (3)$$

where $d_k = d(x_i, y_j)$ indicates distance represented by $w_k = (i,j)$ on W path. A series with a higher level of similarity can be better compared because of alignment and dependencies resulting from dynamic time distance. To determine their relation to real time, the dynamic time wrapping was used. In the proposed procedure, the users' sequences can be compared to distinguish similar behaviour and compared to the pattern of ideal time lapse similar to real time. In the next part the research was conducted in real system, and a particular area of applications was identified.

EXPERIMENTAL RESULTS

The proposed approach was verified on real data and was integrated with online system databases. In the scope of research, the analysis was made of the similarity between real and virtual time during the usage of social service realized in a form of virtual world. 16,165 users who at the time of research created accounts were taken into consideration. The more detailed analysis was made in relation to an input set of 4,410 unique users whose activeness was registered at least for 10 real days. Users were divided into 9 groups depending on the length of log- in sequences. The maximum number of virtual days is equal to the number of real days, and in the analysed case it amounted to 100. Granularity for time and

virtual time factor was adjusted at 1:1, and real day is equal to virtual one. For particular users, changeable dependencies of two dimensions of time can be observed. Chart 1 shows the time dependency for real time R_d and virtual one V_d. On axes x and y, real and virtual days were marked. For u_1 user there is a consistency of measures, and high frequency of system usage is presented. User u_6 in the initial period increased a distance between real and virtual time, and his V_d in point A equals 12, and R_d equals 20. In the following days the distance is slightly changed to $V_d=19$, and there is a stability up to point B where a distance rapidly increases and eventually equals $V_d=49$ and $R_d=100$. For u_5 user, stability occurs from $V_d=7$ up to point C where distance rapidly increases between dimensions. Dependencies among users can be a basis for segmentation depending on behaviour similarities. The analysis was made including similarities and a distance to ideal vector, which represents a state of maximum engagement. In such a situation V_d is consistent with R_d. The exemplary dependency is shown in Chart 2. The distance and similarity to the ideal vector can be an engagement measure, and it enables detection of users with determined characteristics.

Chart 1. Relative users' behaviour characteristics u_1, u_5 and u_6.
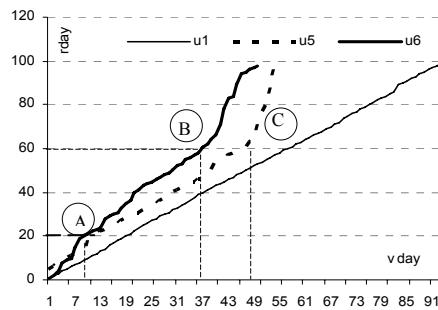
Chart 2. Users' characteristics u7 and u8 in relation to ideal vector

Source: own calculations
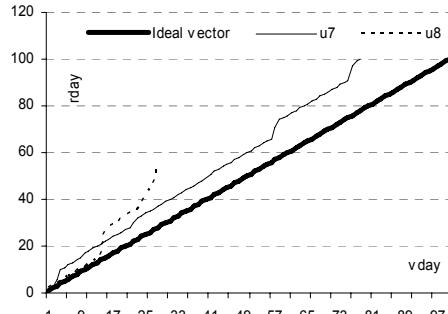
Source: own calculations

A similarity to ideal vector results not only from the sequence length, but also from the character of changes and lag of virtual time vector. Such characteristics are included by dynamic time wrap method that enables, among others, a presentation with comparison of sequence time-lag. Chart 3 illustrates in detail the distance for sequences from 1-1000 (window W_1 from Chart 4) and distances from ideal vector. Point A represents user u92 with dtw=68 and sequence length 74. Point B user u_{183} with distance 574 and sequence length 64/100 elements. Point C shows user u332 with DTW=2875 and sequence length 52. Point D denotes user u719 and dtw=3294 and sequence length= 36. Chart 4 is presented for selected points in relation to the ideal vector representing full consistency of measures V_d and R_d.

Chart 3. Detailed distance characteristics for users u_1-u_{1000}
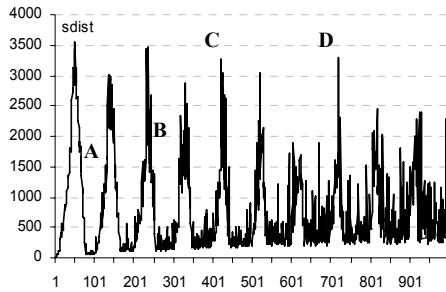
Chart 4. Distance from reference pattern


Source: own calculations
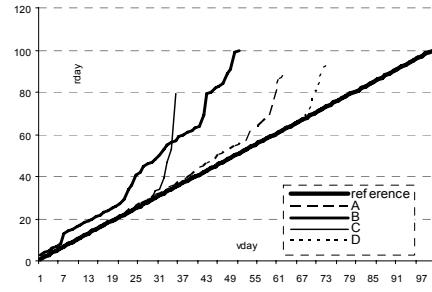
Source: own calculations

In the next step the users were divided into groups depending on the sequence length with incrementing every ten. In group g1 there were identified users with sequence length of 90-100 elements, in the second group from 80 to 90 up to group 9 in which the scope was from 10 to 20 elements. In particular groups the number of users was identified according to $card(g_1)=22$, $card(g_2)=47$, $card(g_3)=63$, $card(g_4)=101$, $card(g_5)=133$, $card(g_6)=241$, $card(g_7)=398$, $card(g_8)=852$, $card(g_9)=2553$. Chart 5 illustrates the similarity alignment between the longest sequence in individual set and other series in set g_2. Chart 6 illustrates the distance of individual groups to ideal vector.

Chart 5. Similarity alignment of series for set g_2
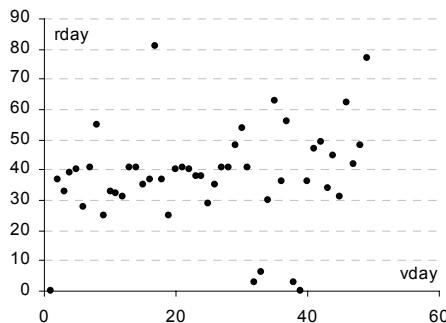
Chart 6. Distance alignment for elements from groups g1-g4

Source: own calculations
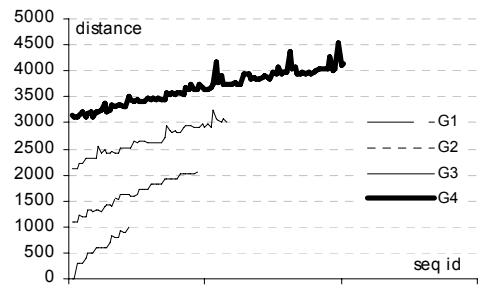
Source: own calculations

Analyses of dependencies of virtual and real time can be used for early detection of active users, and for estimation of the possibility of conversion on permanent users who often participate in the service. The presented idea and conducted research in relation to sequence of visits in the scope of real and virtual

time show the possibility of behaviour pattern analysis with a use of this approach and detection of users with similar characteristics. Introduction of quantitative similarity measures and determination of factors influencing the occurring tendencies can broaden the existing data analysis methods and determination of audiences' parameters. It shows new approach to web data analysis based on dynamic time wrap algorithm and virtual/real time dimensions. In the next phase the research can be extended by time granularity analysis and influence of time patterns on the technical and economic parameters describing users' behaviour in the service. The scope of application of the presented concept is quite broad and enables the implementation of the monitor systems focused on behaviour examination and analysis of current tendencies and changes in audiences' preferences. The knowledge of the behaviour pattern enables the possibility of providing the recipients in a given group with marketing actions and integration of loyalty programs to increase the engagement of users classified in a group with a high risk of withdrawal from service usage, as well as to provide adequate offers to users with the highest levels of engagement.

SUMMARY

Together with the increase of Internet systems' complexness and increasing competition in this sector, there is a bigger demand for introduction of new data analysis methods, and examining behavior of users participating in Internet services. The proper recognition of needs and tendencies provides the basis for making rational decisions and better adaptation of online services to users' needs. High dynamics and changeability of users' behavior show the need for implementation of solutions that takes time characteristics into account. The solutions presented so far in the literature did not include two-dimensional approach towards data character and time series. The presented approach enables quantitative estimation of users' activity with the use of reference sequences and users' segmentation on the basis of time conditioning. The applied methods of dynamic time wrapping enable quantitative presentation of similarity including usage patterns. The presented research proves the possibility of using this approach in different fields and initiating new research areas focused on recognition of audiences' characteristics with bi-temporal representation.

REFERENCES

Acar, A.S., Polonsky, M. (2008) Online Social Networks and Insights into Marketing Communications, Journal of Internet Commerce, Volume 6, Issue 4, pp. 55-72.

Antunes, C.M., Oliveira A.L (2001) Temporal data mining: an overview. In: KDD Workshop on Temporal Data Mining, pp. 1-13, San Francisco.

Beitzel, S.M., Jensen, E.C., Chowdhury, A., Grossman, D., Frieder, O. (2004) Hourly Analysis of a Very Large Topically Categorized Web Query Log. In SIGIR 04, July 25- 29, Sheffield, pp. 321-328, South Yorkshire, UK.

Chau, M., Zeng, D., Chen, M., Huang, M., Hendriawan, D. (2003) Design and evaluation of a multi-agent collaborative Web mining system. Decision Support Systems, vol. 35, no. 1, Elsevier Science Publishers, Amsterdam, The Netherlands.

Chun-Ling, Z., Zun-Feng, L., Jing-Rui, Y. (2010) The Application Research on Web Log Mining in E-Marketing. In: e-Business and Information System Security (EBISS), 2nd International Conference, 22-23 May, pp. 1- 4, Wuhan, China.

Dick, A.S., Basu, K. (1994) Customer loyalty: toward an integrated conceptual framework. Journal of the Academy of Marketing Science Spring , Volume 22, Issue 2, pp. 99-113.

Gomes, M.F., Canuto, A.M. (2006) Carcara: A Multi-agent System for Web Mining Using Adjustable User Profile and Dynamic Grouping. IEEE/WIC/ACM International Conference on Intelligent Agent Technology, pp. 187-190.

Gürsel, A., Sen, S. (2009) Producing Timely Recommendations From Social Networks. Proceedings of 8th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), Decker, Sichman, Sierra and Castelfranchi (eds.), May, 10-15, Budapest, Hungary.

He, J., Chu, W.W. (2010) A Social Network Based Recommender System, Annuals of Information Systems (AoIS). Special Issue on Data Mining for Social Network Data, Volume 12.

Hernández-Ortega, B., Jiménez-Martínez, J., Martín-DeHoyos, M.J. (2008) Differences between potential, new and experienced e-customers: Analysis of e-purchasing behaviour. Internet Research, Volume 18, Issue 3, pp. 248-265.

Herring, S.C., Scheidt, L.A., Wright, E., Bonus, S. (2005) Weblogs as a bridging genre, Information Technology & People. Volume 18, Issue 2, pp.142-171.

Hu, X., Yin, Y., Zhang, B. (2007) Mining Temporal Web Interesting Patterns, International Conference on Computational Intelligence and Security (CIS). pp. 227-231.

Jung, J.J. (2005) Visualizing Recommendation Flow on Social Network. Journal of Universal Computer Science, Volume 11, Issue 11, pp. 1780-1791.

Karahanna, E., Straub D.W., Chervany, N.L. (1999) Information technology adoption across time: a cross-sectional comparison of pre-adoption and post-adoption beliefs. MIS Q., Volume 23, Issue 2, pp. 183-213.

Kazienko, P. (2009) Mining Indirect Association Rules for Web Recommendation. International Journal of Applied Mathematics and Computer Science, Volume 19, Issue 1, pp. 165-186.

Lu, H.P., Wang S. (2008) The role of Internet addiction in online game loyalty: an exploratory study. Internet Research, Volume 18, Issue 5, pp. 499-519.

Marciniak, R., Budnarowska, C. (2009) Marketing Approaches to Pop Up Stores: An Exploration of Social Networking. In: 15th Conference of the European Association of Education and Research in Commercial Distribution (EAERCD), 15-17 July 2009, University of Surrey, England.

Marsili, M., Slanina, F., Vega-Redondo, F. (2004) Dynamics of social networks (the rise and fall of a networked society). Proceedings of the National Academy of Sciences, Volume 101, pp. 1439-1442.

Newman, M.E.J. (2006) Finding community structure in networks using the eigenvectors of matrices. Physical Review, Volume 74, Issue 3, pp. 798-817.

Newman M.E.J., Girvan M. (2004) Finding and evaluating community structure in networks. Physical Review, Volume 69, Issue 2, pp. 1123-1138.

Rabiner, L. R., Juang, B. (1993) Fundamentals of speech recognition. Prentice-Hall.

Roddick, J.F., Myra S. (2002) A Survey of Temporal Knowledge Discovery Paradigms and Methods. IEEE Transactions on Knowledge and Data Engineering, Volume 14, Issue 4, pp. 750-767.

Tsai, H., Huang, H. (2007) Determinants of e-repurchase intentions: An integrative model of quadruple retention drivers, Information & Management. Volume 44, pp. 231-239.

Venkatesh, V., Morris, M. G., Davis, G. B., Davis, F. D. (2003) User acceptance of information technology: Toward a unified view. MIS Quarterly, Volume 27, Issue 3, pp. 425-478.

Wagner, C., Bolloju, N. (2005) Supporting knowledge management in organizations with conversational technologies. Journal of Database Management, Volume 16, Issue 2, April, pp. 1-8.

Wangenheim, F., Bayon, T. (2004) Satisfaction, loyalty and word of mouth within the customer base of utility provider. Journal of Consumer Behaviour, Volume 3, Issue 3, March , pp. 211–220.

Zhang, Y., Jansen, B.J., Spink, A. (2009) Time series analysis of a Web search engine transaction log. Information Processing and Management, Volume 45, Issue 2, pp. 230-245.

ON THE CHOICE OF PARAMETERS
OF CHANGE-POINT DETECTION WITH APPLICATION
TO STOCK EXCHANGE DATA

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Abstract: Our paper is devoted to the study of V-Box Chart method in a parametric model. This algorithm is proposed to be used in the change-point detection in a sequence of observations. The choice of parameters in such an algorithm is heuristic. In our paper we use the mini-max rule for this choice and we control the probability that no signal is given, when the process is out of control as well as the probability of false alarm. We apply this algorithm to the detection of a change in stock exchange data.

Key words: V-Box Chart, mini-max rule, normal distribution

INTRODUCTION

This paper examines a new method of detecting a change in a sequence of observations. The main issue is to design a monitoring procedure that detects the change as quickly as possible. This procedure can be applied in many practical problems relating to signal processing, quality control, finance, clinical medicine.

The literature of various aspects and procedures of the change-point detection problem can be found in [Poor and Hadjiliadis 2009], [Basseville and Nikiforov 1993], [Brodsky and Darkhovsky 2002], [Lai 1995].

In the theory of classical control chart it is assumed that we observe the realization of the model

$$Y_i = \theta 1(i \geq q) + \eta_i, \tag{1}$$

where (η_i) is a noise process, θ is a jump, $q > 0$ is an unknown change point and

$$1(p) = \begin{cases} 1, & \text{if } p \text{ is true} \\ 0, & \text{if } p \text{ is false} \end{cases}.$$

In the parametric approach it is commonly accepted that the noise process (η_i) is Gaussian or has any known distribution of continuous type. Popular detection schemes in this case are CUSUM algorithms and EWMA control charts (see [Lu and Reynolds 1999], [Ritov 1990]).

An interesting algorithm related to the nonparametric model of (1) has been presented in [Rafajłowicz et al 2010]. It was connected with the Vertical Box Control Chart (V-Box Chart). In our work we use the idea of V-Box Chart in the parametric case, together with the mini-max decision procedure. This approach is very similar to the problem of ε-comparison of means of two normal distributions (see [Jaworski and Zieliński 2004]).

The choice of the parameter of the algorithm in [Rafajłowicz et al 2010] is heuristic. We give a simple numerical procedure for choosing the parameters of V-Box Chart in parametric case. For examination of the proposed method we give some examples of detecting changes in stock exchange data.

THEORETICAL BACKGROUND

We consider the following statistical model
$$Y_i = \theta 1(i \geq N+1) + \eta_i, \text{ for } i = 1,...,N+1,$$

where $\theta \in R$, N is the sample size of past observations, Y_{N+1} is a new observation and (η_i) is a noise process. We assume that (η_i) are i.i.d. with symmetric, continuous distribution. The problem is to take one of the following decisions:
$$d_1 : |\theta| \leq \varepsilon \text{ or } d_2 : |\theta| > \varepsilon,$$

for a given $\varepsilon > 0$. We take into account the following class of decision procedures:

$$d(\gamma, H) = \begin{cases} d_1, & \text{if } T(H) > \gamma N \\ d_2, & \text{if } T(H) \leq \gamma N \end{cases},$$

where $\gamma \in (0,1)$, $T(H) = \sum_{j=1}^{N} 1\left(Y_{N+1} - H \leq Y_j \leq Y_{N+1} + H\right)$ and $H > 0$. Notice that $2H$ is the vertical length of box chart and γ is the critical proportion of points in the box chart.

The idea of the optimal choice of parameters γ and H relates to the loss function L given in the Table 1 bellow.

Table 1. Loss function

$L(d_i, \theta)$	$\theta : |\theta| \leq \varepsilon$	$\theta : |\theta| > \varepsilon$
d_1	0	b
d_2	a	0

where a, b are some positive values. For the fixed H we choose γ satisfying the mini-max condition:

$$\max_{\theta} E_{\theta} L\big(d(\gamma, H), \theta\big) = \min_{\gamma^*} \max_{\theta} E_{\theta} L\big(d(\gamma^*, H), \theta\big).$$

In the case when the marginal distributions of the noise (η_i) are symmetric the mini-max condition is equivalent to the following one

$$a P_{\varepsilon}\big(T(H) \leq \gamma N\big) = b P_{\varepsilon}\big(T(H) > \gamma N\big). \tag{2}$$

To prove the above equality, it is enough to note that, for $\theta > 0$, $P_{\theta}\big(T(H) \leq \gamma N\big) = P_{-\theta}\big(T(H) \leq \gamma N\big)$ and the mapping $\theta \mapsto P_{\theta}(T(H) \leq \gamma N)$ is increasing function of θ (for details see [Furmańczyk and Jaworski 2011]). Then (2) follows from the equation

$$E_{\theta} L\big(d(\gamma, H), \theta\big) = a P_{\theta}\big(T(H) \leq \gamma N\big) \mathbb{1}(-\varepsilon \leq \theta \leq \varepsilon) +$$
$$+ b P_{\theta}\big(T(H) > \gamma N\big)\big(1 - \mathbb{1}(-\varepsilon \leq \theta \leq \varepsilon)\big).$$

The equation (2) is a base of numerical consideration of the problem of optimal choice of parameters. But in order to simplify it and to improve numerical stability this equation should be transformed. First we note, that

$$P_{\theta}\big(T(H) \leq \gamma N \mid Y_{N+1} = y\big) = P_{\theta}\left(\sum_{j=1}^{N} \mathbb{1}\big(y - H \leq Y_j \leq y + H\big) \leq \gamma N\right) = P_{\theta}\big(B(H, y) \leq \gamma N\big),$$

where $B(H, y)$ is the binomial random variable with N trials and success probability $\pi(H, y) = F(y + H) - F(y - H)$ and F is the marginal distribution function of the noise process (η_i). Using well known property (see Zieliński [2009]), we obtain

$$P_{\theta}\big(T \leq \gamma N \mid Y_{N+1} = y\big) = P_{\theta}\big(B(H, y) \leq \gamma N\big) = B_{N(1-\gamma), \gamma N + 1}\big(1 - \pi(H, y)\big),$$

where $B_{N(1-\gamma), \gamma N + 1}$ denotes the cumulative distribution function of random variable with beta distribution with parameters $N(1 - \gamma)$ and $\gamma N + 1$. Thus, the equality (2) can be written in the following form:

$$\int_R B_{N(1-\gamma),\gamma N+1}(1-\pi(H,y))f(y-\varepsilon)dy = \frac{b}{a+b}, \tag{3}$$

where f is the marginal density of the noise process (η_i).

Observe that the equation (3) might not have a solution for too small or too large value of H. The left hand side of the equation (3) is equal to $E_\varepsilon P_0(T(H) \le \gamma N \mid Y_{N+1})$. It is clear that if $H \to \infty$, then $\pi(H,y) \to 1$, which implies that $E_\varepsilon P_0(T(H) \le \gamma N \mid Y_{N+1}) \to 0$. On the other side, if $H \to 0$, then $\pi(H,y) \to 0$ and $E_\varepsilon P_0(T(H) \le \gamma N \mid Y_{N+1}) \to 1$.

In the next part of our paper we present the numerical solution of the equation (3) in the case when the marginal distribution of the noise process is the standard normal and we discuss the optimal choice of parameters γ, H. The presented method can be transformed - under some technical assumptions - to any symmetric continuous distribution F of the noise process (see [Furmańczyk and Jaworski 2011]]).

NUMERICAL CONSIDERATION

Let us denote $\delta = \dfrac{b}{a+b}$ and fix $\varepsilon = 1$. We want to solve the equation

$$\int_R B_{N(1-\gamma),\gamma N+1}(1-\pi(H,y))f(y-\varepsilon)dy = \delta,$$

with respect to γ and H. In order to avoid the problem of no solution (i.e., when H is too small or too large), we fix $\delta, \gamma \in (0,1)$ and solve the equation numerically with respect to H. We repeat this procedure for the following grid of parameters: $\delta, \gamma \in \{0.1 \cdot i : i = 1,2,...,10\}$. Thus, we receive a set of solutions $H = H(\delta, \gamma)$, which is presented in Figure 1 as a contour plot.

Figure 1. Solutions of the mini-max equation
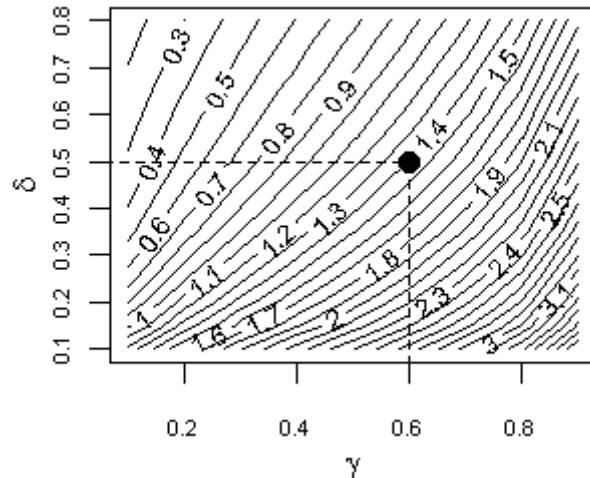
Source: own preparation

In Figure 1 the solution $H = H(0.5,0.6) = 1.4$ is marked. The problem is the solution is not unique and we need the additional criterion to choose the best one. To solve it, we define the following function

$$f_\gamma : \theta \mapsto P_\theta\big(T(H(\gamma,\delta)) \le \gamma N\big) = \int_R B_{N(1-\gamma),\gamma N+1}\big(1 - \pi\big(H(\gamma,\delta), y\big)\big)\phi(y - \theta)dy,$$

where ϕ is the density function of the standard normal distribution. Note that the function is symmetric. In Figure 2 the graph of f_γ for different values of γ and for the fixed $\delta = 0.5$ and $N = 20$ is presented (to notice the differences, we also put next the graph of $f_{0.1} - f_\gamma$).

Figure 2. Chart of functions: f_γ and $f_{0.1} - f_\gamma$.
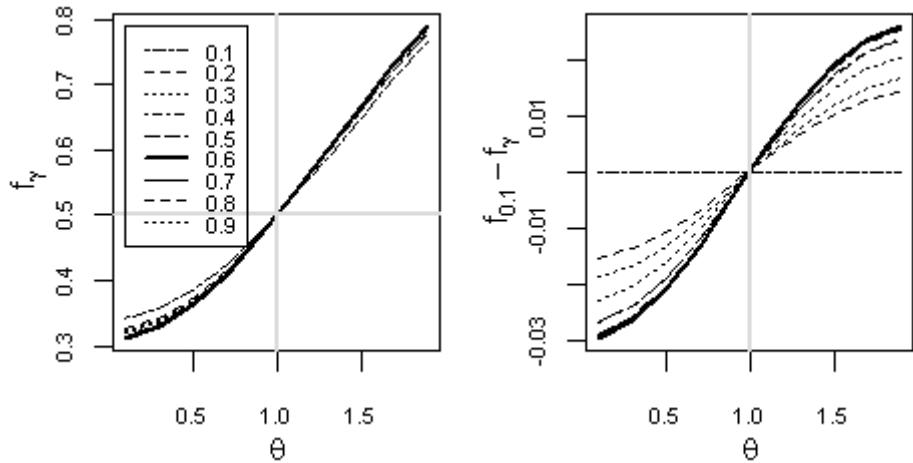
Source: own preparation

It is a natural requirement to have the probability $P_\theta\big(T(H(\gamma,\delta)) \le \gamma N\big)$ (i.e., the probability of a decision $d_2 : |\theta| > \varepsilon$) as high as possible for $|\theta| > \varepsilon$, and as small as possible for $|\theta| \le \varepsilon$. In this meaning, the optimal solution is obtained for $\gamma = 0.6$, which corresponds to $H = H(0.5, 0.6) = 1.4$.

Optimal solution for γ and H can be investigated asymptotically (i.e., when N tends to infinity). We consider $\delta = 0.2$ or $\delta = 0.8$. Figure 3 bellow presents the optimal H in our algorithm for a large sample N.

Figure 3. Asymptotic approach
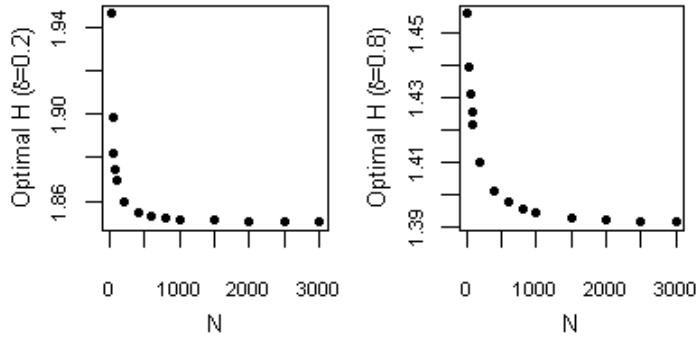
Source: own preparation

From our simulation study we obtain that, when N increases, then the parameters H and γ are getting stabilized. For $\varepsilon = 1$ and $\delta = 0.8$, we have $\gamma = 0.8$ and H is close to 1.3915. In the case when $\varepsilon = 1$ and $\delta = 0.2$, we have $\gamma = 0.5$ and H is close to 1.8504.

APLLICATION TO STOCK EXCHANGE DATA

In the case of the fixed N and ε the procedure can be represented by the value of $\delta = \dfrac{a}{a+b}$. In this chapter we apply our procedure for $N = 50$, $\varepsilon = 2$ and for three values of δ : 0.2, 0.5 and 0.8. We investigate the series of daily closing prices of Polish DTH platform "Cyfrowy Polsat" from 2010.06.25 until 2011.05.10. The value $\delta = 0.2$ relates to the high loss of making the wrong decision that the expected price change is not greater than the chosen level ε. It means we do not care for relatively small and medium price variability. Small and medium changes are more frequent than the large ones. Thus, the procedure is conservative and it does not reflect frequent price changes except for the largest ones. The case of $\delta = 0.8$ corresponds to the procedure sensitive for large changes, as we have high loss of making the wrong decision that the expected price change does not exceed or is less than the predefined level ε.

The value $\delta = 0.5$ expresses the strategy of choosing the more probable case: the new expected price level changed more than ε or not. The loss is equally balanced between the two possible decisions.

The three values of δ relate to different aspects of the time series prices. In the following figures we put two graphs. The first one shows the price changes in time and the second one presents the price levels. The shaded areas in the graphs represent the teaching part of the sample. The teaching observations are fixed. These observations establish a base to investigate each and next observation by the procedure. It means, we apply this procedure sequentially. Thus, the extreme changes are discovered and marked as black bullets in the charts. There are in Figure 4 a few black points. It is worthy to note that positive large changes are followed by the decreasing prices. If it were a rule we would have an excellent investment strategy for the considered time series.

Figure 4. An application of the procedure ($\delta = 0.2$, $N = 50$, $H = 2.88$ and $\gamma = 0.5$)
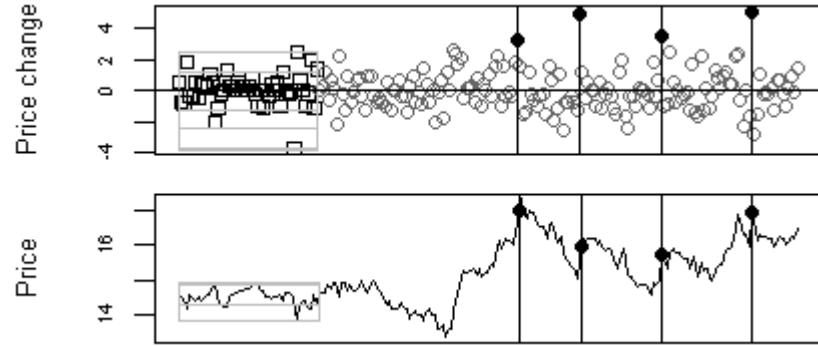
Source: own preparation

Figure 5 represents the case when $\delta = 0.8$. Extreme points are much more exposed. After the teaching part of the sample we have the series of negative changes. Vertical lines cross the extreme points. Note that their compact allocation announce the price level change.

Figure 5. An application of the procedure ($\delta = 0.8$, $N = 50$, $H = 1.44$ and $\gamma = 0.6$)
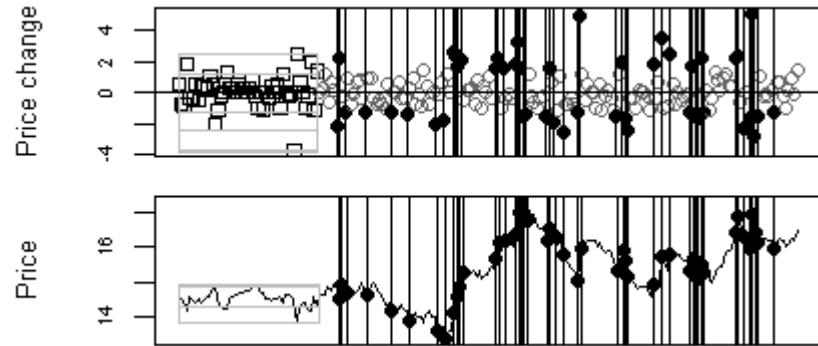
Source: own preparation

Figure 6 represents the case when $\delta = 0.5$. The black points on the first chart of Figure 6 seem more likely to belong to the distribution with the expected value more different from zero than ε.

Figure 6. An application of the procedure ($\delta = 0.5,\ N = 50,\ H = 2.03$ and $\gamma = 0.5$)
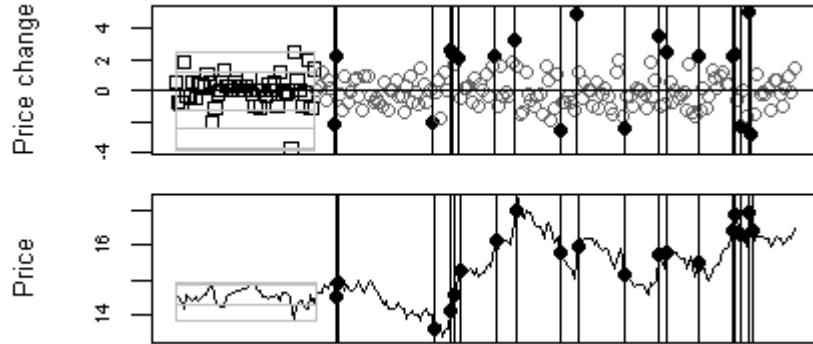
Source: own preparation

In order to apply this procedure we transformed the time series of price changes. The values of the series were divided by the standard deviation of the teaching part of the sample. Simulation study shows that the distribution of the statistic T does not change significantly.

Table 2 presents the results of Monte–Carlo estimation of the probability $P_{\varepsilon}\big(T(H) \le \gamma N\big)$, where $N = 50$ and $\varepsilon = 2$:

Table 2. Monte-Carlo estimation of $P_{\varepsilon}\big(T(H) \le \gamma N\big)$; $N = 50, \varepsilon = 2$; for 100000 of
replications

(δ, γ, H)	Not transformed series	Transformed series
(0.8, 0.6, 1.4404)	0.80009	0.79907
(0.2, 0.5, 2.8706)	0.20033	0.21194
(0.5, 0.5, 2.0002)	0.49832	0.50203

Source: own calculations

SUMMARY

Our paper is devoted to the construction a new algorithm assigned to the change-point detection in a sequence of observations. The choice of parameters in such an algorithm is based on the mini-max rule. Thus, we control both the probability that no signal is given when the process is out of control and the probability of false alarm. We apply this algorithm to detect the change in the stock exchange data. Due to the theoretical assumptions of the procedure, the empirical data must be transformed. Monte Carlo simulation shows that the transformation we used in the application part of the paper does not change the

theoretical results. The utility of the proposed algorithm depends on the specific features of the data. We gave an example of the series producing sudden changes in single observations with reference to its mean level change. This type of variability may be a rule in some time series, but it needs careful consideration. The proposed algorithm gives no indication on the causes from those such behavior occurs, but it responds quickly to large isolated jumps.

REFERENCES

Basseville M. and Nikiforov I. V. (1993) Detection of Abrupt Changes: Theory and Application. Upper Saddle River, NJ: Prentice-Hall.

Brodsky B. E. and Darkhovsky B. S. (2002) Nonparametric Methods in Change-Point Problems. Dordrecht, The Netherlands: Kluwer.

Furmańczyk K. and Jaworski S.. (2011) on internet
http://p117650.users.sggw.pl/VBox/chart_box.pdf

Jaworski S., Zieliński W. (2004) A procedure for epsilon-comparison of means of two normal distribution, Applicationes Mathematicae, 31(2), pp. 155-160

Lai T. L. (1995) Sequential change-point detection in quality control and dynamical systems, J. Roy. Statist. Soc. Ser. B, vol. 57, pp. 613–658.

Lu C. W. and Reynolds M. R (1999) EMWA control charts for monitoring the mean of autocorrelated process, J. Quality Technol., vol. 3, pp. 166–188.

Poor H. V. and Hadjiliadis O. (2009) Quickest Detection, Cambridge, U.K. Cambridge Univ. Press.

Rafajłowicz E., Pawlak M., Steland A. (2010) Nonparametric sequential change-point detection by a vertically trimmed box metod, IEEE Transactions on Information Theory, 56(7), pp. 3621-3634.

Ritov Y. (1990) Decision theoretic optimality of the CUSUM procedure, Ann. Statist., vol. 18, pp. 1464–1469.

Zieliński W. (2009) A nonparametric confidence interval for At-Risk-of-Poverty-Rate, Statistics in Transition new series, 10(3), pp. 437-444.

SOME APPLICATIONS OF RANK CLOCKS METHOD

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Abstract: A method first used to search for events leading to the changes of sizes of major US, UK, and world cities is applied to investigate the presence of events influencing incomes of commercial companies. Top 100 US companies from the period of 1955-2005 are analyzed. Distributions of incomes are found to be stable but the changes in rank positions of companies lead to discovery of some instabilities. Parameters describing changes in the rank positions are calculated, discussed, and compared to the results of previous studies.

Key words: companies' incomes, rank clocks, rank shifts

INTRODUCTION

Many objects could be characterized by one parameter. Most works are focused on studying such objects in single instant of time. Sizes of cities, turnovers of companies, heights of cloud scrapers, network traffic etc. scale with time in the upper tail of their distributions [Blanc *et al*. 2000]. On the other hand, while such objects are growing sometimes violent changes of other nature could appear. Such evolutions could be present as changes of rank position of studied objects and were observed for largest cities in US, UK and whole world. Such micro dynamics could be manifest of the global changes influencing most of the studying objects. It has been shown that an investigation of micro dynamics for largest cities could exhibit such events like the fall of the Constantinople or migration of whites to the west of United States. The latter made some large eastern cities being partly abandoned and some very big ones appeared on the west [Turchin 2003]. However,

distributions of population at single instant of time are stable and scale with time, especially in their upper tail. It has been shown that such events could cause rapid changes in the cities' rank positions, while their aggregate distributions appear stable. In this work we performed studies of such micro dynamics and adopted graphical representation of rank position. for the top 100 US companies. Income of companies and changes in their rank positions could be influenced by an invention of new technology or an introduction of new products and services in a global range. Similarly, changes in the rank structure of cities could be determined by other, but also global, kind of events sometimes related to new technologies, level of life, wars, destruction, or migration of people.

DATA SELECTION AND ANALYSIS

We analyze income of 100 top US companies according to the Fortune magazine. Companies have been selected based on their annual income in 1955 to 2005 only, to avoid effects related to the recent global financial crisis. We investigated changes in rank position for companies noted in top 100 for two successive years, disregarding companies which fell below or just appeared on the list. We followed the procedure described in [Batty 2006]. As could be predicted [Stanley *et al*. 1996], the companies grow according to constantly increasing annual income (Figure 1) showing exponentially decreasing upper tail.

Figure 1. Distributions of income for 100 top US companies for years (from bottom to top): 1955, 1975, 1990, 2005
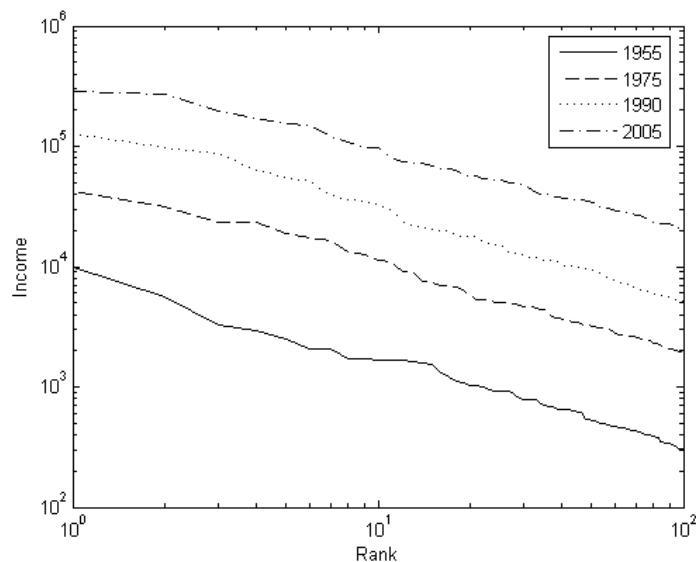
Source: own preparation

First we searched for micro dynamics by looking for changes in rank of one company at a time. Of course, one company is influenced by specific financial phenomena and sector specific economy. Thus results could be different according to the field the company is active in. In order to search for global effects we calculated average change in rank position over all companies selected for each year. That procedure would made our results more comparable to other works as well. Another step in the analysis was looking at the growth rates. Following [Batty 2006], we defined total growth rate as weighted average of growth rates of individual companies, as described below.

- growth rate (i − th company):

$$\lambda_i(t) = \frac{P_i(t)}{P_i(t-1)} \tag{1}$$

- share:

$$p_i(t) = \frac{P_i(t)}{P(t)}, \text{ where } P(t) = \sum_i P_i(t) \tag{2}$$

- total growth rate (all companies):

$$\lambda(t) = \sum_i p_i(t)\lambda_i(i) = \left(\frac{P(t)}{P(t-1)}\right)\left(\sum_i p_i(t)\frac{P_i(t)}{P_i(t-1)}\right), \tag{3}$$

where i = 1 to N (N − number of companies), $P_i(t)$ − income of i-th company in year t.

As could be seen from the above equations, total growth rate is expressed as the product of two terms. The first one depends only on the increase of total income of all companies and thus could be related to the global economy or progress of US market itself. The other term depends on the shares of separate companies in total income. Thus being related to the changes of individual companies as compared to all. The latter term could be influenced by changes in individual sectors of market or new technology invented what would "push" some of the companies up in the rank position. Both terms are given below:

$$\Gamma(t) = \frac{P(t)}{P(t-1)} \tag{4}$$

$$\Theta(t) = \sum_i p_i(t)\frac{p_i(t)}{p_i(t-1)} \tag{5}$$

After averaging both parameters over years we compared them to the results from the paper [Batty 2006].

RESULTS

Figure 2 contains changes in rank position for selected 5 companies: Boeing, Exxon Mobil, IBM, Microsoft, PepsiCo.

Figure 2. Rank position of separate 5 sample companies. Both plots present the same data in different ways
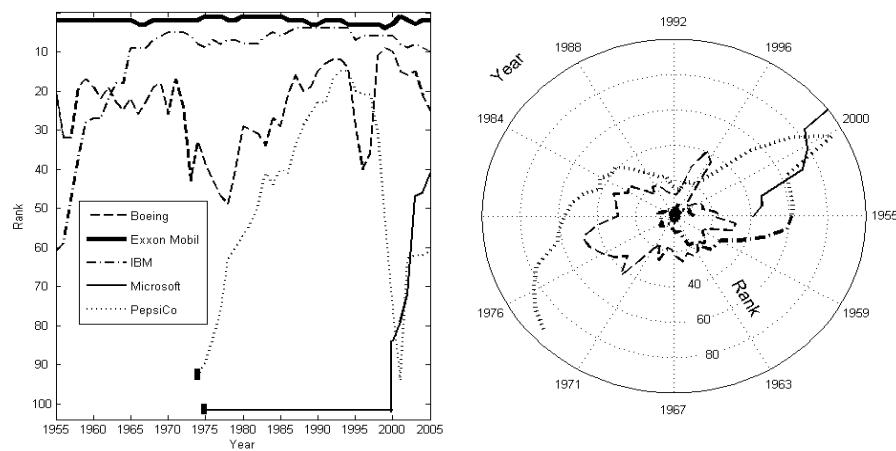
Source: own preparation

As follows from an inspection of Figure 2 the most stable position in the rank have IBM and Exxon Mobile. The IBM has been in the top 10 since 1965, what could be related to the rapid growth of the computer technology. IBM is known worldwide for its inventions in that field. The growing importance of IT in the US as well as economy ensured IBM a stable position in the ranking. Even more stable is Exxon Mobile which for the whole period is in the top 5. That is for sure related to the constant demand for traditional petroleum-based fuels. However, that situation could be changed in the future, due to the increasing funds on the invention of technologies related to natural sources of energy. Big changes in the rank position are observed in the case of Boeing – the one of the largest aircraft manufacturers. The reason of that behavior is harder to explain. Such a big changes in the rank position could be related to the construction of new aircraft models by Boeing or its competitors. The other two companies were founded between 1995 and 2005, what was indicated by small rectangles on the figure. PepsiCo was formed in the 1965 with the merger of the Pepsi-Cola Company and Frito-Lay. Since then PepsiCo expanded from its namesake product Pepsi to a broad range of food and beverage brands, by acquiring another companies. In 2001 PepsiCo merged with company Quarker Oats. Maybe that decision was the reason its position was so low then. Another company, which originated during analyzed

period of years was Microsoft. However, the company has not been noted on the top 100 list till the year 2000, what was indicated in the figure by horizontal line. After the 2000 the rapid improvement in its rank position could be observed. Microsoft is well known of its strong marketing politics and its expansion into new sectors of technology as well as forcing their solutions towards industry standard. The change in rank position could be due to the launch of new version of MS Windows, which always is a big marketing event.

We analyzed the average rank position change for every year. Figure 3 contains results, where one can observe strong stability from the beginning of data till the year 1995. The average range from 3 to 7 regardless of year.

Figure 3. Average change in the rank position for all companies and years analyzed
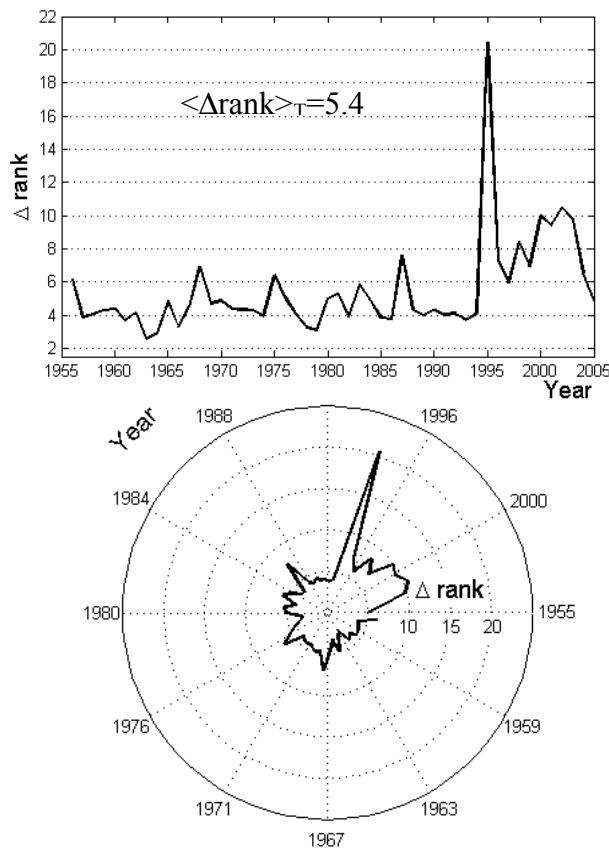
Source: own preparation

One shall keep in mind we are talking about the 40-year-period, where a lot of companies emerged while the others disappeared. New technologies were developed and new products were invented. After this 40 years there is a high peak

at 1995, meaning that the average change in the rank position has raised from 4 to 20. This could hardly be related to the invention of new technologies, because implementation of new products and services usually takes at least few years. A possible explanation of the observed effect could be some kind of actions which opened new markets or made possible to sell new products with no or drastically smaller limits. It is important to note that after the 1995 the studied parameter is consistently larger than before 1995.

At the last step we average studied parameter over all years and compare it to the results for top cities [Batty 2006]. Table 1 contains results of our work and the results of the same analysis for US, UK and World cities. For completeness, we also compared average $\lambda(t)$, $\Gamma(t)$ and $\Theta(t)$.

Table 1. Parameters averaged over all analyzed years

<Param>	US Cities	UK Cities	World's cities	US Companies
Δrank	4.67	4.22	14.28	5.45
$\lambda(t)$	1.38	1.00	1.13	1.06
$\Gamma(t)$	1.31	0.99	0.99	1.09
$\Theta(t)$	1.05	1.05	1.06	0.98

Source: own calculations

The average change in rank position is larger than for the US and UK cities but is smaller when compared to world's cities. On the other hand average growth rate is rather small being only slightly bigger than for UK cities. The growth rate is more related to the overall progress than to increase of separate companies ($\Gamma(t) > \Theta(t)$). The same behavior has been observer previously for US cities.

CONCLUSIONS

Utilizing the rank clocks methodology we studied micro dynamics behind the ranking of biggest US companies for the period of fifty years starting from 1955 till the end of 2005. Income of companies constantly grew up because of change in value of money and development of companies. The distribution of companies' income scaled with time showing well known exponential upper tail. On the other hand we studied micro dynamics which could show influences of other factors on the income of companies. Such factors could force changes in rank position what could not be observed by studying typical aggregated income distributions.

The average change in rank position was very stable for 40 years being about 5, what was of the same order of magnitude like for the US and UK cities studied previously. On the other hand that value was almost three times smaller than for world's cities. We observed very strong rise in the average rank position change (<Δrank>) in year 1995 as well as increased <Δrank> value thereafter. That may

indicate that some important events or changes took place in US economy in 1995. Moreover effects of those changes are visible in subsequent years. Total growth rate being slightly more influenced by overall increase of income than by progress of separate companies. For future studies we plan to analyze a nature of changes in more details by looking at the separate sectors of economy and comparing values of $\lambda(t)$, $\Gamma(t)$, $\Theta(t)$ on the year by year basis.

LITERATURE

Batty M. (2006) Rank Clocks, Nature, 444, pp. 592-596.

Blank A., Solomon S. (2000) Power laws in cities population, financial markets and internet cities: Scaling and systems with variable number of components, Physica A, 287, pp 279-288.

Stanley M.H.R. *et al*. (1996) Scaling behavior in the growth of companies, Nature, pp. 379, 804-806.

Turchin P. (2003) Historical dynamics: Why States Rise and Fall, Princeton University Press, Princeton, New Jersey.

EFFICIENCY OF INDIRECT WAYS OF INVESTING
IN COMMODITIES IN CONDITIONS
OF POLISH CAPITAL MARKET

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Summary: Within last couple of years one could observe record levels of commodity prices and now commodity investments grow in popularity. Although there are possible direct and indirect ways of investing in commodities, the paper focuses on indirect investing through commodity-related stocks and commodity funds. The research aims at assessing their efficiency in comparison to stock market, so the main index of Warsaw Stock Exchange (WIG) is a benchmark. There were calculated basic characteristics of considered assets and there were tested hypotheses of normality of their logarithmic returns and equality of their means and deviations from mean in relation to the benchmark. As no time series followed normal distribution, Mann-Whitney U test was applied.

INTRODUCTION

High volatility of financial markets, poor results on stock markets, growing inflation and risk from global crisis incline investors to seek alternative ways of investing capital. Investment in tangible assets is becoming a more and more popular solution. There are many classes of possible alternative investments. Jajuga [2007] in the first place mentioned real estate and then commodities[1]. In this group special attention should be paid to: precious metals, industrial metals, agricultural

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[1] Other alternative forms of investing include: investing in art (painting, photography, graphics, design, sculpture or antique) and collector's items (coins, stamps, cars, watches, wine, precious stones, etc.). A more detailed discussion on alternative forms of investing was provided by Jagielnicki (2011), as well as Borowski (2008) and Niedziółka (2008).

goods and energy raw materials. However, taken recession on American mortgage market into account, investing in real estate is not as popular as it used to be. In this situation investors have become interested in commodity markets.

The main reason behind such a state of affairs is the fact that commodities provide one with portfolio diversification, hedge against inflation and generate attractive profit when interest rates are low. Prices of many commodities have recently shown an upward tendency, regardless of correction periods. Investors may choose between direct and indirect forms of investing in commodities. In the case of the former, i.e. purchase of the physical commodity on the cash market, it may be difficult to provide proper storage conditions. The only exception here, are precious metals that do not require special conditions. On the contrary, agricultural goods and energy raw materials are really bothersome in this respect. Hence, individual investors rather prefer indirect investments, e.g. purchase of stocks of companies specialising in commodity production or purchase of participation units in specialized commodity funds. Polish investors also have such an opportunity as such funds operate on Polish market. As a matter of fact, they have recently grown in number.

The present paper is therefore aimed at determining the effectiveness of indirect forms of investing in commodities on Polish capital market. The analysis covers stock quotations of selected companies listed on the Warsaw Stock Exchange and operating on commodity markets. Furthermore, attention is also paid to quotations of participation units in commodity mutual funds. The main index of Warsaw Stock Exchange – WIG, commonly considered a substitute of market portfolio, is a benchmark.

EMPIRICAL DATA AND RESEARCH METHODS

Empirical data used for the purpose of the analysis covers the period from 17 December 2008 to 31 March 2011. These are daily participation unit quotations of selected commodity mutual funds operating on Polish market as well as stock quotations of companies listed on the Warsaw Stock Exchange and functioning on commodity markets. Time horizon was limited due to data availability. Despite the fact that a great number of commodity mutual funds currently function on Polish market, most of them have emerged relatively recently and hence they do not have long quotation records. Detailed analysis of available data allowed selecting the following 8 funds: Idea Surowce Plus, Investor Gold Otwarty, Investor Agrobiznes, Skarbiec Rynków Surowcowych, BPH Globalny Żywności i Surowców, Pioneer Surowców i Energii, PZU Energia Medycyna Ekologia, Opera Substantia.pl. As far as commodity-related companies listed on the Warsaw Stock Exchange are concerned, the analysis covered these belonging to WIG20 index at the moment of the examination as well as companies with the largest shares in sub-index portfolios: WIG Chemia (WIG Chemical), WIG Energia (WIG Energy), WIG Paliwa (WIG Oil&Gas), WIG Spożywczy (WIG Food),

WIG Surowce (WIG Basic Materials), provided that they had not already been included in WIG 20. At the same time, it turned out that three of these companies had been listed for too short period and thus they could not be subject to analysis (Bogdanka since 25 June 2009, PGE since 6 November 2009 and TAURON PE since 30 June 2010). Eventually, the following seven companies were taken into consideration: CEZ, Kernel, KGHM, Lotos, PGNiG, PKN Orlen and Synthos.

In the first step of research there were calculated logarithmic returns becoming a base to evaluate the following characteristics for considered assets: expected rate of return, standard deviation, range, skewness, kurtosis, coefficient of variability and correlation. Then, normality of distributions was tested. In the literature, there are discussed several tests of normality, e.i. chi-squared, Shapiro-Wilk, Lilliefors, Jarque-Bera or Kolmogorov-Smirnov tests. Here, just the two of them were used: Shapiro-Wilk and Jarque-Bera tests.

In order to answer the question whether indirect investing in commodities was an attractive alternative to traditional investments in stock markets in considered period, WIG index was taken as a benchmark. Then hypotheses of equality of expected rates of return and deviations from mean in relation to WIG index were tested for several assets. Due to the fact that such time series usually do not fit normal distribution[2], following Filipowicz [2010], non parametric Mann-Whitney U test was applied. This test, based on ranks, is the most useful in testing equality of means in two populations and is recommended when assumption of distribution normality is not fulfilled as the only assumption demanded is that all observations from both groups are independent to each other and were selected in a random way [Aczel 2000].

First, one should arrange all observations into a single ranked series. If there are any equal values, they receive arithmetic averages of their ranks. Then one should add up the ranks for the observations coming from sample 1. The sum of ranks is denoted by R_1. U is then given by:

$$U = n_1 n_2 + \frac{n_1(n_1 + 1)}{2} - R_1,$$
(1)

where n_1 is the sample size for sample 1, and n_2 is the sample size for sample 2. For large samples, when both n_1 and n_2 are larger than 10, U is approximately normally distributed. In that case, the standardized value Z is given by:

$$Z = \frac{U - \mu_U}{\sigma_U},$$
(2)

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[2] Results of research for financial market in Poland reveal that distributions of logarithmic rates of return and logarithmic prices of stocks and indices listed at the Warsaw Stock Exchange do not follow normal distribution (Tarczyński 1997, Kowalewski, Kuziak 2000, Tarczyński 2002, Osińska 2006, Witkowska, Matuszewska, Kompa 2008, Winkler-Drews 2009)

where μ_U and σ_U are the mean and standard deviation of U given by:

$$\mu_U = \frac{n_1 n_2}{2}, \tag{3}$$

and

$$\sigma_U = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}} . \tag{4}$$

It is worth to notice that these calculations are not based on values of separate characteristics. They are based on sample sizes [Mynarski 2006].

RESEARCH RESULTS

On the base of 573 quotations of considered assets, there were calculated logarithmic returns used to evaluate basic characteristics given in tables 1 and 2, for considered stocks and funds respectively. These are: minimal and maximal observed values, range, expected rate of return (mean), standard deviation, skewness, kurtosis and coefficient of variability. Values of Pearson correlation coefficients for all investigated assets are reported in table 3 (bold type denotes values that did not differ significantly from zero at 0,05 level).

Table 1. Basic characteristics of logarithmic returns obtained for considered stocks and WIG index

Measure	Asset							
	CEZ	Kernel	KGHM	Lotos	PGNiG	Orlen	Synthos	WIG
Min	-0,11630	-0,16058	-0,10754	-0,10536	-0,06147	-0,08601	-0,08069	-0,06881
Max	0,06454	0,16058	0,12078	0,17076	0,07719	0,12866	0,14732	0,05799
Range	0,18084	0,32116	0,22833	0,27612	0,13866	0,21466	0,22801	0,12680
Mean	0,00031	0,00299	0,00319	0,00226	0,00012	0,00122	0,00371	0,00101
Std. dev.	0,01800	0,02987	0,02911	0,02578	0,01753	0,02475	0,02778	0,00134
Skewness	-0,30650	0,36032	0,17836	0,84914	0,29007	0,15789	0,76033	0,03073
Kurtosis	3,1466	4,6347	1,4639	5,6283	1,7589	1,6444	2,7123	2,4080
Coeff. of variability	58,2310	9,9741	9,1177	11,4000	139,95	20,354	7,4901	13,895

Source: own calculations

Analysis of results, given in tables 1 and 2, allows to state that in the studied period all considered stocks and funds produced low expected rates of return. In the case of stocks, the highest observed rate of return was that obtained for Synthos (0,37%), and then that for KGHM (0,32%). The lowest noted value was that obtained for PGNiG (0,01%).

Table 2. Basic characteristics of logarithmic returns obtained for considered commodity funds

Measure	Fund							
	Idea Surowce Plus	Investor Gold	Investor Agrobiz-nes	Skarbiec Rynków Surowco-wych	BPH Globalny Żywności i Surow-ców	Pioneer Surowców i Energii	PZU Energia Medycyna Ekologia	Opera Substan-tia
Min	-0,06551	-0,04220	-0,05278	-0,03854	-0,04002	-0,03425	-0,04270	-0,05163
Max	0,06084	0,04562	0,06878	0,05258	0,04976	0,04416	0,05452	0,03782
Range	0,12635	0,08781	0,12156	0,09112	0,08978	0,07841	0,09723	0,08945
Mean	0,00139	0,00069	0,00093	0,00093	0,00092	0,00082	0,00027	0,00066
Std. dev.	0,01415	0,01067	0,01198	0,01048	0,00981	0,01050	0,00717	0,01082
Skewness	-0,17744	0,04037	-0,04865	-0,01071	-0,10480	-0,01833	0,71661	-0,12601
Kurtosis	2,3849	1,5933	3,3771	1,8682	2,3150	1,2664	10,9690	2,4776
Coeff. of volatility	10,145	15,372	12,883	11,268	10,710	12,272	26,097	16,472

Source: own calculations

In the case of funds, Idea Surowce Plus was the one with the highest rate of return equal to 0,14%. However, this result is more than two times lower than the best result obtained for stocks. On the other hand it is worth to notice that the lowest rate of return observed in the group of funds (0,03% for PZU Energia Medycyna Ekologia) was actually higher than the worst result in the group of stocks, generated by PGNiG. While considering standard deviation, the highest value of the measure obtained for stocks was this of Kernel, and the lowest one was this of Synthos. Funds considered in the research were characterised by much lower values of standard deviation in comparison to stocks, as the highest value 1,4% for Idea Surowce Plus was lower than all standard deviations in the group of stocks (it was higher only than the lowest standard deviation of all which was observed in the case of WIG index). The lowest standard deviation among funds equaled 0,7% and was obtained for PZU Energia Medycyna Ekologia.

Values of coefficients of variability revealed higher variability of rates of return of investigated stocks. Nevertheless investing both in considered stocks and funds should be considered risky due to the fact that values of expected rates of return are much lower than values of standard deviations. In all cases there was observed hightened kurtosis (values greater than zero), thus analysed time series are leptokurtic. Hence, the distributions have fatter tails than the normal distribution, indicating a higher occurrence of extreme events. In the case of investigated stocks in most cases we have positive skewness (only for CEZ skewness is negative), which is favorable as indicates that in the considered period there were many more positive rates of return than negative rates of return.

However, two funds only (Investor Gold and PZU Energia Medycyna Ekologia) had positive skewness.

On the base of data reported in table 3 one may notice that the highest positive value of correlation coefficient was that for WIG index and PKN Orlen and then for WIG index and KGHM. It should not be a surprise while remembering that the largest share in WIG portfolio is that of KGHM (9,94%) and the fifth share is that of PKN Orlen (5,485%). The second, third and fourth shares are those of PKO BP, PZU and PeKaO SA – not included in the research due to their activity profiles unrelated to commodity sector. The highest negative correlation (statistically significant) was observed for the following pairs: PZU Energia Medycyna Ekologia – BPH Globalny Żywności i Surowców and PZU Energia Medycyna Ekologia – PGNiG. Generally, the results obtained confirm that between commodities and stocks or bonds there is usually negative or positive weak correlation [Jensen at al. 2000; Gorton, Rouwenhorst 2006; Geman 2007; Schofield 2007; Stockton 2007].

In the next step of research, with the use of Shapiro-Wilk and Jarque-Bera tests, there were verified hypotheses that considered logarithmic returns were normally distributed. The results obtained are reported in table 4. There are also displayed probability values called critical significance levels. If they are higher than the prespecified significance level α, then the null hypothesis on distribution normality cannot be rejected. Additionally, figures 1 and 2 show histograms for logarithmic returns of all investigated assets. On the base of results given in table 4, one can state that distributions of logarithmic returns of all considered assets do not follow normal distribution. Taken significance level $\alpha=0,05$, allows to notice that in all cases it is higher than p-value, which obliges to reject the null hypothesis that variable under consideration is normally distributed. The decision seems to be unmistakable as it can be changed to the opposite (not rejecting the null hypothesis) at extremely low significance level.

Next, in order to assess efficiency of indirect ways of investing in commodities in the conditions of Polish market, Mann-Whitney U test was used to verify the following hypothesis:

H_0: mean values of logarithmic returns series of X asset and WIG index are equal against

H_1: mean values of logarithmic returns series of X asset and WIG index differ significantly.

At the significance level 0,05, critical value of standardized normal distribution equals $\pm 1,96$, so critical areas for hypothesis formulated above are the following: $(-\infty, -1,96)$ and $(1,96, +\infty)$. Results of verification, presented in table 5, let us conclude that in the considered period expected rates of return of separate assets and WIG index did not differ significantly as in all cases the null hypothesis cannot be rejected.

Table 3. Correlation coefficients for logarithmic returns of considered stocks, funds and WIG index.

	CEZ	Kernel	KGHM	Lotos	PGNiG	Orlen	Synthos	Idea S.P.	Inv. G.	Inv. Agr.	Skarbiec	BPH	Pioneer	PZU	Opera	WIG
CEZ	1															
Kernel	0,0558	1														
KGHM	0,1961	0,1576	1													
Lotos	0,2061	0,0942	0,5148	1												
PGNiG	0,1508	0,0653	0,3592	0,3394	1											
Orlen	0,2314	0,1528	0,5797	0,6349	0,4522	1										
Synthos	0,1680	0,1692	0,3780	0,3041	0,1116	0,3850	1									
Idea S.P.	0,0457	0,1097	0,3419	0,2574	0,1371	0,204	0,1203	1								
Inv. G.	-0,0551	-0,0773	0,0252	-0,0063	0,004	0,0393	-0,0116	0,0448	1							
Inv. Agr.	0,0811	0,0628	0,0934	0,1002	0,0291	0,0746	0,082	0,2405	-0,0390	1						
Skarbiec	0,0696	0,0759	0,4002	0,2935	0,1907	0,2768	0,1179	0,5238	0,2397	0,1289	1					
BPH	0,1079	0,1094	0,4381	0,3186	0,1951	0,3350	0,2029	0,3292	0,1418	0,0677	0,6762	1				
Pioneer	0,0144	0,0918	0,0961	0,0323	0,0499	0,0494	-0,0164	0,6150	0,0355	0,2288	0,3582	0,1658	1			
PZU	-0,0195	-0,0096	0,0084	-0,0497	-0,1156	-0,0658	0,0508	-0,0355	0,0677	-0,0087	-0,0333	-0,1667	-0,0011	1		
Opera	0,0709	0,0779	0,2476	0,1749	0,1202	0,1893	0,0445	0,5218	0,1378	0,1283	0,4222	0,3496	0,4986	0,0066	1	
WIG	0,3005	0,2296	0,7181	0,6575	0,5337	0,7912	0,4622	0,3371	-0,0322	0,1408	0,3264	0,3501	0,1074	-0,0116	0,2090	1

Source: own calculations

Table 4. Results of testing normality of logarithmic returns of considered assets

Asset	Statistic	
	Shapiro-Wilk	Jarque-Bera
CEZ	0,976 (3,514E-26)	244,939 (6,487E-54)
Kernel	0,928 (9,197E-0,45)	524,321 (1,397E-114)
KGHM	0,984 (5,931E-6)	54,111 (1,778E-12)
Lotos	0,943 (4,559E-14)	823,720 (1,354E-179)
PGNiG	0,976 (3,725E-8)	81,758 (1,764E-18)
PKN Orlen	0,986 (2,021E-5)	66,821 (3,090E-15)
Synthos	0,953 (1,784E-12)	230,447 (9,099E-51)
Idea Surowce Plus	0,973 (1,161E-8)	139,043 (6,414E-31)
Investor Gold	0,980 (6,223E-7)	60,661 (6,725E-14)
Investor Agrobiznes	0,963 (1,002E-10)	272,043 (8,446E-60)
Skarbiec Rynków Surowcowych	0,981 (9,227E-7)	83,196 (8,593E-19)
BPH Globalny Żywności i Surowców	0,972 (5,362E-9)	128,757 (1,09864E-28)
Pioneer Surowców i Energii	0,986 (3,472E-5)	38,269 (4,897E-9)
PZU Energia Medycyna Ekologia	0,979 (11,166E-20)	2916,690 (0,000)
Opera Substantia	0,966 (2,562E-10)	148,046 (7,115E-38)
WIG	0,969 (1,459E-9)	128,283 (9,382E-31)

Source: own calculations

Apart from the expected rate of return, another basic characteristic of every investment is its level of risk. In the case of commodity investments, it is believed that their returns are less volatile and thus less risky [Akey 2005]. In order to verify the hypothesis with regard to indirect commodity investments, there were

calculated absolute values of returns deviations from the mean. Then, Mann-Whitney U test was applied again, but this time observations were ordered in descendent way (higher absolute deviations received lower rank numbers).

Figure 1. Histograms for logarithmic returns of considered stocks: CEZ (a), Kernel (b), KGHM (c), Lotos (d), PGNiG (e), PKN Orlen (f), Synthos (g) and WIG index (h)
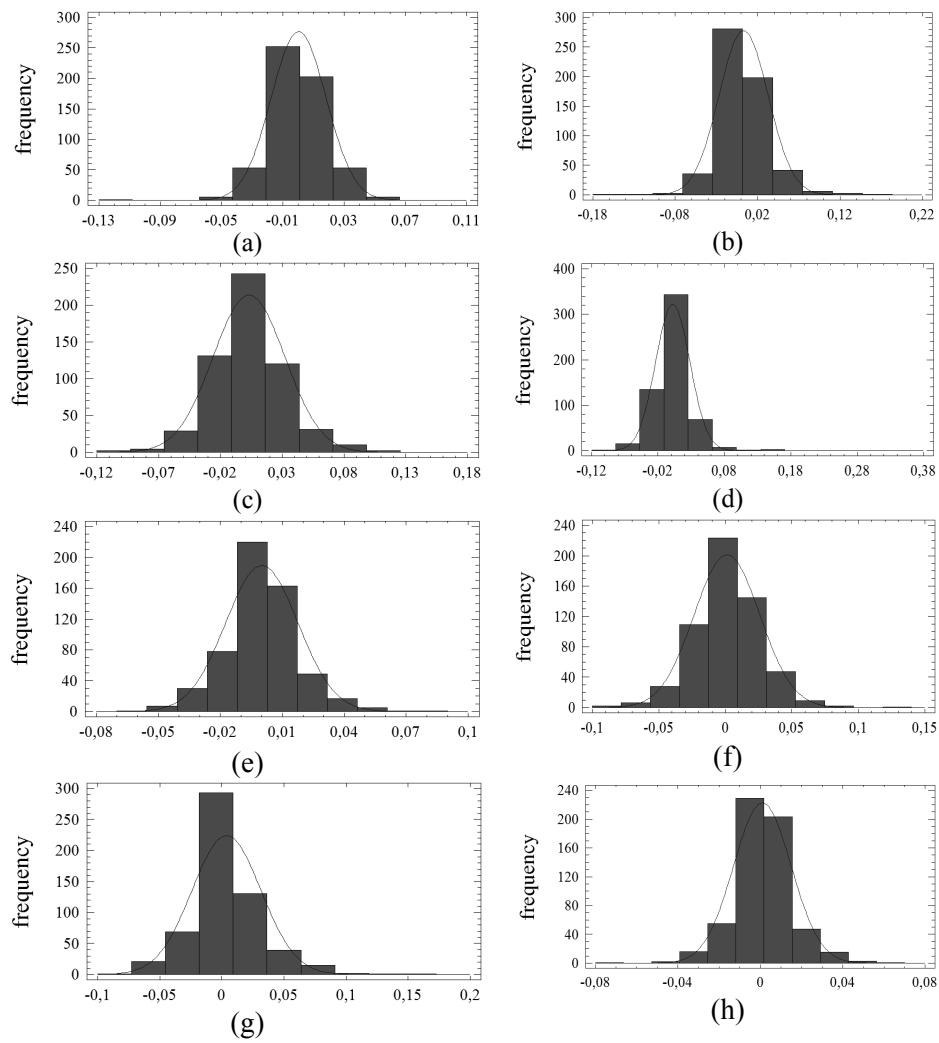
Source: own elaboration

Figure 2. Histograms for logarithmic returns of considered commodity funds: Idea Surowce
    Plus (a), Investor Gold (b), Investor Agrobiznes (c), Skarbiec Rynków
    Surowcowych (d), BPH Globalny Żywności i Surowców (e), Pioneer Surowców
    i Energii (f), PZU Energia Medycyna Ekologia (g) and Opera Substantia (h)
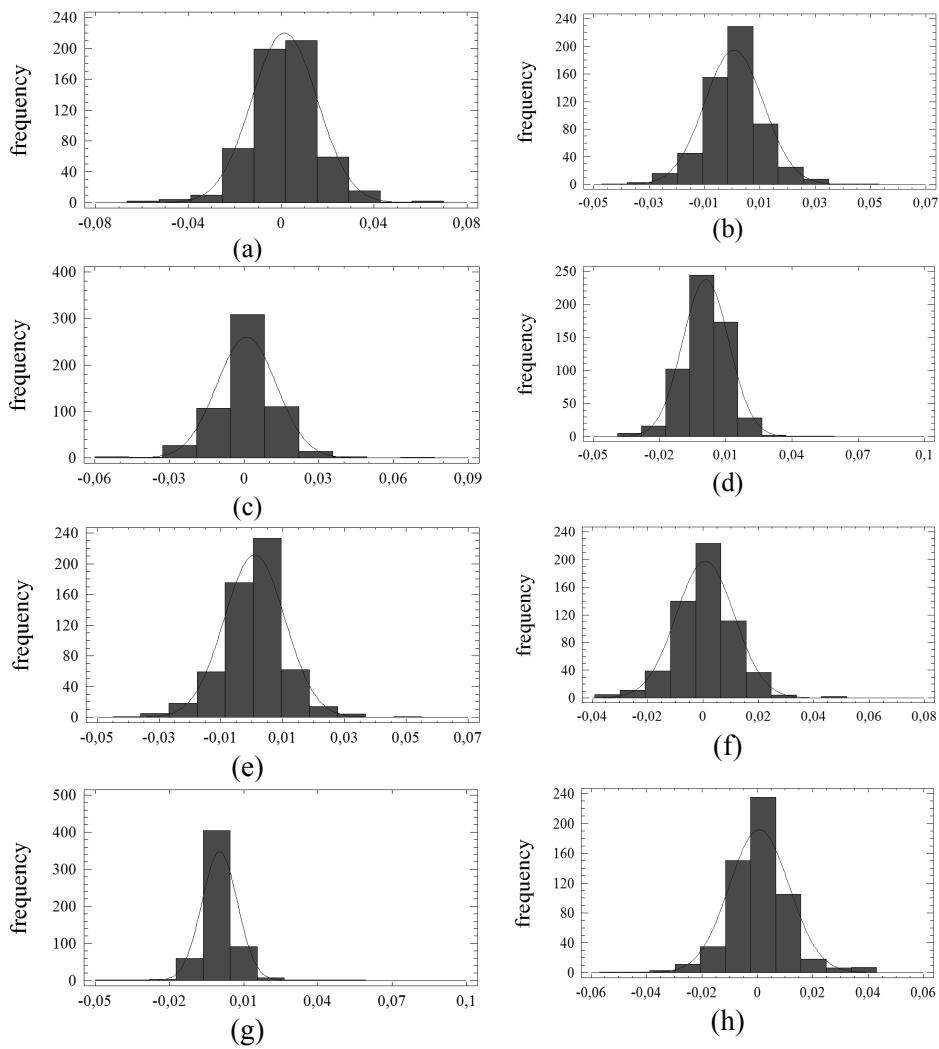
Source: own elaboration

The following hypothesis was formulated and then verified:

H_0: deviations of X asset rates of return from the mean are identical as deviations
of WIG index rates of return

against

H_1: deviations of X asset rates of return from the mean differ significantly from
deviations of WIG index rates of return.

Results are reported in table 6.

Table 5. Mann-Whitney U test results for logarithmic returns of separate assets and WIG index

Pair	Z statistic	Pair	Z statistic
CEZ-WIG	-0,8525	Inv. Gold-WIG	-0,3655
Kernel-WIG	-0,5042	Inv. Agrob.-WIG	-0,3281
KGHM-WIG	-1,2943	Skarbiec-WIG	-0,2225
Lotos-WIG	-0,4712	BPH-WIG	-0,2668
PGNiG-WIG	-1,2911	Pioneer-WIG	-0,0668
PKN Orlen-WIG	-0,4222	PZU-WIG	-1,3070
Synthos-WIG	-0,8145	Opera-WIG	-0,2966
Idea S. -WIG	-0,8349	×	×

Source: own calculations

Table 6. Mann-Whitney U test results for deviations from mean for separate assets and WIG index

Pair	Z statistic	Pair	Z statistic
CEZ-WIG	-5,5459	Inv. Gold-WIG	-4,3641
Kernel-WIG	-7,8842	Inv. Agrob.-WIG	-2,8860
KGHM-WIG	-11,9122	Skarbiec-WIG	-3,2901
Lotos-WIG	-9,1540	BPH-WIG	-4,9882
PGNiG-WIG	-2,9947	Pioneer-WIG	-3,7103
PKN Orlen-WIG	-9,8481	PZU-WIG	-12,8273
Synthos-WIG	-9,5408	Opera-WIG	-4,2265
Idea S. -WIG	-0,3714	×	×

Source: own calculations

On the base of results given in table 6, one may state that only in the case of pair: Idea Surowce Plus – WIG, the null hypothesis cannot be rejected. In all other cases one should reject the null hypothesis, which implies that deviations of separate assets rates of return from their means differed significantly from the deviations of WIG index rates of return. More detailed analysis leads to the conclusion that in the case of considered stocks, deviations from their means are higher than in the case of WIG index. On the contrary deviations of investigated commodity funds are lower than deviations of WIG index. Thus it is legitimate to conclude that indirect investing in commodities through commodity funds may be attractive to investors due to the lower risk in comparison to the stock market in Poland.

CONCLUDING REMARKS

Although investing in commodities grows in popularity among investors, some experts say that commodity sector still remains undervalued, particularly

when compared with the financial markets and the prices will continue to rise. Within last couple of years we have observed record levels of commodity prices, but Balarie [2007] claims it is not a bubble – it is a reasonable tendency which is going to be continued. Akey [2005] listed several reasons for interest in commodity investments. The most important are the following: growth of consumption of raw materials in developing economies (China, India, Russia, Brasil) creating demand for commodities across all sectors; commodities are valuable hedging instrument for investors with economic view on inflation and weakened currencies; commodities produce similar returns to equities with less historical volatility, commodity returns are not correlated to financial assets like stocks and bonds, so adding commodities to a traditional portfolio can enhance returns and decreases volatility. That is why investors view commodities as a source of both portfolio diversification and investment return.

The research presented here focuses on investments through commodity-related stocks listed at the Warsaw Stock Exchange and commodity funds operating in Poland. On the base of logarithmic returns of selected assets there were calculated basic characteristics, such as expected rate of return, standard deviation, kurtosis, skewness or correlation. Then the normality of distribution of investigated time-series was tested. In all cases one should reject the hypothesis of distribution normality. Thus in the next step Mann-Whitney U test was used to verify the hypotheses of equality of means and deviations from mean of logarithmic returns of analysed assets. The research revealed that in the considered period those commodity-related investments generated returns no different from the stock market (the benchmark was the main index of Warsaw Stock Exchange - WIG). Analysed stock returns were more volatile, while commodity funds returns were less volatile in comparison to the benchmark, so the second investment form generated lower risk. Nevertheless, one could draw the conclusion that those commodity-related investments did not seem much more attractive in comparison to the Polish stock market. Author's earlier research [Krawiec 2010] for German capital market, based on Deutsche Bank Liquid Commodity Index and DAX index led to the similar conclusion. This corresponds with Gilbert's opinion that commodity investments are generally justified more in terms of their contribution to overall portfolio returns than as attractive stand alone investment [Gilbert 2008]. However it is worth to remember that the study presented here covers a specific period when one could observe financial instability in numerous markets all over the world. That undoubtedly influenced Polish capital market. Moreover, the period analysed is quite short due to the fact that many commodity funds have occurred on the Polish market within last 2 – 3 years, so in most cases the track record does not go beyond 3 years.

REFERENCES

Aczel A. D. (2000) Statystyka w zarządzaniu, WN PWN, Warszawa.

Akey R. P. (2005) Commodities: a case for active management, www.dnb.co.in

Borowski K. (2008) Nowe kierunki bankowości inwestycyjnej – rynek przedmiotów kolekcjonerskich, Inwestycje alternatywne, praca zbiorowa pod red. I. Pruchnickiej-Grabias, CeDeWu, Warszawa, 217-266.

Filipowicz E. (2010) Inwestycje w złoto jako alternatywna forma lokowania kapitałów na polskim rynku, Zeszyty Naukowe Uniwersytetu Szczecińskiego Finanse, Rynki Finansowe, Ubezpieczenia nr 29, 329-347.

Geman H. (2007) Commodities and commodity derivatives, John Wiley&Sons, New Jersey.

Gilbert Ch. L. (2008) Commodity speculation and commodity investment, Discussion Paper No. 20, Universita degli Studi di Trento, Dipartimento di Economia, www.unitn.it

Gorton G., Rouwenhorst K. G. (2006) Facts and fantasies about commodity futures, Financial Analyst Journal, 62, 47-68.

Jagielnicki A. (2011) Inwestycje alternatywne. Pierwsze kroki na rynku pozagiełdowym, Helion, Gliwice

Jajuga (2007) On some tendencies in alternative investments. Inwestycje i ubezpieczenia – tendencje światowe a polski rynek, Prace naukowe AE we Wrocławiu nr 1176, Wrocław, 119-128.

Jensen G. R., Johnson R. R., Mercer J.M. (2000) Efficient use of commodity futures in diversified portfolios, The Journal of Futures Markets, Vol. 20, No 5, 489-506.

Kowalewski G., Kuziak K. (2000) Badanie rozkładów stóp zwrotu i cen akcji – badania empiryczne dla GPW w Warszawie, Metody ekonometryczne i statystyczne w analizie rynku kapitałowego, praca zbiorowa pod red. K. Jajugi, Wydawnictwo AE we Wrocławiu, Wrocław, 76-90.

Krawiec M. (2010) Weryfikacja efektywności pośrednich form inwestowania w towary na przykładzie Deutsche Bank Liquid Commodity Index, Metody ilościowe w badaniach ekonomicznych, tom XI, Nr 2, 161-170.

Mynarski S. (2006) Analiza danych rynkowych i marketingowych z wykorzystaniem programu Excel, Wydawnictwo Akademii Ekonomicznej w Krakowie, Kraków.

Niedziółka P. (2008) Inwestycje w sztukę na świecie oraz perspektywy rozwoju rynku sztuki w Polsce, Inwestycje alternatywne, praca zbiorowa pod red. I. Pruchnickiej-Grabias, CeDeWu, Warszawa, 197-215

Osińska M. (2006) Ekonometria finansowa, PWE, Warszawa.

Schofield N. C. (2007) Commodity derivatives, John Wiley&Sons, Chichester, West Sussex.

Stockson K. A. (2007) Understanding alternative investments: the role of commodities in a portfolio, www. vanguard.com

Tarczyński W. (1997) Rynki kapitałowe – metody ilościowe Vol. II, Agencja Wydawnicza Placet, Warszawa.

Tarczyński (2002) Fundamentalny portfel papierów wartościowych, PWE, Warszawa.

Winkler-Drews T. (2009) Zarządzanie ryzykiem zmiany ceny. PWE, Warszawa.

Witkowska D., Matuszewska A., Kompa K. (2008) Wprowadzenie do ekonometrii dynamicznej i finansowej, Wydawnictwo SGGW, Warszawa.

APPLICATIONS OF PRODUCTION FUNCTION
IN AGRICULTURE

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Abstract: The article describes some possibilities of using the production function for the theoretical analysis of agricultural economics. It's proposed a some approaches to the analysis of Agricultural economy in the Poland as national economy part and Agricultural economy in the Republic of Buryatia1 as regional economy part. The mathematic models are made for both cases, the results is obtained, which show the production functions efficiency for macroeconomic analysis.

Key words: Production function, Agricultural economy, analyze, forecast

INTRODUCTION

The production function shows the dependence between the production result and the volume of resource use. Production function is a function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost of all macroeconomic theories, like macroeconomic theory, real business cycle theory, neoclassical growth theory presupposes production function. In this sense, production function is one of the key concepts of neoclassical macroeconomic theories. It is also important to know that there is a subversive criticism on the very concept of production function.

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[1] The Republic of Buryatia is a region of the Russian Federation.

THEORETICAL BACKGROUND

The subject of study on production function is found interesting because of following reasons:

1. Production functions seem to be a group functions widely used to analyze economic processes since being highly efficient they are not difficult to study, and they require estimated capacities. Unfortunately, today production functions are no longer used to analyze the Agricultural economy in Russia and Poland due to a range of circumstances.

2. Students being specialized in economics study production functions as the theoretical discipline as well as applied science. Study on production functions is carried out within a single discipline since it is not special one.

3. Production functions are widely spread in the World where our colleagues are actively using them. But we need to come to an agreement about calculation methods and their application.

Thus, there is a precedent to apply production functions in practical analysis of economy development in the region. At the same time there are some factors restricting their use. We have mentioned them before.

To have a good understanding of the essence of production functions we should study some theory first, and then pass on to some practical decisions.

As economic process production functions seem to demonstrate dependence of production capacity on tangible resources. As a factor of production of macro level we focus on capital (usually fixed assets) K and labour L, as for the outcome it can be distinguished as follows.

- Gross product (GP) – X.
- Gross domestic product (GDP) – Y;
- National revenue (NR) – N.

In this case we shall focus on gross output in economy – X.

Production assets are introduced in the form of basic and floating, production and non-production. The choice of this or that factor K should be defined by the purpose of study.

IMPLEMENTATION OF PRODUCTION FUNCTION IN
AGRICULTURE – CASE STUDY

The results of agricultural economy development analysis of the Poland and the Republic of Buryatia are presented in this article. The model in the form of nonlinear production function is substituting the Agricultural economy.

$$X = F(K, L) \tag{1}$$

It means that production output is a function resulted from costs of resources including production assets and labour. The most widely spread are multiplied forms of production functions.

$$X = AK^{\alpha_1}L^{\alpha_2}, \qquad \alpha_1 > 0, \ \alpha_2 > 0 \text{,} \tag{2}$$

where A is total factor productivity (a ratio of neutral technical progress that is distinguished as dimension ratio).

α_1, α_2 are the output elasticities of capital and labour. These values are constants determined by available technology.

The Cobb-Douglas function can be found as particular case of this function.

$$X = AK^{\alpha}L^{1-\alpha}, \qquad \alpha_1 = \alpha, \ \alpha_2 = 1-\alpha \tag{3}$$

Two types of functions have being distinguished:

1. Constant Elasticity of substitution (CES).

In this case resource elasticity of substitution doesn't depend on neither K nor L, that is why it is constant.

$$X = A[(1-a)K^{-b} + aL^{-b}]^{-\frac{c}{b}} \tag{4}$$

2. Variable Elasticity of substitution (VES).

$$X = Ae^{\alpha t} \cdot K^{\alpha} \cdot L^{\beta} \cdot \exp[c(\frac{K}{L})] \tag{5}$$

Let's focus on the production function once again. There is following information for making analytical calculations:

Table 1. – Gross output, capital (fixed assets) and labour (number of agricultural employers) in Poland.

Years	Gross output (X), mln. PLN	Capital - Fixed assets (K), mln. PLN	Labour (L), thousands
2000	55985,4	106831,4	4304,6
2001[2]	55845,7	108655,5	3232,9
2002	55706,0	110479,5	2161,1
2003	56263,6	110707,5	2138,3
2004	69747,7	110935,4	2139,5
2005	63337,2	112375,7	2134,1
2006	65083,4	114669,3	2140,6
2007	81509,2	117377,2	2138,2
2008	83126,5	119921,4	2128,3
2009	79706,6	122570,0	2124,9

Source: Statistical Yearbook of Agriculture, Warsaw, 2002 -2010, own

[2] 2001 data are estimated.

Table 2. – Gross output, capital (fixed assets) and labor (number of agricultural employers) in Republic of Buryatia .

Years	Gross output (X), mln RUB	Capital - Fixed assets (K), mln RUB	Labour (L), thousands
2000	4481,7	3799,0	60,9
2001	5164,2	4655,0	60,3
2002	5638,1	5031,0	58,0
2003	6344,7	4715,0	71,7
2004	7471,4	7133,0	67,3
2005	8036,6	6676,0	56,2
2006	8993,5	8335,0	54,1
2007	10546,2	8103,0	54,9
2008	11745,6	8944,0	55,8
2009	12086,3	9523,0	55,9

Source: Russian Statistical Yearbook, Moscow, 2010, Regions of Russia: Social and Economic Indicators, Moscow, 2010.

Production function (3) is used for the analysis in both cases. Multiplicative production function is determined from a time series of production result and resource costs (X_t, K_t, L_t), $t = 1,...,T$, where T is the length of time series. It is assumed that there is T number of relationship $X_t = \delta_t A K_t^{\alpha_1} L_t^{\alpha_2}$ where δ_t is correction coefficient, which aligns the actual and estimated production results, and shows the result fluctuation under the influence of other factors. At the same time $M\delta_t = 1$. This function is linear in logarithms

$$\ln X_t = \ln A + \alpha_1 \ln K_t + \alpha_2 \ln L_t + \varepsilon \tag{6}$$

where $\varepsilon_a = \ln \delta_t$, $M\varepsilon_t = 0$. It is obtained a model of linear multiple regression. Function parameters A, α_1, α_2 can be determined by the method of least squares.

This model can be used to analyze and forecast the Agricultural economy in the Poland.

$$X' = 1 \cdot 10^{-13} K^{3,493} L^{0,0458} \tag{7}$$

From the analysis of coefficients α_1, α_2 in the function (7) shows that. $\alpha_1 + \alpha_2 > 1$. Therefore, the function (7) be called a disproportionate and quickly growing. Coefficient $\alpha_1 = 3,493$ means that the increase in fixed assets by 1% causes an increase in gross output of agriculture in the Poland by 3,5%. And coefficient $\alpha_2 = 0,0458$ shows the number of employers increase by 1% causes the gross agricultural output growth by only 0.05%. Hence, we can conclude that Agricultural economy growth in the Poland is highly dependent on the fixed assets development and their high productivity. The coefficient of neutral technical

progress A is close to zero and significantly reduces the value of gross output in Polish agriculture.

For describing the development of the agricultural economy in the Republic of Buryatia has received the following production function:

$$X' = 4,4476 \cdot 10^{-1} K^{1,0688} L^{0,1046}.$$

(8)

Analysis of the elasticity coefficients for the function (8) shows that $\alpha_1 + \alpha_2 > 1$, i.e. function (7) is increasing disproportionately. Increase in fixed assets by 1% causes the increase in gross output in agriculture in the Republic of Buryatia by more than 1.06%. But the increase in the number of employers per 1% increase in gross output by 0,1%.. Therefore one can say, that a qualitative indicator of capital productivity in agricultural economics is more important than labour productivity.

CONCLUSION

One can conclude that agricultural economy growth in the Poland is highly dependent on the fixed assets development and their high productivity. Production growth in the Republic of Buryatia largely depends on the value of assets growth in agriculture, rather than growth in the number of employers.

REFERENCES

Cohen, A.J. and Harcourt, G.C. (2003) Retropectives: Whatever Happend to the Cambridge Capital Theory Controversies? /Cohen, A.J. and Harcourt, G.C. //Journal of Economic Perspectives. – №17(1). – pp.199-214.

Дареев Г.Е. (2003) Использование производственных функций в макроанализе сельскохозяйственного производства в регионе /Г.Е. Дареев //Состояние и перспективы развития агропром. комплекса Забайкалья: Материалы ежегод. науч.-практ. конф. (4-6 февр. 2003 г.) /Бурят. гос. с.-х. акад. им. В.Р. Филиппова. – Улан-Удэ,– С.272 – 276.

Дареев Г.Е. (2003) Производственные функции /Г.Е. Дареев //Развитие агробизнеса и сельских территорий в Байкал. регионе: Материалы междунар. семинара (9-13 сент. 2002 г.) /Бурят. гос. с.-х. акад. им. В.Р. Филиппова, Швед. ун-т с.-х. наук. и др. – Улан-Удэ, – С.96 – 101.

Дареев Г.Е. (2006) Производственные функции в моделировании экономического роста регионального АПК /Г.Е. Дареев //Инновации молодых ученых – сел. хоз-ву: Всерос. конф.: Сб. материалов/ МСХ РФ. – М., ФГНУ «Росинформагротех», – ч.1. – С.21-27. – Библиограф. с.27 (3 назв.)

Колемаев В.А. (1998) Математическая экономика: Учебник для вузов. – М.: ЮНИТИ, – 240 с.

Moroney, J. R. (1967) Cobb-Douglass production functions and returns to scale in US manufacturing industry /Moroney, J. R. //Western Economic Journal. vol 6, №1, December – pp 39-51.

REACTION OF THE INTEREST RATES IN POLAND
TO THE INTEREST RATES CHANGES
IN THE USA AND EURO ZONE[1]

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Abstract: Behavior of interest rates is of key importance for understanding the functioning of an open economy. The simplest models usually assume equal interest rates in individual countries, while the international arbitrage serves as a mechanism of their equalization. In our study an attempt has been made to determine whether and to what extend the interest rates in the Polish market are linked to the USA and the euro zone exchange rates. The analyses have been carried out for rates of different maturity terms, using the integration and co-integration concept.The analyses indicate that differences between the Polish interest rates, and those in the USA and the euro zone have strongly diminished. Cointegration analyses show the existence of a long-term linkages between the domestic and foreign interest rates, in particular with those in the euro zone. The nature of co-integrating relationships was different in the period 2001-2004 as compared with that after 2004, when we see a stronger impact of the euro zone rates than those of the USA. It may be assumed that the Polish accession to the EU had certain influence in the change of the above mentioned relationships.

Key words: interest rates, world markets, cointegration analysis, Error Correction Model (ECM)

INTRODUCTION

Behavior of interest rates is of vital importance for understanding the functioning of an open economy and its significance in pursuing of macroeconomic policy must not be underestimated. Usually in the simplest models the real interest rates in individual countries are assumed equal with their level equalized by the mechanism of international arbitrage[2]. In accordance with the rule of unsecured interest parity the domestic interest rate for the same term of maturity should differ by the anticipated rate of foreign exchange modification and a specific for a given country risk premium[3].

Analyses made for the periods of 60's and 70's indicated a significant differentiation of interest rates[4]. In the last 30 years however an alignment of interest rates has been observed at least in the countries with the developed financial markets[5]. Nevertheless it's still rather difficult to talk about full markets' integration. Many are the reasons of such incomplete integration to mention only not fully coordinated macroeconomic policy in individual countries, still existing restrictions in transfer of capital in some countries or the so called „home bias".

Many analyses have been focused on interest rate linkages in individual countries and most of them are based on the integration and cointegration methods. Their findings however do not present an unambiguous result. Some works confirm the existence of cointegration among markets[6], while other do not show a full integration of interest rates between countries[7]. In case of some countries the

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[2] I.A. Moosa, R.H. Bhatti, Testing the effectiveness of arbitrage and speculation under flexibile exchange rates. Economica Internazionale, 47. 1994. pp. 392-408.

[3] M. Beenstock, J. Longbotom.. The term structure of interest rates in a small open economy. Journal of Money. Credit and Banking. 13. 1981pp. 44-59.

[4] F. Mishkin, Are real interest rates equal across countries? An empirical investigation of international parity conditions. Journal of Finance. 39. 1984. pp. 1345-1357.

[5] H. Sasaki. S. Yamaguchi. H. Takamasa. The globalisation of financial markets and monetary policy. BIS Conference Papers no.8. 2000. pp. 57-78.

[6] G. Quiro-Romero. S. Sosvilla-Rivero. Do short-term rates influence long-term interest rates? Empirical evidence from some EMS countries. Applied Economics Letters. 4. 1997. s. 449-451. B.T. Ewing, J.E. Payne, S.M. Forbes, Co-movements of the prime rate. cd rate. and the S&P financial stock index. Journal of Financial Research. 21. 1998. pp. 469-482.

[7] S. Kleimeier. H. Sander. Regionalisation versus globalisation in European financial market integration: evidence from cointegration analyses. Journal of Banking and Finance. 24. 2000. pp. 1005-1043. E. Waller, H. Kiymaz, Cointegration between

existence of cointegration has been found while in the case of some a hypothesis of such relationship has been rejected.

In our study an attempt has been made to determine whether and to what extend the interest rates in the Polish market are linked to those in the USA and in the Euro zone. We have also been interested in identification of long- and short term relationships. Analyses have been made for the interest rates with different maturities using the concept of integration and cointegration.

METHODOLOGY

The very straightforward methods of analyzing relations between interest rates are based on calculations of correlation coefficients. They are technically the simplest, but due to certain characteristics of time series (non-stationarity, heteroscedacity) they may yield results of little reliability. More analytical possibilities are provided by auto regression models referring to the concepts of integration and cointegration and in our study the latter have been used. Our analyses encompassed the following stages:

- analysis of the integration degree of individual variables (interest rates time series). To this end the ADF test has been used while the selection of lags number in testing has been determined on the basis of the Akaike information criterion;

- analysis of cointegration. It has been carried out by construction of a cointegration vector between the level of interest rates in Poland and those in the USA and the euro zone and also by testing the stationarity of residuals from cointegrating regression;

- analysis of transmission based on the autoregression model. A specific shape of the model depended on results obtained in the first two stages. In the case when stationarity of exchange rates has been established, an autoregression model based on variables on their levels should be applied; while for the stationary variables in degree one (stationary first differences of variables) and mutually cointegrated we use the autoregressive model based on first differences of variable with the error correction mechanism, as shown below:

$$\Delta y_t = c + \sum_{i=1}^{k-1} \theta_i \Delta y_{t-i} + \sum_{i=0}^{k-1} \gamma_i \Delta x_{t-i} + \alpha ECM_{t-1} + \varepsilon_t \tag{1}$$

where:
ECM_{t-1} - residuals of cointegration equation.

international short-term interest rates. Journal of Accounting and Finance Research. 6. 2004. pp. 11-21

In the case of change in the non-integrated variables it would seem more appropriate to use the autoregression model overriding the error correction mechanism.

Source material for our analyses has been based on data on the interest rates level calculated for 1-year, 1-month and 1-week treasury bonds in the period 2001-2009. Data on interest rates level were obtained from the National Bank of Poland. In our analyses of integration, cointegration and transmission we used data on the weekly interest rates quotes.

TRENDS IN THE INTEREST RATE CHANGES IN THE MARKETS UNDER STUDY

Difficulties in modeling relationships between the Polish interest rates and those in the USA and the euro zone may be a result of not only the levels of interest rates, but also of different statistical properties of these series (table 1). In the beginning of the period under examination the Polish interest rates were at the higher level than those in the USA and in the euro zone. It is only since 2005 when there has been a significant drop in the Polish interest rates that we can talk about the interest rates alignment.

The interest rates time series in Poland are marked by the highest variability (standard deviation about 4%), while in the USA it has been significantly lower (standard deviation about 1.5%) and the lowest in the euro zone (standard deviation below 1%). All the interest rates time series are marked by the right bias asymmetry i.e. in general the low interest rates prevail and the average level of interest rates is significantly higher than that for the most often observed, however the magnitude of this phenomenon varies. For the Polish interest rates the asymmetry coefficient was about 1.5, while for the USA and the euro zone the asymmetry level is weaker. Besides, the time series in Poland display a very strong kurtosis while for the USA and the euro zone this measure is very weak.

Table 1. Descriptive characteristics of the interest rates time series

Interest rate	Mean	Std.deviation	Skewness	Kurtosis
POLAND 1Y	7.24	3.71	1.55	1.33
USA 1Y	3.28	1.52	0.14	-1.48
EURO 1Y	3.06	0.81	0.39	-1.26
POLAND 1M	7.51	4.27	1.50	1.04
USA 1M	2.96	1.61	0.37	-1.42
EURO 1M	2.91	0.85	0.75	-0.50
POLAND 1W	7.55	4.38	1.49	0.95
USA 1W	2.94	1.62	0.39	-1.40
EURO 1W	2.90	0.86	0.79	-0.41

Source: own studies

ANALYSIS OF THE INTEREST RATES RELATIONSHIPS

Different statistical properties of the interest rates time series produce different results of the interest rates stationarity tests in Poland, the USA and the euro zone (table 2). As evidenced by the tests, the interest rates time series in Poland are marked by stationarity. Non-stationarity has only been shown by the ADF test with the 1-week constant. Besides, due to the close to zero values of the ADF statistics, the test for first differences of interest rates has also been made and statistical values turned out to be much lower. The results indicate stationarity of the first increments of interest rates. In the case of the USA and the euro zone interest rates non-stationarity of the observation series has been found, as well as stationarity of the first differences which means the 1 degree integration of variables.

Table 2. Stationarity tests of the interest rates time series

Interest rate	ADF without constant				ADF with constant			
	I(0)		I(1)		I(0)		I(1)	
	ADF	p	ADF	p	ADF	p	ADF	p
POLAND 1Y	-3.8730	0.0001	-20.9989	0.0000	-3.2835	0.0164	-21.5309	0.0000
USA 1R	-0.5397	0.4827	-17.9385	0.0000	-0.9577	0.7688	-17.9112	0.0000
EURO 1R	-0.2467	0.5968	-15.6917	0.0000	-0.9541	0.7700	-15.6674	0.0000
POLSKA 1M	-4.3811	0.0000	-21.8390	0.0000	-3.4191	0.0110	-22.5688	0.0000
USA 1M	-0.5016	0.4985	-12.5497	0.0000	-0.7555	0.8296	-12.5302	0.0000
EURO 1M	-0.8154	0.3622	-12.6076	0.0000	-1.4297	0.5680	-12.6005	0.0000
POLSKA 1T	-3.5396	0.0004	-23.3005	0.0000	-2.5845	0.0972	-23.7656	0.0000
USA 1T	-0.6676	0.4274	-22.7245	0.0000	-0.8695	0.7971	-22.6930	0.0000
EURO 1T	-0.9204	0.3171	-21.2561	0.0000	-1.5689	0.4974	-21.2393	0.0000

Explanations: I(0) - level stationarity testing. I(1) - first increments stationarity testing, ADF - empirical value of test, p - significance level of test.
Source: own studies

Results of stationarity tests may indicate a lack of long-term linkages between national and foreign interest rates. However, attention must be drawn to a different nature of the interest rates changes in Poland in the years 2001-2004 and 2005-2009, as expressed by taking an account of a zero-one variable in the cointegrating equation (table 3). All the parameters of determined cointegrating equations have proven to be of statistical significance and also provide a very good explanation of the interest rates shaping in Poland - determination coefficients are formed at the level close to 0.9.

The obtained results show the existence of cointegration between the Polish and foreign markets. One exception to the above is a linkage of 1-year interest rates in Poland with the USA interest rates where the cointegrating equation residuals turn out to be non-stationary. In other cases, the hypothesis about nonstationarity of the cointegrating equation residuals may be rejected in favor of the residuals stationarity at the significance level below 0.1.

Table 3. Cointegration tests

Independent variable	Model					ADF without constant		ADF with constant	
	a1	a2	a3	c	R2	ADF	p	ADF	p
Dependent variable: POLAND 1YEAR									
USA 1YEAR	0.3705	0.2619	-0.0058	2.3150	0.9226	-1.2172	0.2050	-1.2072	0.6723
EURO 1YEAR	0.7141	0.5132	-0.0056	1.7590	0.9221	-3.0407	0.0024	-3.0363	0.0326
Dependent variable: POLAND 1MONTH									
USA 1MONTH	0.3247	0.2266	-0.0062	2.5027	0.9515	-1.7976	0.0688	-1.7888	0.3858
EURO 1MONTH	0.6927	0.4200	-0.0054	1.8527	0.9320	-2.6692	0.0076	-2.6652	0.0812
Dependent variable: POLAND 1WEEK									
USA 1WEEK	0.3287	0.2052	-0.0061	2.5089	0.9501	-2.2605	0.0232	-2.2510	0.1889
EURO 1WEEK	0.7148	0.3975	-0.0053	1.8272	0.9298	-2.6411	0.0082	-2.6370	0.0866

Explanations: parameters of the cointegrating equation are denoted in accordance with formula 1.
Source: own studies

Taking into account the previously obtained results for describing linkages between the national and foreign interest rates we used here the model with an error correction mechanism. In table 4 we present detailed characteristics for individual categories of interest rates. In the models presented below the following parameters remained:

- parameters of explanatory variables for statistically significant,
- parameters of explanatory variables with the significance level not meaningly divergent from 0.1,
- parameters of current independent variable increments and parameters of residuals from cointegrating equation irrespective of their significance.

At the aforesaid conditions, it will be possible to determine the reaction of the interest rates in Poland to the current changes of interest rates in the USA and the euro zone, as well as the process of coming to a long-term equilibrium.

The model of interest rates linkages shows that changes in the 1 Y Polish interest rates are significantly influenced by changes in the level of current interest rates in the USA and the euro zone. Impact of the euro interest rates is stronger

than that of the USA. It appears that the increase of the USA interest rate by 1% generates an average increase of the Polish interest rate by 0.1275%, while the same interest rate increase in the euro zone generates an average increase of the interest rate in Poland by 0.2270%.

Table 5. Interest rates transmission models

Independent variable	c	d(yt-1)	d(yt-2)	d(yt-3)	d(yt-4)	d(yt-5)	d(xt)	d(xt-1)	d(xt-2)	d(xt-3)	d(xt-4)	d(xt-5)	ECMt-1	R2	DW
Dependent variable: d(POLAND 1Y)															
d(USA 1Y)	-0.0033	-0.1684	0.0958	0.1155	0.1308		0.1275						-0.0244	0.1172	2.0123
	0.0267	0.0019	0.0775	0.0331	0.0138		0.0001						0.0419		
d(EURO 1Y)	-0.0033	-0.1780	0.0877	0.1051	0.1471		0.2270						-0.0206	0.1084	2.0120
	0.0268	0.0012	0.1094	0.0538	0.0058		0.0004						0.0886		
Dependent variable: d(POLAND 1M)															
d(USA 1M)	-0.0057	-0.2415	-0.1160		0.0947		0.0982	-0.1011		0.2092			-0.0201	0.1235	1.9981
	0.0002	0.0000	0.0356		0.0727		0.1026	0.0903		0.0002			0.1535		
d(EURO 1M)	-0.0056	-0.2236	-0.1464		0.0917		0.2044	-0.2930	0.2020			0.2555	-0.0332	0.1427	2.0000
	0.0002	0.0000	0.0067		0.0829		0.0512	0.0081	0.0541			0.0094	0.0040		
Dependent variable: d(POLAND 1W)															
d(USA 1W)	-0.0086	-0.4719	-0.4328				0.2309						-0.0664	0.3222	1.9946
	0.0033	0.0000	0.0000				0.0006						0.0227		
d(EURO 1W)	-0.0084	-0.4633	-0.4002				0.1393	-0.1690		0.2226			-0.0719	0.3222	2.0156
	0.0041	0.0000	0.0000				0.1983	0.1195		0.0391			0.0023		

Explanations: the first figure in a box denotes the value of model's parameters, while the next a significance of this parameter. Source: own calculations

Behavior of the 1-month interest rates is quite similar. Here also, the strongest influence on the short term interest rate fluctuations is exerted by the euro zone interest rates rather than by those in the USA. 1 per cent increase of the USA interest rate results in an average of 0.0982 per cent increase in the Polish interest rate, while a similar increase in the euro zone brings about in Poland an increase by 0.2044 per cent.

A reverse relationship has been observed with regard to the 1-Week interest rates. In this case, the impact of the USA interest rates on the short-term changes of interest rates in Poland is stronger that of the euro zone interest rates 1 per cent increase of the USA interest rate results in an average of 0.2309 per cent increase in the Polish interest rate, while a similar increase in the euro zone generates in Poland an increase by 0.1393 per cent.

The process of attaining the long-term equilibrium between the interest rates in Poland and those in the USA and the euro zone is a slow one, as evidenced by very low values of the ECM parameters. The slowest rate is observed for the 1 year and 1 month interest rates and only slightly stronger for the 1 week rates of interest.

CONCLUDING REMARKS

Our studies indicate a strong decrease in the differences between the Polish interest rates and the interest rates in the USA and the euro zone. Thus, the trend observed earlier in other open economies is noticeable now also in Poland.

Cointegration analyses show the existence of long-term linkages between the domestic and foreign interest rates in particular with those of the euro zone. The nature of cointegrating relations was different in the period 2001-2004, as compared with the years after 2004. It may be thus assumed that such a change in linkages was influenced by the Polish accession to the EU.

The transmission models show an increase of the Polish interest rates as a reaction to the interest rates growth abroad, although such a response has not been strong. Also, the very process of reaching the long-term equilibrium between the Polish and foreign interest rates has been slow. In the light of the above it would seem advisable to extend the interest rates quote interval from one week to one month.

REFERENCES

Beenstock M., Longbotom J. (1981) The term structure of interest rates in a small open economy. Journal of Money. Credit and Banking. 13.

Ewing B.T., Payne J.E., Forbes S.M. (1998) Co-movements of the prime rate. cd rate. and the S&P financial stock index. Journal of Financial Research. 21.

Kleimeier S., Sander H. (2000) Regionalisation versus globalisation in European financial market integration: evidence from cointegration analyses. Journal of Banking and Finance. 24.

Mishkin F. (1984) Are real interest rates equal across countries? An empirical investigation of international parity conditions. Journal of Finance. 39.

Moosa I.A., Bhatti R.H. (1994) Testing the effectiveness of arbitrage and speculation under flexibile exchange rates. Economica Internazionale. 47.

Quiro-Romero G., Sosvilla-Rivero S. (1997) Do short-term rates influence long-term interest rates? Empirical evidence from some EMS countries. Applied Economics Letters. 4.

Sasaki H., Yamaguchi S., Takamasa H. (2000) The globalisation of financial markets and monetary policy. BIS Conference Papers no.8.

Waller E., Kiymaz H. (2004) Cointegration between international short-term interest rates. Journal of Accounting and Finance Research. 6.

ON APPLICATION OF NEWTON'S METHOD
TO SOLVE OPTIMIZATION PROBLEMS
IN THE CONSUMER THEORY. EXPANSION'S PATHS
AND ENGEL CURVES

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Abstract: The paper continues investigations concerned with numerical modeling of the neoclassical theory of consumption begun in our previous paper [Strasburger et al. (2009)] This time we are interested in expansion's paths and Engel curves for separable utility functions.. The used numerical procedure is based on Newton's approximation method, implemented in the MATLAB environment. Based on the prior determination of the approximate solutions of the optimum problem we determine in this paper the expansion paths and Engel curves and compare them with their standard form studied in econometrics.

Key words: neoclassical theory of consumption, numerical approximations, multidimensional Newton method, expansion path, Engel curve

INTRODUCTION

The first mention of "Engel Laws", whose formulation preceded introduction of "Engel curves", dates back to the year 1857. Thus had started an important research line of what later would be called "Econometrics". This line in short can be described as a study of dependence of consumer demand on his/her income (expenditures). The subsequent development of those studies was split into two parallel directions. The first one was an empirical determination of the shape

of what was assumed to be the functional dependence of the quantity of a particular good on total expenditures − the price of that good considered constant. The resulting curve has became to be known as Engel curve. In the second one researchers became interested in a theoretical determination of this functional relation based on assumptions regarding the formation of consumer demand and its dependence on the total income (expenditures). It is with the latter line that our paper is concerned.

In our previous paper, *On application of Newton's method to solve optimization problems in the consumer and production theories,* [Strasburger, Zembrzuski (2009)] hereafter referred to as **I,** we have dealt with the problem of determining the demand functions by means of a numerical procedure based on the Newton's approximation method. Here, extending that study further, we show that by the same approximation procedure we can derive a tractable numerical approximation of expansion paths and Engel curves from the assumed form of the utility function, and then by using the Ordinary Least Squares (OLS) method we can determine resulting functional form of those curves.

We now briefly present indispensable notions and results from the classical theory of consumer demand underlying this note and summarize essentials of the approximation method elaborated in **I**.

RESUME OF THE CONSUMER THEORY

SETTING

Following the standard microeconomical approach to the consumer theory we assume that the consumer preferences may be described by means of a utility function $u : x \in X \rightarrow u(x) \in R$ defined on the consumption set X, taken for simplicity to be the nonnegative orthant of the Euclidean space R^k of k dimensions. Here k is the number of commodities considered for the problem, while vectors $x = (x_1, \ldots, x_k)$ represent commodity bundles in such a way that the quantity of an l-th commodity is given by the number x_l, $l = 1, \ldots, k$. We let $p = (p_1, \ldots, p_k) \in R_+^k$ to denote a price vector, whose components p_l have the meaning of the amount paid in exchange for one unit of the l-th commodity. Thus the inner product $\langle x, p \rangle = \sum_{i=1}^{k} x_i p_i$ is the value of a commodity bundle x with respect to the given price system p. Further, we denote by I the total income (wealth) at the consumer 's disposal, so that the set by $D(p, I) = \{ x \in X \mid \langle x, p \rangle \leq I \}$ is the budget set of the consumer. It represents the

set of all consumption bundles available to the consumer. In the case $X = R_+^k$ the budget set is the solid compact k-dimensional simplex in R_+^k.

MARSHALIAN DEMAND FUNCTION

This fundamental object of the consumer theory is obtained as the solution of the problem of maximization of the utility function within the budget set.

Roughly speaking, the solution of the following maximization problem

$$\max\{u(x) \mid x \in X\}$$

under conditions   (1)

$$x_i \geq 0, \quad \text{for } i = 1, \ldots, k, \quad \text{and} \quad \langle x, p \rangle = \sum_{i=1}^{k} x_i p_i \leq I$$

regarded in its relation to prices p and income I constitutes **the demand correspondence**.

Under suitable assumption on the utility function (monotonicity and strict quasi-concavity) and positivity of the price vector p it can be proved that the problem is solved in a unique way leading to the demand function rather than the demand correspondence in a manner described by the following theorem (cf. Barten and Böhm in [Arrow, Intriligator 1982, p. 409]).

Theorem 1 *Given a positive price vector p and a positive wealth I the maximization problem above is uniquely solved by a certain vector $x^0 \in D(p, I)$ with positive coordinates, so that:*

There exists a unique positive constant (Lagrange multiplier) λ^0 such that the vector x^0 is the unique solution of the system of equations

$$\frac{\partial u}{\partial x_i}(x) = \lambda^0 p_i, \quad \text{for } i = 1, \ldots, k,$$

$$\langle x, p \rangle = I,$$ (2)

with $k+1$ unknowns $x_1, \ldots, x_k, \lambda$.

Unfortunately, this system of equations is in general highly nonlinear and therefore only in rare cases its solution can be given explicitly. In order to apply our approximation procedure we recast the system into vectorial formulation.

Let us denote the equations of the system (2) by $\phi_i(x_i, \lambda, p_i, I) = 0$, where $i = 1, \ldots, k+1$, and the functions ϕ_i are given by

$$\phi_i(x, \lambda, p, I) = \frac{\partial u}{\partial x_i}(x) - \lambda p_i, \quad \text{for } i = 1, \ldots, k,$$

$$\phi_{k+1}(x, \lambda, p, I) = I - \langle x, p \rangle.$$ (3)

Introducing the vector function $\Phi : (x, \lambda, q, I) \in R_+^{k+1} \times R_+^{k+1} \rightarrow$ $\rightarrow \Phi(x, \lambda, q, I) \in R^{k+1}$ defined by

$$\Phi(x, \lambda, p, I) = \begin{pmatrix} \phi_1(x, \lambda, p, I) \\ \vdots \\ \phi_{k+1}(x, \lambda, p, I) \end{pmatrix} \tag{4}$$

enables us to write down the equations (2) in a form of one vector equation

$$\Phi(x, \lambda, p, I) = 0, \tag{5}$$

which is to be solved for the unknown (x, λ) for any given wealth I and the price system p.

Although the above theorem guarantees existence of a unique solution for any data (price system and wealth) so that we obtain an implicitly defined function $(x, \lambda) = Q(p, I)$, it says nothing about the functional form of this solution. Apart from the Lagrange multiplier λ this function represents the Marshalian Demand Function written in the vector form as $x = Q(p, I)$. Fixing the prices and varying the income I only, this function describes what is called an Expansion Curve []. Referring to a single commodity one at the time we have the demand functions for individual goods

$$x_i = Q_i(p, I), \quad i = 1, \ldots, k.$$

If the prices are absorbed into the functional form, these functions take up the form

$$x_i = q^*_i(I), \quad i = 1, \ldots, k, \tag{6}$$

which is regarded as the conventional form describing *the Engel curves,* [Deaton, Muellbauer, 1980]

THE NEWTON'S APPROXIMATION METHOD

In the formulation given above this clearly is an instance of the "Implicit Function Problem" in its general form, so we can employ the machinery of the multivariable approximation methods for obtaining and studying its solution, as indicated e.g. in [Krantz, Parks 2002]. We would like to point out that there were attempts to use approximation methods to describe the solution to the optimization problem (1), cf. e.g. [Panek E., (ed.) (2001), pp], but in our opinion they can hardly be considered satisfactory. More satisfactory method to obtain this goal, which uses the multivariable Newton's Method has been described in our preceding paper **I**, and we present briefly its basic results here.

If in the equation (5) we consider (p, I) fixed and absorb it in the functional form, the question reduces to a solution of an equation of the form $\Phi(x, \lambda) = 0$.

Denoting by $\Phi'\left(x\,,\lambda\,\right)^{-1}$ the inverse of the full derivative (Jacobian matrix) of $\Phi(x,\lambda)$, we define the recursive sequence

$$\left(x^{(n+1)},\,\lambda^{(n+1)}\right)=\left(x^{(n)},\,\lambda^{(n)}\right)-\Phi'\left(x^{(n)},\,\lambda^{(n)}\right)^{-1}\cdot\Phi\left(x^{(n)},\,\lambda^{(n)}\right) \tag{7}$$

with an initial value $\left(x^{(0)},\,\lambda^{(0)}\right)$ suitably chosen. It can be shown in general that for a large class of functions this sequence is convergent and that the limit is a solution of (5) (see e.g. [Fortuna et al (2005)]).

NUMERICAL CALCULATIONS

ALGORITHM

In **I** this method was tested for some choice of the utility functions, depending on a varied number of variables (varied number of commodities) and certain range of parameters (p,I). To be more specific, we assumed the utility functions to be separable, that is given by the following formula,

$$u(x_1,x_2,\ldots,x_k)=x_1^{a_1}+x_2^{a_2}+\ldots+x_k^{a_k},\quad\text{where }0<a_l<1. \tag{8}$$

It is well known that those functions satisfy the above formulated assumptions, in particular the maximization problem is uniquely soluble.

The calculations were performed for a few exemplary functions with the number of commodities ranging from $k=2$ to $k=20$.

The discussed algorithm starts from some arbitrary initial values $\left(x^{(0)},\,\lambda^{(0)}\right)$ and in following steps (see eq. 7) it seeks the optimal solution. We have found that the algorithm is numerically stable and the vector $x^{(n)}=\left(x_1^{(n)},x_2^{(n)},\ldots,x_k^{(n)}\right)$ converges to the optimal solution very quickly, just in several steps of iterations in the presented examples. The results do not depend on the assumed initial values, which may or may not satisfy the budget limitations $\langle x,p\rangle\le I$.

CHOICE OF UTILITY FUNCTION

In the present paper we extend previous considerations by studying dependence of the solution of the optimization problem (1) on the income of the consumer. This gives us the so called expansion paths and Engel curves, primarily depending on the utility function used in our optimization problem. We look at the separable utility function (8) with two commodities only, $k=2$. For the sake of the numerical calculations we assume:

$$u(x_1,x_2)=x_1^{0,2}+x_2^{0,5}, \tag{9}$$

and choose the prices as $p_1 = 10$ (*numeraire*) and $p_2 = 2, 5, 10$. The budget bound has the form $x_1 p_1 + x_2 p_2 \leq I$, where various values of the income I are taken into consideration.

RESULTS

The numerical calculations were performed with the use of a program written in the Matlab language (see e.g. [Pratap (2006), Zalewski, Cegieła (2002)]).

Applying the Newton's method we solve the optimization problem and determine the demand by computing values x_1 and x_2 any given value of the income I. Varying subsequently the value of I with fixed p_1 and p_2 we determine the expansion paths in this model, i.e. graphs of the demand (x_1, x_2) as a function of I, which show how the income is distributed between two commodities. The results are shown in Fig. 1 where different curves correspond to different values of $p_2 = 2, 5,$ and 10 and $p_1 = 10$ kept fixed. The shape of so obtained curves depends on the assumed form of the utility function (9). In particular eliminating the parameter I we may express the expansion path in the functional form $x_1 = x_1(x_2)$, what indicates that the functions $x_1 = x_1(x_2)$ are concave and also that the second commodity is more desirable than the first one, since $x_2 > x_1$.[1] This of course is due to the fact that in the expression for the utility function the exponent of x_1 is smaller than that of x_2.

In Fig. 1 we also show points corresponding to some chosen values of the wealth, namely $I = 100$, 500 and 1000. In general, the plots confirm that the demand is an increasing function of the income, as expected from theoretical considerations. Note also, that we have tested the presented method of calculations for various input data and we have found that it is numericaly stable in a wide range of the income from less than $I = 10^{-3}$ up to more than $I = 10^9$ (not shown).

The numerical results are shown once again in Fig. 2, this time in the form of Engel's curves, i.e. x_1 and x_2 are plotted separately as the functions of the income I in the range $0 \leq I \leq 1000$. Basing on the shapes of the plots, one can draw some qualitative conclusions regarding the model. For example, the plots indicate

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[1] Strictly speaking the values of x_2 are higher than the values of $x_1 p_1 / p_2$ for $x_1 > t/p_1$ and $x_2 > t/p_2$, where t is the solution of the equation $\frac{d}{dt}\left(\frac{t}{p_1}\right)^{0,2} = \frac{d}{dt}\left(\frac{t}{p_2}\right)^{0,5}$. Taking for instance $p_1 = p_2 = 10$ we obtain $t \approx 0,5$, so x_2 is higher than x_1 for $x_1, x_2 > 0,05$.

that the demand x_2 depends on I almost linearly while $x_1 = x_1(I)$ is a concave function and grows slower than x_2. Moreover the distribution of the expenditures between two goods depends on the assumed values of prices. For the price of the second commodity equal to $p_2 = 10$ (dotted lines in Fig. 2) the demand x_2 is obviously greater and x_1 is lower than for lower prices $p_2 = 5$ (dashed lines) and $p_2 = 2$ (solid lines).

Our analysis of Engel curves is purely theoretical and the results presented here are just a sample of what could be obtained from numerical solutions of the optimization problem. They are derived basing on a specific choice of a utility function (9), but a similar procedure is possible for other choices as well.

On the other hand, the empirical approach to the demand theory had produced a large number of model curves containing free parameters, which were used to describe observed patterns of demand for specific commodities, see e.g. [Tomaszewicz, Welfe (1972), Dudek (2011), Stanisz (1993)]. We list below a few examples — many more can be found in the quoted sources.

$$x(I) = \alpha_1 + \alpha_2 \cdot I , \qquad (10)$$

$$x(I) = \alpha_1 + \alpha_2 \cdot \ln I , \qquad (11)$$

$$x(I) = \alpha_1 + \alpha_2 / I , \qquad (12)$$

$$x(I) = \exp\left(\alpha_1 + \alpha_2 \ln I + \alpha_3 \ln^2 I\right). \qquad (13)$$

The parameters α_i are usually fitted through statistical analysis of the data.

Our goal now is to determine a functional form of the numerical results presented in Fig. 2. It seems to be especially interesting to know if the results of numerical simulations can be described with use of the same analytical formulae as in analyses based on empirical data for real commodities, so in the following discussion we consider the functions given by eqs. (10-13). The parameters α_i are determined with the OLS method. For simplicity we assume $p_1 = p_2 = 10$.

The results are shown in Figs. 3-6. The numerical solutions of the optimization problem are plotted with dotted lines. The points used in OLS method to determine α_i parameters are marked with circles. The evaluated parameters have in general different values for the first (Figs. 3a, 4a, 5a, 6a) and the second commodity (Figs. 3b, 4b, 5b, 6b). The solid lines are plots of the considered functions (eqs. 10-13) with the values of the parameters α_i determined.

First we consider the linear function (10). Since the dependency x_2 versus I obtained from the optimization problem is almost linear, the parameters can be chosen in such a way, that there are no visible difference between the dotted and solid line (Fig. 3b). On the other hand, the linear function can not describe properly

the dependence of x_1 on I, which is nonlinear (Fig. 3a). We note that the linear function (10) would be the proper model for both x_1 and x_2 if the two exponents in the utility function were equal: $u(x_1, x_2) = x_1^a + x_2^a$.

The best possible fits of the functions (11) and (12) are presented in Figs. 5 and 6, respectively. One can see, that none of those functions describe properly the behavior of either x_1 or x_2,

Finally we take the last of the considered functions given by the formula (13). Using the OLS method we obtain $\alpha_1 = -2,935$, $\alpha_2 = 0,708$, $\alpha_3 = -0,005$ for the first commodity (Fig. 6a) and $\alpha_1 = -2,863$, $\alpha_2 = 1,133$, $\alpha_3 = -0,009$ for the second one (Fig. 6b). In both cases the analytical functions (solid lines) agree very well with the numerical results (dotted lines). We can conclude that a proper functional form of our numerical results has been determined and it is given by the equations:

$$x_1(I) = \exp\left(-2,935 + 0,708 \cdot \ln I - 0,005 \cdot \ln^2 I\right), \tag{14}$$

$$x_2(I) = \exp\left(-2,863 + 1,133 \cdot \ln I - 0,009 \cdot \ln^2 I\right). \tag{15}$$

The obtained functions correspond to the utility function assumed in the separable form (9). However the utility functions are purely a theoretical concept and they can not be measured or determined from empirical data. Nevertheless it seems possible to obtain some conclusions about these functions from studies of empirical Engel's curves. If an empirical dependency between the demand and income for some real commodities is well described by the functions (11) or (12) then it means, that the consumers' preferences can not be described properly by the utility function in the form (9), since this formula leads to a different dependency between the demand and the wealth. On the other hand, if the empirical Engel's curves have an analytical form given by eq. (13), in such a case one can not exclude the utility function in separable form, which was assumed in our paper (9).

Figure 1. The expansion paths obtained for the prices p_1=10 and p_2=2 (solid), p_2=5 (dashed), p_2=10 (dotted line). The points corresponding to chosen values of the income, I=100 (circles), I=500 (triangles) and I=1000 (squares), are shown.
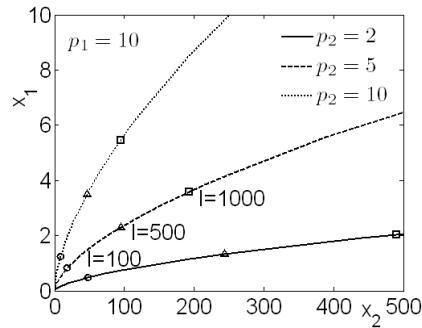
Source: own preparation

Figure 2. The demand x_1 (a) and x_2 (b) presented as a functions of the income I. The prices p_1=10, p_2=2 (solid), p_2=5 (dashed) and p_2=10 (dotted line) are assumed.
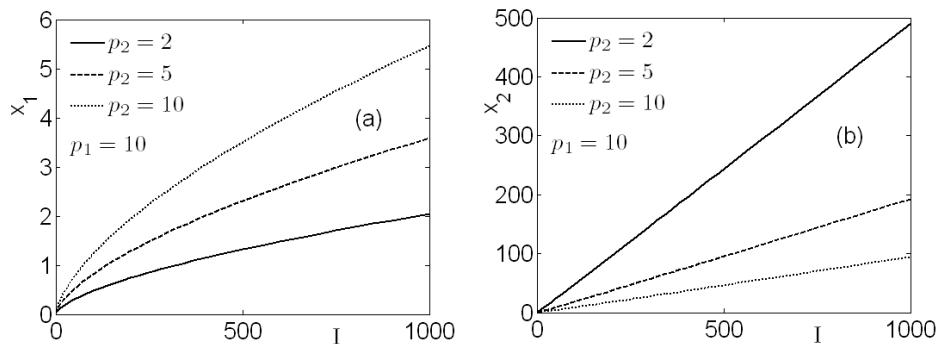

Source: own preparation

Figure 3. The demand x_1 (a) and x_2 (b) shown as a functions of the income I. The dotted lines correspond to the solutions of the optimization problem with $p_1=p_2=10$, while the solid ones are plots of the linear functions $x = \alpha_1 + \alpha_2 \cdot I$ with the parameters α_i obtained using the OLS method.
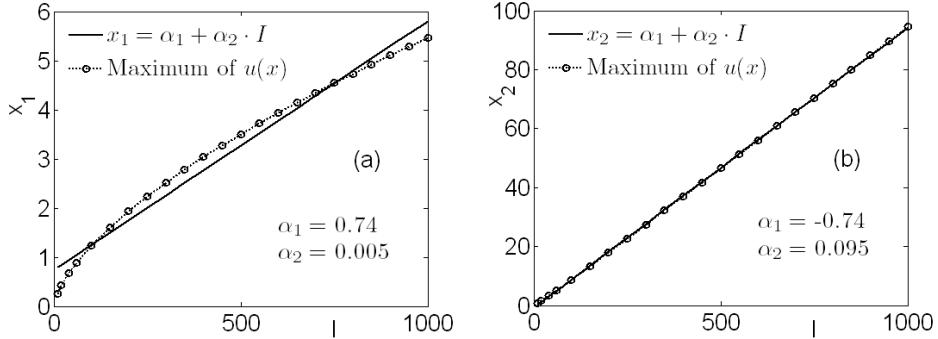

Source: own preparation

Figure 4. The same as in Fig. 3 for the function $x = \alpha_1 + \alpha_2 \ln I$ (solid line).
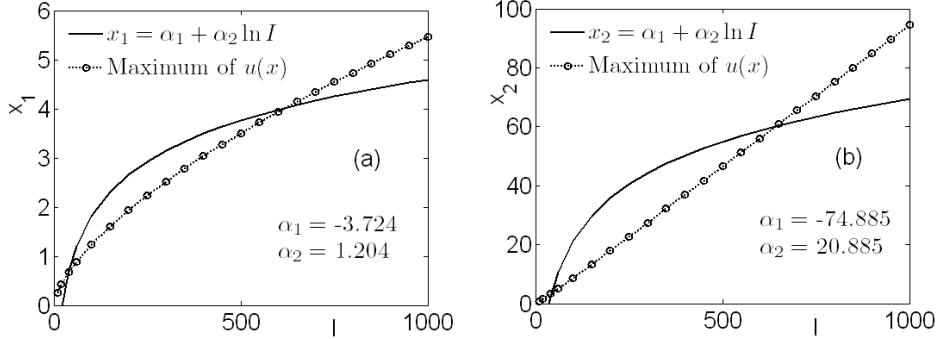
Source: own preparation

Figure 5. The same as in Fig. 3 for the function $x = \alpha_1 + \alpha_2/I$ (solid line).
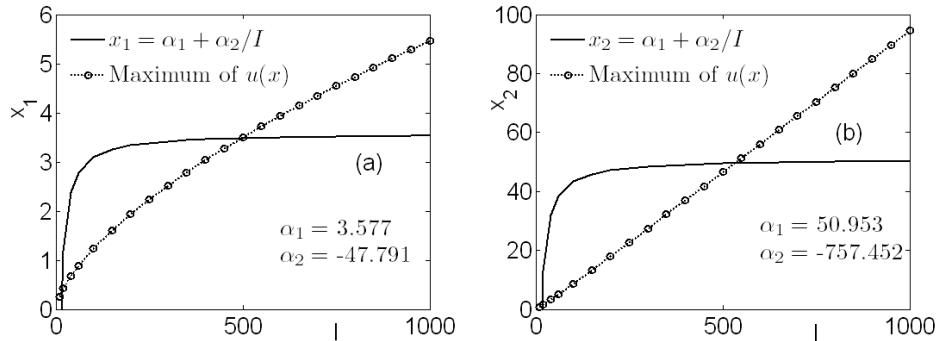
Source: own preparation

Figure 6. The same as in Fig. 3 for the function $x = \exp(\alpha_1 + \alpha_2 \ln I + \alpha_3 \ln^2 I)$ (solid line).
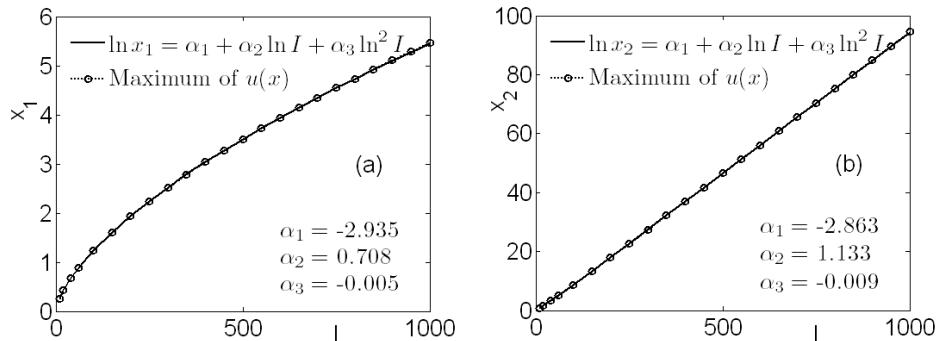
Source: own preparation

REFERENCES

Arrow K. J., Intriligator M. D., ed. (1982) Handbook of Mathematical Economics, North-Holland, Vol. II.

Deaton, A., Muellbauer, J., (1980) Economics and consumer behavior, Cambridge University Press, Cambridge.

Dudek H., (2011) Skale ekwiwalentności − estymacja na podstawie kompletnych modeli popytu, (in Polish) Wydawnictwo SGGW, Warszawa.

Fortuna Z., Macukow B., Wąsowski J. (2005) Metody numeryczne (in Polish), Wydawnictwo Naukowo-Techniczne, Warszawa.

Krantz S. G., Parks H. R. (2002) The Implicit Function Theorem: History, Theory, and Applications, Birkhäuser, Boston.

Panek E. (2000) Ekonomia matematyczna (in Polish), Akademia Ekonomiczna w Poznaniu, Poznań.

Panek E. (red.) (2001) Podstawy ekonomii matematycznej, Materiały do ćwiczeń, Wyd. trzecie, Akademia Ekonomiczna w Poznaniu, Poznań.

Pratap R. (2006) Getting started with Matlab 7, Oxford University Press Inc, Oxford.

Stanisz T.(1993) Funkcje jednej zmiennej w badaniach ekonomicznych, PWN, Warszawa.

Strasburger, A, Zembrzuski, A (2009) Polish J. of Environmental Studies, 2009, Vol. 18, 5B, pp. 198– 202.

Tomaszewicz A., Welfe W. (1972) Studia Prawno-Ekonomiczne, Vol. IX, pp. 193-313.

Zalewski A., Cegieła R. (2002) Matlab - obliczenia numeryczne i ich zastosowania, (in Polish), Wydawnictwo NAKOM, Poznań.

COINTEGRATION SINCE GRANGER:
EVOLUTION AND DEVELOPMENT

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Abstract: This paper is an attempt to give a subjective overview of evolution and development of cointegration concept since the first paper by C.W.J. Granger in 1991, Johansen's reduced rank method of 1987 and Engle and Granger 1987 paper. Various generalizations are rather diversified and find many applications in macroeconomics and financial econometrics. After 30 years the concept is still quite important in theory and in applied work.

Key words: cointegration, fractional cointegration, nonstationarity, Engle-Granger metod; Johansen method; seasonal cointegration; nonlinear cointegration

INTRODUCTION

The aim of this paper is to present a (subjective) overview of evolution of cointegration[1]. The reason of this is double anniversary of two important papers – one by C.W.J. Granger, published in 1981 and introducing a concept of cointegration, the other by S. Johansen, published in 1991 and introducing now well-known reduced rank method. Those and first historical method for cointegration estimation and testing [Engle and Granger 1987], are now the usual scope of general lectures on econometrics.

But the cointegration concept in 30 years since the seminal paper by Granger underwent an impressive evolution. Many a new method and concepts have evolved, such as stochastic and deterministic integration and cointegration, polynomial cointegration, with time-varying parameters, for fractional integrated

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and cointegrated time series, in nonlinear framework, hidden cointegration [Granger and Yoon 2002], Bayesian version[2] of those concepts, and so on and so forth.

GRANGER AND COINTEGRATION IN ECONOMETRICS

[Gonzalo 2010, p.174] starts his paper reminiscing about earlier work on cointegration with words: „The eighties were very good for music as well as econometrics. In time-series econometrics, the first half of that decade was dominated by research on unit roots while cointegration was the queen of the second half.

Indeed it seems so. [Granger 1981] is still widely quoted nowadays as the first published paper introducing the concept of cointegration. In his Nobel lecture[3] Granger emphasized connection between error correction model ECM (by Sargan) and cointegration analysis: "I am often asked how the idea of cointegration came about; was it the result of logical deduction or a flash of inspiration? In fact, it was rather more prosaic. A colleague, David Hendry, stated that the difference between a pair of integrated series could be stationary. My response was that it could be proved that he was wrong, but in attempting to do so, I showed that he was correct, and generalized it to cointegration, and proved the consequences such as the error-correction representation."

The next famous publication [Engle Granger 1987] have had a story of its own, which has been described by Granger himself in the following way[4]. The first version of the paper, submitted by Granger to *Econometrica*, was rejected for several reasons: lack of empirical application, need of rewriting proof of theorem, etc. Granger then started to work on improved version of his Representation Theorem, and accepted help of Robert Engle in empirical work. New version "first became Granger and Engle, next Engle and Granger" during his leave on a sabbatical. Second version was again rejected by *Econometrica* so they contemplated sending it to other publishers when *Econometrica* asked them to publish it because "they get so many papers on cointegration that they needed this one for a reference".

Soon after the first Granger and Engle papers appeared applications of the concept, e.g. [Bossaerts 1988] to stock prices, which seems to be quite natural field of applications. [Stock Watson 1988] described a variant of cointegration testing.

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[2] For Bayesian approach to integration and cointegration see e.g. [Koop et al. 1997] and papers listed on Prof. Koop web page http://personal.strath.ac.uk/gary.koop/research.htm. For multivariate cointegration applications in economics see e.g. [Majsterek 2008].
[3] http://ideas.repec.org/p/ris/nobelp/2003_007.html
[4] See Granger, Clive W.J. (2010) Some thoughts on the development of cointegration, Journal of Econometrics, 158(1), pp. 3–6.

Next decade, 1990's, also seems to be not so bad for the concept. [Johansen 1991] gave a strong impulse for further development, with his reduced rank method and cointegration tests which give all elements of the cointegration space in multivariate case. There are several important papers by Katarina Juselius and Søren Johansen, e.g. [Juselius Johansen 1990, 1992], with application to long-run equilibrium relationships of purchasing power parity and uncovered interest parity. Such parities are often a subject of cointegration tests – see e.g. [Corbae, Ouliaris 1988][5]; or [Cheung Lai 1993] test of PPP with use of fractional cointegration. There were also other methods of cointegration testing. [Bewley and Yang 1995] introduced a method a bit similar to Johansen's but perhaps less known.

Those methods were usually applied to series such as annual values of macroeconomic variables or daily quotes of financial instruments. But quite early cointegration methods for seasonal variables were introduced – e.g. [Hylleberg, Engle, Granger, Yoo 1990].

Start of the new century was marked by a Prize in Economic Sciences in memory of Alfred Nobel, awarded to R.F.Engle and C.W.J. Granger by the Royal Swedish Academy in 2003. In his Nobel lecture Granger gave an interesting overview of concept of cointegration and other topics of time series econometrics[6].

COINTEGRATION – FORMAL DEFINITIONS

We shall remind several important definitions and methods. First of all, we need concept of stationarity and nonstationarity of a series. Time series is stationary if its expected value and variance are constant in time, and covariance

$$C_\tau = \frac{1}{n-\tau} \sum_{t=1}^{n-\tau} (x_t - \bar{x})(x_{t-\tau} - \bar{x})$$

depends only on τ.

Engle and Granger method

Definition of cointegration [Engle, Granger 1987] is the following. Assume that x, y are nonstationary, but integrated, i.e. with stationary first differences. If there is a linear combination which is stationary, then they are cointegrated, and more general: if there is a set of integrated variables $y_t, x_{1t}, x_{2t}, ..., x_{kt} \sim$ I(d), but

$$y_t - \beta_1 x_{1t} - \beta_2 x_{2t} - ... - \beta_k x_{kt} \sim \text{I(d-b)},$$

where $y_t, x_{1t}, x_{2t}, ..., x_{kt}$ – variables of interest, $[1, -\beta_1, -\beta_2 ... -\beta_k]$ – cointegrating vector, we say that they are cointegrated.

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[5] Corbae and Ouliaris are the authors of the COINT addition to the GAUSS programming language.
[6] See http://nobelprize.org/nobel_prizes/economics/laureates/2003/granger-lecture.pdf

Granger has shown[7] that cointegration is equivalent to an error correction mechanism (ECM in short), i.e. that for cointegrated variables one can build the model capturing both short- and long-run features:

$$\Delta y_t = c_0 + c_1 \Delta x_t + \alpha(y_{t-1} - \beta x_{t-1}) + u_t$$

where $e_{t-1} = y_{t-1} - \beta x_{t-1}$ denotes the error from the previous period, $[1, -\beta]$ is a cointegrating vector. The error correction mechanism holds if an estimate of α has negative sign.

The Engle-Granger (1987) method of testing and estimating a cointegration relationship consists of OLS estimation of regression

$$y_t = \beta x_t + u_t$$

and checking whether u_t is stationary (or, in general case, of order of integration lower than the variables). Note that if errors are stationary, then variables are cointegrated. If errors are nonstationary, then we conclude only that the OLS estimates $[1 - \hat{\beta}]$ are not a cointegrating vector. [Maddala and Kim 1998] gave an excellent overview on literature concerning properties of cointegrating vectors:

1) For two variables cointegrating vector is unique; this does not hold for more variables.
2) [Stock 1987] proved that the OLS estimator of β is superconsistent, i.e., converges to β at the rate T instead of $\sqrt{T}$
3) Engle-Granger two-step procedure, i.e. estimating β by OLS, substituting into ECM and estimating ECM by OLS gives the same asymptotic distribution for ECM parameters as if β were known[8].

Triangular system and Phillips-Hansen method

Another way of formulating cointegrating relationship is with triangular system (see [Maddala and Kim 1998]):

$$y_{1t} = \beta' y_{2t} + u_{1t}$$
$$\Delta y_{2t} = u_{2t}$$

where y_{2t} denotes all I(1) variables other than y_{1t}, by assumption have one unit root and are not cointegrated. Vector $u_t = [u_{1t} \quad u_{2t}]^T$ is strictly stationary with zero mean and covariance matrix $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$; $\sum_{t=0}^{n} u_t$ is a multivariate random walk with limiting Wiener process $W(r) = [W_1(r) \quad W_2(r)]^T$. Its covariance

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[7] This is the Granger Representation Theorem.
[8] [Maddala, Kim 1998], p. 156.

matrix is $\Omega = \lim_{T\to\infty}\dfrac{1}{T}\sum_{t=1}^{T}\sum_{t=0}^{n}E(u_t\,u'_s) = \Sigma + \Lambda + \Lambda'$ where $\Sigma = E(u_0\,u'_0)$,

$\Lambda = \sum_{t=1}^{\infty}E(u_0\,u'_t), \Lambda = \sum_{t=1}^{\infty}E(u'_t\,u_0)$.

Denote $\Omega = \begin{bmatrix} \omega_{11} & \omega'_{21} \\ \omega_{21} & \Omega_{22} \end{bmatrix}$ and $\delta = \sum_{t=0}^{\infty}E(u_{1t}\,u_{20})$ then OLS estimator

$\hat{\beta} = (Y'_2\,Y)^{-1}Y'_2\,y_1$ has asymptotic distribution:

$$T(\beta - \hat{\beta}) \Rightarrow (\int_0^1 W_2 W_2')(\int_0^1 W_2\,dW_{1.2} + \omega_{12}\Omega_{22}^{-1}W_2 + \delta)$$

where $W_1, W_2, W_{1.2}$ denote particular Wiener processes.

[Phillips, Hansen 1990] gave a semiparametric estimator[9] with correction for endogeneity:

$$\hat{y}_t^+ = y_{1t} - \hat{\omega}_{12}\hat{\Omega}_{22}\Delta y_{2t}$$
$$\hat{u}_{1t}^+ = u_{1t} - \hat{\omega}_{12}\hat{\Omega}_{22}\Delta y_{2t}$$

and serial correlation correction term $\hat{\delta}^+$ as a consistent estimator of $\sum_{k=0}^{\infty}(u_{1t}^+ u_{21}^+)$.

Hence the fully modified estimator, FM–OLS, by Phillips and Hansen has the following form: $\tilde{\beta} = (Y'_2\,Y_2)^{-1}(Y'_2\,\hat{y}_1^+ - T\hat{\delta}^+)$; $\hat{\Omega}$ is estimated with use of Newey-West estimator.

System estimation – Johansen method

The Johansen reduced rank method[10] is based on maximum likelihood approach applied to VAR model, assuming that errors are Gaussian: in a way similar to ECM, from a VAR model for variables $\mathbf{Y}_t = [Y_{1t}, Y_{2t},...,Y_{mt}]'$:

$$\mathbf{Y}_t = \mathbf{A}_1\mathbf{Y}_{t-1} + ... + \mathbf{A}_k\mathbf{Y}_{t-k} + \mathbf{u}_t, \quad t = 1,2,...,T$$

a VECM model can be built:

$$\Delta\mathbf{Y}_t = \Pi\mathbf{Y}_{t-1} + \mathbf{B}_2\Delta\mathbf{Y}_{t-1} + ... + \mathbf{B}_k\Delta\mathbf{Y}_{t-k+1} + \mathbf{u}_t, \quad t = 1,2,...,T$$

where

$$\Pi = -\mathbf{I} + \sum_{i=1}^{k}\mathbf{A}_i, \quad \mathbf{B}_j = -\sum_{i=j}^{k}\mathbf{A}_i, \quad j = 2,...,k$$

[9] P.C.B.Phillips, B.E.Hansen (1990), Statistical inference in Instrumental Variables Regression with I(1) Process, *Review of Economic Studies*, 57, 99–125.
[10] For description see e.g. [Maddala Kim 1998] or original texts by[Johansen 1991, 1995].

Π has rank $r < k$ (is not a full rank) and can be represented as $\Pi = \alpha\beta'$, where $r = rank(\Pi)$, matrices are $m \times r$, $\beta'\mathbf{Y}_{t-1}$ constitute r cointegrating vectors, α is a matrix of error-correction terms (adjustment matrix). Construction of the Johansen test is as follows. Regress $\Delta\mathbf{Y}_t$ on its lags; $\mathbf{R}_{0t}$ – residuals; regress $\mathbf{Y}_{t-1}$ on $\Delta\mathbf{Y}_{t-1},\ldots,\Delta\mathbf{Y}_{t-k}$; $\mathbf{R}_{1t}$ – residuals: $\mathbf{R}_{0t} = \alpha\beta'\mathbf{R}_{1t} + \mathbf{u}_t$

Let $\begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix}$ – matrix of sums of squares and sums of products of $\mathbf{R}_{0t}$ and $\mathbf{R}_{1t}$.

Then the asymptotic variance of $\mathbf{R}_{0t}$ is Σ_{00}, of $\beta'\mathbf{R}_{1t}$ is $\beta'\Sigma_{11}\beta$; asymptotic covariance of $\beta'\mathbf{R}_{1t}$ and $\mathbf{R}_{0t}$ is $\beta'\Sigma_{10}$. Maximize the likelihood function with respect to α holding β constant, then maximize with respect to β in second step: $\hat{\alpha}' = (\beta'S_{11}\beta)^{-1}\beta'S_{10}$; conditional maximum is $| S_{00} - S_{01}\beta(\beta'S_{11}\beta)^{-1}\beta'S_{10} |$; solve $| S_{10}S_{00}^{-1}S_{01} - \lambda S_{11} | = 0$ or $| S_{10}S_{00}^{-1}S_{01} - \lambda S_{11} |$ for maximum eigenvalue.

The Johansen tests statistics are based on properties of eigenvalues, e.g.

- determinant of the matrix = product of its eigenvalues;
- rank of the matrix = number of non-zero eigenvalues:

$$\Pi_{i=1}^{n}(1 - \lambda_i) = | \mathbf{I} - S_{11}^{-1}S_{10}S_{00}^{-1}S_{01} |, \quad L_{\max}^{-2/T} = | S_{00} | \cdot \Pi_{i=1}^{n}(1 - \lambda_i)$$

If there are r cointegrating vectors, then the m–r smallest eigenvalues are zero. Hence one has to find all eigenvalues, order them by size, then the eigenvectors corresponding to r greatest eigenvalues are cointegrating vectors and form columns of the matrix β.

There is a distinction between stochastic and deterministic cointegration: A vector $\mathbf{Y}_t$ of I(1) variables is said to be stochastically cointegrated with cointegrating rank r, if there are r linearly independent combinations of the variables that are I(0). These combinations may have nonzero deterministic trends.

Variables $\mathbf{Y}_t$ are said to be deterministically cointegrated, with cointegrating rank r, if there are combinations of $\mathbf{Y}_t$ that are I(0) are stationary *without* deterministic trends. The original definition of Engle and Granger excluded deterministic trends.

Interpretation of cointegration is as follows. Cointegrating relationships represent long-term equilibria between nonstationary variables:

If for $y_{1t}, y_{2t}, \ldots, y_{mt}$ there are r cointegrating relationships, then there are m–r integrated time series, $u_{j,t}$ called common trends, such that every series $y_{it} = \sum_{j=1}^{m-r} \gamma_j u_{j,t} + \varepsilon_{it}$, with stationary ε_{it}.

[Johansen 1995], p. 39, notes: "One motivation for the [ECM] model is to consider [cointegrating] relation as defining the underlying economic relations, and assume that the agents react to the disequilibrium error through the adjustment coefficient α, to bring back the variables of the right track, that is, such that they satisfy the economic relations"[11].

"The process is pushed along the attractor set by the common trends and pulled towards it by the adjustment coefficients" (ibidem, p. 41).

Polynomial (dynamic) cointegration

Classical cointegrating relationships are static; polynomial cointegration introduces a small number of lags into a cointegrating relationship: "In other words, cointegration reduces the order of integration by applying linear regressions between variables; dynamic cointegration reduces the order of integration by applying autoregressive modeling. A VAR model with *n* lags

$$\mathbf{Y}_t = \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + ... + \mathbf{A}_n \mathbf{Y}_{t-n} + \varepsilon_t$$

exhibits dynamic cointegration if there exists a stationary autoregressive combination of the variables of the type $\alpha' \mathbf{Y}_t + \Delta \mathbf{Y}_t$" ([Focardi, Fabozzi 2004, p. 541)[12]. [Johansen 1995, p. 39] defines the I(2) process $\mathbf{Y}_t$ as polynomially cointegrated, if there exist β_0, β_1 such that $\beta'_0 \mathbf{Y}_t + \beta'_1 \Delta \mathbf{Y}_t$ are stationary. First term reduces I(2) variables to I(1) linear combination, and second term ensures that in turn its combination with I(1) differences is stationary.

[Focardi, Fabozzi 2004] explains that variables can be cointegrated *and* dynamically cointegrated: e.g., log prices of assets are nonstationary, log returns are stationary – factor models for returns and cointegrating models for prices can coexist; in addition, linear combination of prices and returns can also be stationary.

Cointegration and canonical correlation analysis

[Bossaerts 1998] applied canonical correlation analysis to tests of cointegration: used a model $\Delta \mathbf{Y}_t = \mathbf{HCY}_t + \varepsilon_t$, applied canonical correlation analysis for $\Delta \mathbf{Y}_t$ and $\mathbf{Y}_t$, and checked if the canonical variates are nonstationary. [Bewley and Yang 1995] improved the method, including deterministic trends, developed asymptotic theory, new tests and computed critical values for number of cointegrating vectors. The LCCA (*level canonical correlation analysis*) is similar in spirit to the Johansen method:

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[11] Soren Johansen (1995) *Likelihood-based inference in cointegrated vector autoregressive models*, Oxford University Press, Oxford.
[12] Sergio Focardi, Frank J. Fabozzi (2004) *The mathematics of financial modeling and investment management*, Wiley, Hoboken, New Jersey.

1. eliminate additional variables by regression of $\mathbf{Y}_t$ and $\mathbf{Y}_{t-1}$ on those variables. For residuals $\mathbf{R}_{0t}$, $\mathbf{R}_{1t}$ of these regressions form $\mathbf{R}_{0t} = \mathbf{B}\,\mathbf{R}_{1t} + \mathbf{u}_t$.

2. determine the canonical correlation between $\mathbf{R}_{0t}$, $\mathbf{R}_{1t}$, solve the eigenvalue problem: $|\mathbf{S}_{10}\,\mathbf{S}_{00}^{-1}\mathbf{S}_{01} - \lambda\mathbf{S}_{11}| = 0 \qquad \mathbf{S}_{ij} = \dfrac{\mathbf{R}_i\mathbf{R}_j}{T}, i,j = 1,2$

   Interpretation is different: here canonical correlations are for levels only, in the Johansen method –for levels and differences.

Cointegration and state-space models

Another path of development is formulating cointegration theory for state-space models. According to [Fabozzi et al. 2006], who give an overview of Bauer and Wagner methods[13], D. Bauer and M. Wagner provided systematically developed theory of cointegration in this framework, tests for order and cointegrating rank of state-space models. An extensive description of Bauer and Wagner results can be found in [Wagner 2010][14].

Seasonal cointegration

One of seasonal cointegration tests, known as HEGY test, was introduced by [Hylleberg, Engle, Granger, Yoo 1990]. Unit root, causing nonstationarity of a series, corresponds to a root (of modulus one) of autoregressive polynomial A(L), where $A(L)y_t = c_0 + \varepsilon_t$ is a representation of a series in question. Difference of the series is stationary. For quarterly data, seasonal differencing operator may be necessary to obtain stationarity: this operator (defined with use of lag operator L)

$$\Delta_4 y_t = (1-\mathrm{L}^4)y_t; (1-\mathrm{L}^4) = (1-\mathrm{L})(1+\mathrm{L})(1+\mathrm{L}^2) = (1-\mathrm{L})(1+\mathrm{L})(1+i\mathrm{L})(1-i\mathrm{L})$$

has four roots – one corresponds to zero frequency (a trend), one to two cycles per year, and a pair of complex roots – to one cycle per year. If a pair of series has unit root at frequency ω and their linear combination does not have a unit root at this frequency, they are called cointegrated with frequency ω. [Hylleberg et al. 1990] give tests for seasonal cointegration, provide (Monte Carlo) critical values for seasonal integration and cointegration, and give an example for consumption function of the UK[15].

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[13] For description of state space model estimation, see [Fabozzi et al. 2006], p. 539-544.

[14] Available at http://www.ihs.ac.at/publications/eco/es-248.pdf

[15] For example of seasonal cointegration applied to Polish economy see e.g. Kotłowski, J. *Money and Prices in the Polish Economy, Seasonal Cointegration Approach* (in Polish), Warsaw School of Economics, or shorter version – Working Paper in English: http://ideas.repec.org/p/wse/wpaper/20.html

Fractional cointegration

Distinction between I(1) and I(0) series is an important one, but this is too crude a description for actual economic series. More accurate tool is a fractional differencing, see e.g. [Granger and Joyeux 1980], [Hosking 1981], and fractional integration and cointegration. Fractional difference or fractional filter is defined as:

$$(1\text{-}L)^d = \sum_{k=0}^{\infty}\binom{d}{k}(-L)^k = \sum_{k=0}^{\infty}\frac{\Pi_{a=0}^{k-1}(d-a)(-L)^k}{k!} = 1 - dL + \frac{d(d-1)}{2}L^2 + ...$$

where d denotes a fractional integration parameter, L – the lag operator.

Fractional integrated processes are intermediate between integrated of order 1 and 0. They exhibit long memory, i.e., their autocorrelation function decays very slowly. There are several methods of estimating fractional integration parameters: [Geweke and Porter-Hudak 1993] periodogram regression method, appropriate for $-1/2 < d < 1/2$ processes; generalization by P.C.B. Phillips[16] and [Robinson 1994, 1995]; algorithms using fast Fourier transform or wavelet transform; direct maximum likelihood method by [Sowell 1990, 1992], see extensive paper by [Baillie 1996] and also his later papers.

Fractional cointegration, suggested by [Engle, Granger 1987] and developed by [Cheung, Lai 1993] and [Marinucci, Robinson 2001], deals with linear common long-memory persistence feature (see [Marinucci, Robinson 2001], [Dittmann 2004]). This framework is routinely applied to tests of long-run equilibrium relationships, such as purchasing power or interest rate parities. It can be used as a tool of financial econometrics, e.g. in portfolio building.

Nonlinear cointegration

Nonlinear cointegration requires more sophisticate definitions and multistep procedures compared to the two-step Engle-Granger procedure or even the Johansen's procedure. Definition of nonlinear cointegration is not unique, as well as formulation of error correction model; empirical procedure requires testing several features of variables and of deviations from an equilibrium (i.e., an attractor).

Detailed overview of nonlinear cointegration is [Dufrénot, Mignon 2002]. They comment on unit root and stationarity tests adequacy in nonlinear case, describe several versions of nonlinear series and models, nonlinear measures of persistency, define equilibration, nonlinear cointegration, nonlinear error correction mechanism, and nonlinear cotrending. Last part contains asymmetric and threshold nonlinear cointegration models – those are important for applied work, where adjustment process starts only when the distance from equilibrium is greater than a particular threshold, and also some asymmetries of behaviour may occur.

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[16] See [Phillips 199a,b] and [Shimotsu, Phillips 2002].

An example of nonlinear cointegration and nonlinear error-correction models (NECM) applied for oil and stock markets is research by [Arouri, Jawadi, Nguyen 2010]. They first give an overview of nonlinear cointegration and NECM. To an equilibrium state for linear cointegration here corresponds an attractor representing long-run equilibrium (attractor for nonlinear dynamical systems). The authors present three definitions of nonlinear cointegration, based on concept of a mixing series: long-range dependence corresponds to non-mixing, short-range dependence to mixing.

They also give a description of empirical strategy of a single equation NEC estimation (subsection 9.2.3, pp. 180 – 181). First step is to check integration of series, second: to estimate an equation and test residuals of the long-run attractor for stationarity. Absence of unit root suggests linear cointegration, in this case mixing properties of the series should be checked. Third step consists of testing null of linear cointegration against nonlinear. Four: if linear cointegration is not rejected in step 2 and rejected in step 4, this may mean nonlinearity in the short-run dynamics (p. 181). The fifth step is testing for dynamic nonlinearity and estimation of short-run models. Rejection of linear cointegration in step 3 leads to Step 6: estimate a NECM by nonlinear least squares (NLS), and estimate NECM allowing for nonlinear mean-reversion process. Step 7: check the mixing hypotheses for residuals of the NECM, apply several diagnostic and misspecification tests.

Their example of application is to the relationship between world stock market index and world oil market index, based on monthly data since May 1987 until January 2009. Both indices are not stationary in levels, their differences are stationary. Log returns of the indices seem "to show some linkages between variables" and returns have downward trends and negative values during crisis (p. 183). The Dickey and Fuller test and the Zivot and Andrews test lead to rejection of linear cointegration hypothesis. Next they apply the KPSS test and modified R/S test [Lo 1991], thus show that there is a nonlinear cointegration relationship between oil and equity markets, which is confirmed by [Keenan 1985] and LM3 test [Van Dijk et al. 2002]. They use three variants of NEC model, and show that "lagged oil market returns significantly affect the dynamics of world market returns" (p. 187). Next they build the smooth transition Error Correction Model with logistic transition function and show that adjustment towards long-run equilibrium is activated when a shock (e.g., "the 1987 stock crash, the 1991 first Gulf War, the 2003 second Gulf War and the 2007-09 financial crisis", p. 190) affects one of the markets.

[Bruzda 2006] applies nonlinear cointegration analysis to money demand models. She also uses the KPSS test and the modified R/S analysis to check integration of series, and next (instead of NLS mentioned above) applies the Phillips-Hansen FM-OLS estimator to several versions of the money demand models. The KPSS test and [Breitung 2001] cointegration tests are then applied, checking not only cointegration in levels, but also long memory in information for residual series. Specification of models is based on a general equilibrium model

with maximization of multiperiod utility function of a representative household, using budget constraint. A CES form of utility function leads to the double-logarithmic functional form; if liquidity preferences of agents depend on a monetary regime (levels of inflation and nominal interest rate), then preferred is a semilogarithmic function in which "the interest rate elasticity is an increasing function of the interest rate spread" [Bruzda 2006, p. 115]. Third form, a log-inverse specification, reflects a discontinuity of money demand of a household for certain nominal interest rates. Another three versions of models contain a linear trend in long-run relationship. Results of estimation of the first three models are consistent with economic theory, for models with trend signs are not interpretable, but the parameters are also stable. The Breitung's test is "in favour of the presence of (possibly non-linear) cointegration without the trend component" (p. 120), "the Hansen's test and tests for significance of mutual information coefficients distinguished the semilogarithmic model, while the KPSS test and the modified R/S analysis singled out the double-logarithmic functional form" (p. 121).

An example of nonlinear fractional cointegration with financial application is Andersen, Bollerslev, Diebold and Wu, concerning persistence of realized beta[17]. The authors study stock-systematic risk to check whether it is time-varying ("time varying betas from the CAPM model"). They explain that use of quarterly data did not allow for direct estimation of fractional integration parameters, hence they have adopted estimates from other papers, based on higher frequency data. Their estimates of quarterly betas are based on nonparametric realized quarterly market variances and individual equity covariances with the market.

[Andersen et al. 2006] are interested in nonlinear copersistence function – nonlinearity caused by the fact that the betas are ratios. They are not able to estimate and test fractional integration parameter for a function of betas, due to small number of observations. Instead, they check behaviour of autocorrelation function of fractional differences of betas (for a stationary function, the ACF should decay at higher rate than for fractionally integrated series) and are able to formulate conclusions about financial properties of studied instruments.

IMPORTANCE OF COINTEGRATION

We can visualize the influence of [Granger 1981] and [Johansen 1991] with use of diagrams from the http://ideas.repec.org website, which provides access to and statistics for economic and econometric publications. Both authors (and R. F. Engle) are among 5% most cited economists. [Granger 1981], [Johansen 1991] and [Engle Granger 1987] papers are among 1‰ most cited papers in economics.

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[17] T.G. Andersen, T. Bollerslev, F.X. Diebold, G. Wu (2006) Realized beta: persistence and predictability, in: *Econometric Analysis of Financial and Economic Time Series, Part B. Advances in Econometrics*, Vol. 20, p. 1–39, Elsevier. The paper is dedicated to Clive W.J. Granger.

It seems that the interest in the papers is still growing, with some local maxima[18] (see Fig. 1a,b,c). General interest in the concept of cointegration is reflected in number of Google search for reference in Internet (see Fig. 2), which is growing according to a increase in search diagram.

Figure 1a. Abstract and paper downloads of [Granger 1981]

Source: http://ideas.repec.org
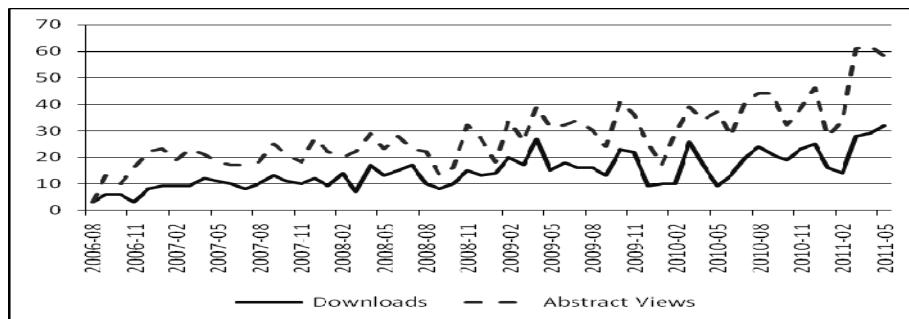
Figure 1b. Abstract and paper downloads of [Engle, Granger 1987]

Source: http://ideas.repec.org

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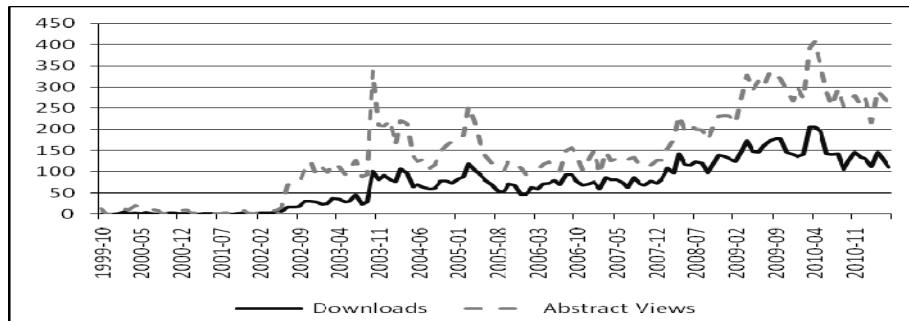
[18] One of them around the timing of Prize in Economics award.

Figure 1c. Abstract and paper downloads of [Johansen 1991]
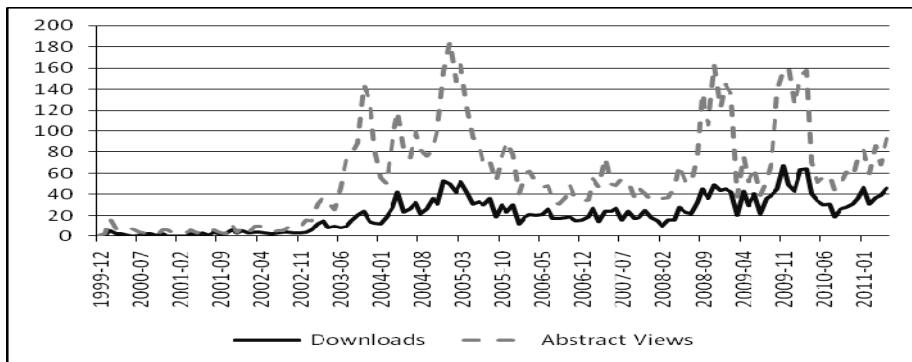
The first diagram in Fig. 2 reflects general search for cointegration in all meaning of the word. The second diagram is more informative about detailed formulations – users search both for methods and for software applications. The last diagram shows increase in number of searches – greatest for the Johansen method, cointegration tests, software applications and for general descriptions.

Figure 2. Google statistics on search for reference on cointegration in Internet

a)   General search number, by week, 2004 – 2011.
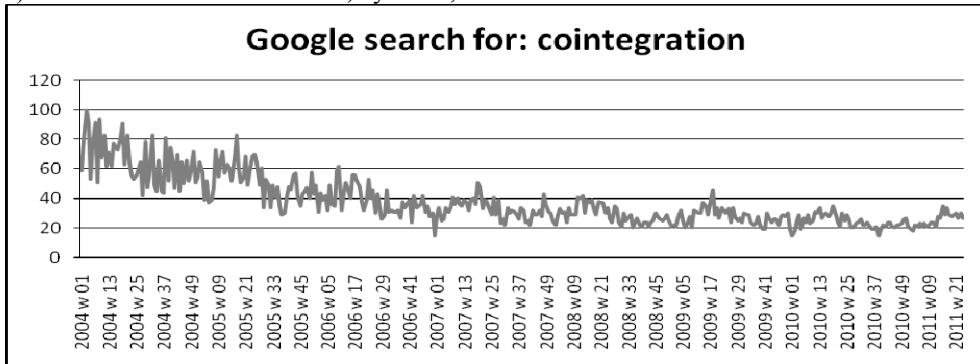
b)   Most popular searches
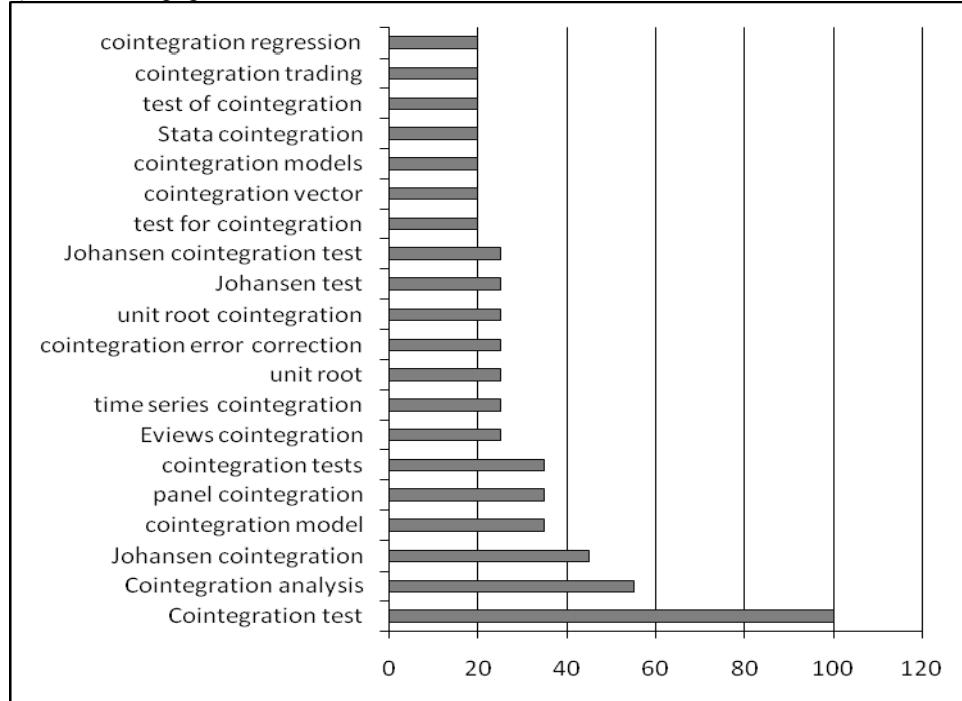
c)   Increase in search numbers
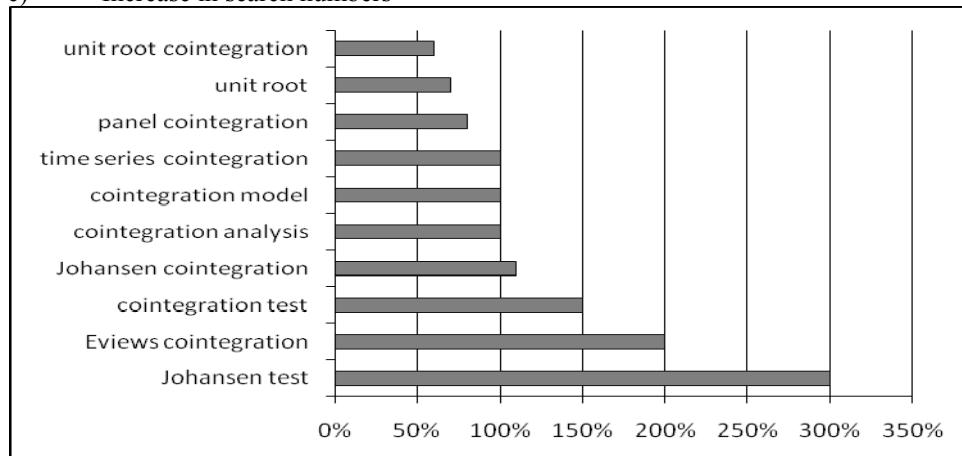
Source: Google Statistics

Let us turn to Johansen and Juselius for comments on importance of cointegration – and of careful approach to its applications. In "Conversation with Søren Johansen and Katarina Juselius", published in [Rosser et al. 2010] and

available at Prof. Juselius web page[19], they talk on history of their cooperation, development of econometric methods, state of economic research and publishing, and give comments on the topic of cointegration:

Søren Johansen: "I think there was a time when all the theoretical econometricians were working on topics related to this methodology and many were applying it. Now many econometricians work on other topics – panel data, financial econometrics, factor models, analysis of large data sets, and so on. But the people who make a living on analyzing the usual monthly or quarterly data sets – they routinely apply cointegration methods, and that is possibly because the programs are there."

Katarina Juselius: "Unfortunately, this is not a method that can be applied routinely using standard cointegration software; it requires interaction between the analyst and the data; it is a powerful tool for an expert to use, not a tool for someone who doesn't understand the methodology. […] A good cointegration analysis is when you structure the information in the data, so that the complexity of the empirical reality can be grasped and better understood. […] Serious data analysis is a long process that requires a very systematic study; continually working with the data, trying different specifications until reaching the point where one can say: now I understand the basic features of the data (statistically as well as economically). This is a baseline model that it makes sense to continue working with. You have to carefully check for misspecifications […] These steps are enormously important."

REFERENCES

Andersen T.G., Bollerslev T., Diebold F.X., Wu G. (2006) Realized beta: persistence and predictability, in: Econometric Analysis of Financial and Economic Time Series, Part B. Advances in Econometrics, 20, pp. 1–39.

Andrews D.W.K., Zivot E. (1992) Further Evidence on the Great Crash, the Oil Price Shock and the Unit Root Hypothesis, Journal of Business and Economic Statistics, 10(3), pp. 251–270.

Arouri M. E. H., Jawadi F., Nguyen D. K. (2011) Nonlinear Cointegration and Nonlinear Error-Correction Models: Theory and Empirical Applications for Oil and Stock Markets, chapter 9 in: [Gregoriou, Pascalau 2011], pp. 171-193.

Baillie R. T. (1996) Long Memory Processes and Fractional Integration in Econometrics, Journal of Econometrics, 73(1), pp. 5–59.

Bailie R. T., Bollerslev T. (1994) Cointegration, Fractional Cointegration, and Exchange Rate Dynamics, The Journal of Finance, 79, 737–745.

Balke N. S., Fomby T.B. (1997) Threshold cointegration, International Economic Review, 38, pp. 627–645.

Bauer D. (2001) Order Estimation for Subspace Methods, Automatica 37, pp. 1561–1573.

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[19] The interview is available on Prof. Juselius web page: http://www.econ.ku.dk/okokj/#bio

Bauer D., Wagner M. A Canonical Form for Unit Root Analysis in the State Space Framework, VWI Diskussionschrift, University of Bern, Switzerland.

Bewley R., Yang M. (1995) Tests for Cointegration Based on Canonical Correlation Analysis, Journal of the American Statistical Association, 90, pp. 990–996.

Bossaerts P. (1988) Common Nonstationary Components in Stock Prices, Journal of Economic Dynamics and Control, 12, pp. 347–364.

Breitung J. (2001) Rank Tests for Nonlinear Cointegration, Journal of Business and Economic Statistics, 19, 331 –340.

Bruzda J. (2006) Empirical Verification of Money Demand Models: Non-linear Cointegration Analysis, Dynamic Econometric Models, Vol. 7, pp. 113–123.

Cheung Y.–W., Lai K.S. (1993) A Fractional Cointegration Analysis of Purchasing Power Parity, Journal of Business and Economic Statistics, 11, pp. 103–112.

Christensen, B.J., M.O.Nielsen (2002) Semiparametric Analysis of Stationary Fractional Cointegration and the Implied-Realized Volatility Relation, Working Paper No. 2001-4, Centre for Dynamic Modelling in Economics, Department of Economics, University of Aarhus, Aarhus, Denmark, ftp://ftp.econ.au.dk/afn/wp/01/wp01_4.pdf

Conversation with Søren Johansen and Katarina Juselius (2010), in: [Rosser et al. 2010], and at: http://www.econ.ku.dk/okokj/papers/

Corbae D., Ouliaris S. (1988) Cointegration and Tests of Purchasing Power Parity, The Review of Economics and Statistics, MIT Press, 70(3), pp. 508–11.

Dittman I. (2004) Error Correction Models for Fractionally Cointegrated Time Series, Journal of Time Series Analysis, Wiley Blackwell, 25(1), pp. 27–32.

Dittman I., Granger C. W. J. (2002) Properties of Nonlinear Transformations of Fractionally Integrated Processes, Journal of Econometrics, 110(2), pp. 113–133.

Dufrénot G., Mignon V. (2002), Recent Developments in Nonlinear Cointegration with Applications to Macroeconomics and Finance, Kluwer Academic Publishers, Dordrecht.

Engle R.F., Granger C.W.J. (1987) Cointegration and error correction mechanism: Representation, Estimation and Testing, Econometrica, 55, pp. 251–276.

Fabozzi F.J., Focardi S.M.,. Kolm P.N (2006) Financial Modeling of the Equity Market: from CAPM to Cointegration, Wiley, Hoboken, New Jersey.

Focardi S., Fabozzi F.J. (2004) The mathematics of financial modeling and investment management, Wiley, Hoboken, New Jersey.

Geweke J., Porter-Hudak S. (1983), The Estimation and Application of Long Memory Time Series Models, Journal of Time Series Analysis, 4, pp. 221–238.

Gonzalo J. (2010) The Making of „Estimation of Common Long-Memory Components in Cointegrated Systems", Journal of Financial Econometrics, Oxford University Press, vol. 8(2), pages 174–176.

Granger C.W.J. (1981) Some Properties of Time Series Data and Their Use in Econometric Model Specification, Journal of Econometrics 16, pp. 121–130.

Granger C.W.J., Joyeux R. (1980), An Introduction to Long-Memory Time Series Models and Fractional Differencing, Journal of Time Series Analysis, 1, pp. 15–29.

Granger C.W.J., Yoon G. (2002) Hidden Cointegration, Economic Working Paper Series, No. 539384, Department of Economics, University of California at San Diego.

Gregoriou G.N., Pascalau R. (ed.) (2011) Nonlinear Financial Econometrics: Markov Switching Models, Persistence and Nonlinear Cointegration, Palgrave Macmillan, Houndmills, Basingstoke.

Hosking J.M.R. (1981), Fractional differencing, Biometrika,68, pp. 165–176.

Hylleberg S., Engle R.F., Granger C.W.J., Yoo S.B. (1990) Seasonal integration and cointegration, Journal of Econometrics, 44, pp. 215–238.

Johansen S. (2011) An Extension of Cointegration to Fractional Autoregressive Processes, CREATES Research Papers 2011–06, School of Economics and Management, University of Aarhus.

Johansen S. (1991) Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian VAR Models, Econometrica, 59, pp. 1551–1580.

Johansen S. (1995) Likelihood-based inference in cointegrated vector autoregressive models, Oxford University Press, Oxford.

Juselius K., Johansen S. (1990) The full information maximum likelihood procedure for inference on cointegration – with applications, Oxford Bulletin of Statistics and Economics, 52(2), pp. 169–211.

Juselius K., Johansen S. (1992) Testing Structural Hypotheses in a Multivariate Cointegration Analysis of the PPP and the UIP for UK, Journal of Econometrics, 53, pp. 211–244.

Keenan D.M. (1985) A Tukey Nonadditivity-Type Test for Time Series Nonlinearity, Biometrika, 72(1), pp. 39–44.

Koop G., Ley E., Osiewalski J. and Steel M.F.J. (1997) Bayesian analysis of long memory and persistence using ARFIMA models, Journal of Econometrics 76, pp. 149-169.

Kotłowski J. Money and Prices in the Polish Economy, Seasonal Cointegration Approach (in Polish), Warsaw School of Economics, or shorter version – Working Paper in English: http://ideas.repec.org/p/wse/wpaper/20.html

Lasak K. (2008) Likelihood Based Testing for No Fractional Cointegration, CREATES Research Papers 2008–52, School of Economics and Management, University of Aarhus.

Lasak K. (2008) Maximum Likelihood Estimation of Fractionally Cointegrated Systems, CREATES Research Paper 2008–53, School of Economics and Management, University of Aarhus.

Lo A.W. (1991) Long Term Memory in Stock Market Prices, Econometrica, 59(5), pp. 1279–1313.

Maddala G.C., In-Moo Kim (1998) Unit Roots, Cointegration, and Structural Change, Cambridge University Press, Cambridge.

Majsterek M. (2008) Wielowymiarowa analiza kointegracyjna w ekonomii (Multivariate cointegration analysis, in Polish), Wydawnictwo Uniwersytetu Łódzkiego, Łódź.

Marinucci D., Robinson P.M. (2001) Semiparametric Fractional Cointegration Analysis, Journal of Econometrics, 105, pp. 225–247.

Phillips P.C.B. (1999a) Discrete Fourier Transforms of Fractional Processes, Cowles Foundation Discussion Paper No. 1243, Cowles Foundation for Research in Economic, Yale University, New Haven, http://cowles.econ.yale.edu/P/cd/d12a/d1243.pdf

Phillips P.C.B. (1999b) Unit Root Log Periodogram Regression, Cowles Foundation Discussion Paper No. 1243, Cowles Foundation for Research in Economic, Yale University, New Haven, http://cowles.econ.yale.edu/P/cd/d12a/d1244.pdf

Phillips P.C.B., B.E. Hansen (1990) Statistical Inference in Instrumental Variables Regression with I(1) Process, Review of Economic Studies, 57, 99–125.

Robinson P.M. (1994) Semiparametric Analysis of Long-memory Time Series, Annals of Statistics, 22, pp. 515–539.

Robinson P.M. (1995) Log-periodogram Regression of Time Series with Long Range Dependence, Annals of Statistics, 23, pp. 1048–1072.

Rosser J.B., Holt R.P.F., Colander D. (eds.) (2010), European Economics at a Crossroads, Edward Elgar, Cheltenham.

Shimotsu K., Phillips P.C.B. (2002) Exact Local Whittle Estimation of Fractional Integration, Cowles Foundation Discussion Paper, 1367.

Sowell F. (1992) Maximum likelihood estimation of stationary univariate fractionally integrated time series models, Journal of Econometrics, 53(103), pp. 165–188.

Sowell F. (1990) The Fractional Unit Root Distribution, Econometrics, 58(2), 495–505.

Stock J.H. (1987) Asymptotic Properties of Least Squares Estimators of Cointegrating Vectors, Econometrica, 55, pp. 277–302.

Stock J.H., Watson M.W. (1988) Testing for Common Trends, Journal of the American Statistical Association, 83, pp. 1097–1107.

Van Dijk D., Teräsvirta T., Franses P.H. (2002) Smooth transition autoregressive models - a survey of recent developments, Econometric Reviews 21, 1-47.

Wagner M. (2004) A Comparison of Johansen's, Bierens' and the Subspace Algorithm Method for Cointegration Analysis, Oxford Bulletin of Economics and Statistics, 66(3), pp. 399-424.

Wagner M. (2010) Cointegration Analysis with State Space Models, Economics Series, 248, Institute for Advanced Studies, Vienna.

INTERNET REFERENCES

Google

Gary Koop web page: http://personal.strath.ac.uk/gary.koop/research.htm

Katarina Juselius web page: http://www.econ.ku.dk/okokj/#bio

C.W.J. Granger Nobel prize lecture:

http://nobelprize.org/nobel_prizes/economics/laureates/2003/granger-lecture.pdf

http://ideas.repec.org/p/ris/nobelp/2003_007.html

BEHAVIOR OF THE CENTRAL EUROPE EXCHANGE RATES TO THE EURO AND US DOLLAR[1]

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Abstract: Our objective has been to measure an impact of the two main global currencies - Euro and USD on shaping of exchange rates in countries of Central Europe. We have also endeavored to measure whether and to what extent a different approach to the Euro introduction as well as differentiated macroeconomic situation of these countries influenced the behavior of their exchange rates. The hitherto analyses indicate that the PLN rate of exchange was until 2004 strongly tied to the USD, but since 2004 links with the EUR exchange rate have become stronger. However the exchange rates of other countries in the region had been tied to the EUR earlier than the PLN exchange rate as they already had strong such links in the whole period of our analysis. Currency integration of the Central European countries is very strong although they are formally outside the euro zone and formation of their exchange rates should be perceived through trends of the EUR exchange rate versus other currencies, the USD.

Key words: exchange rate, Central European, cointegration analysis, euro zone

INTRODUCTION

Exchange rates belong to the most important variables in an open economy model. A mechanism for attaining a short-term equilibrium as well as effectiveness of fiscal and monetary policies depend to a large extend on mechanisms shaping exchange rates. An exchange rate may serve as a buffer for external shocks absorption[2]. At the same time the forex market is one of markets of crucial importance. Trading in this market is continuous, irrespective of time of the day, in any place in the world.

The exchange rate of any currency depends on many factors of complex nature, among which of great significance are those related to economic activity of business entities, movements of capital, migration of population, or to speculation. However, in last years an increasing role was played by the process of integration among states and resulting thus integration of economies. It is particularly noticeable in Europe where the Euro is a dominating currency. The rates of exchange formation in the global market are linked to the monetary policy of the state[3], and ongoing empirical studies indicate only a short-term influence of macroeconomic factors, even if the latter are subject to considerable fluctuations[4]. In practice this may signify a strong influence of currencies used by well developed economies on the exchange rates of currencies of less developed countries. In the case of non-euro E.U states in may indicate subordination of their exchange rates to the European currency[5].

This work is aimed at defining the impact of the two major global currencies i.e. the Euro and the U.S. dollar on formation of exchange rates in the countries of Central Europe: Czech Republic, Poland, Slovakia and Hungary. An attempt was also made to assess whether and to what extent different approaches to the introduction of euro, and differentiated macroeconomic situation in these countries impacted the behavior of exchange rates.

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[2] C. Goodhart, Money, Information and Uncertainty, Macmillan Press, London 1985, p. 430.

[3] J. Frenkel, A Monetary Approach to the Exchange Rate: Doctrinal Aspects and Empirical Evidence, Scandinavian Journal of Economics, 1978, Vol. 78, No. 2, p. 219 - 220.

[4] R. Flood, A. Rose, Fixing the Exchange Rate Regime: A Virtual Quest for Fundamentals, Journal of Monetary Economics, 1995, Vol. 36, p. 33.

[5] F. Ravenna, The European Monetary Union as a Commitment Device for New EU Member States, European Central Bank, Working Paper No. 516, Frankfurt/Main 2005, p. 26.

METHOD OF ANALYSIS

In this study we analyze foreign exchange rate formation of Central Europe currencies, i.e. the Polish zloty (PLN), Czech crown (CZK), Slovak crown (SKK), Hungarian forint (HUF) to the selected major global currencies: the US dollar (USD) and euro (EUR). To this end we use daily FOREX quotations in the years 2002-2008. The rates of exchange of the above mentioned currencies to the US$ are presented in fig.1. and a similar trend in behavior of their value is noticeable. Subsequent peaks and bottoms fell on similar moments, the overall trend however indicates a regular appreciation of the analyzed currencies and depreciation of the US$. In the last few months we observe the strongest adjustment - a reversal of the trend, i.e. a strengthening of the US$ position against all currencies under study.

Figure. 1. Currency quotes 2002-2008 - 1 USD rate of Exchange 1 USD rate of exchange; Sty - Jan (January)
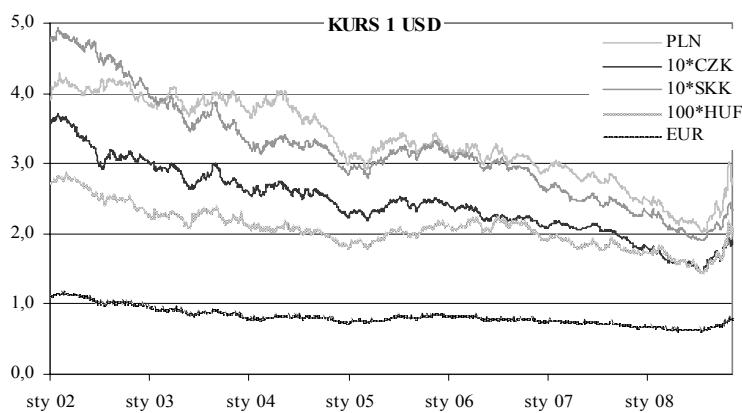
Source: FOREX quotes

Fig.2 shows annual volatility coefficients ($V = s / \bar{x}$) of the exchange rates of analyzed currencies against 1 USD. Some divergences can be observed here and they concern mainly the PLN versus other currencies. We see that in the initial period - years 2002 and 2003 volatility of the PLN exchange rate to the USD was much lower than that of the other currencies of the region. In 2004 however, we witness a reverse phenomenon - volatility of the PLN exchange rate to the USD is much higher than that of other currencies. In years 2005-2007 the exchange rates of all the analyzed currencies were relatively stable, while in 2008 rates of exchange volatility significantly increased, as a result of the USD depreciation at the beginning of that year, and appreciation of this currency in the last month of that period.

Figure. 2. CV (volatility/variance) coefficient values- quotations 1 USD
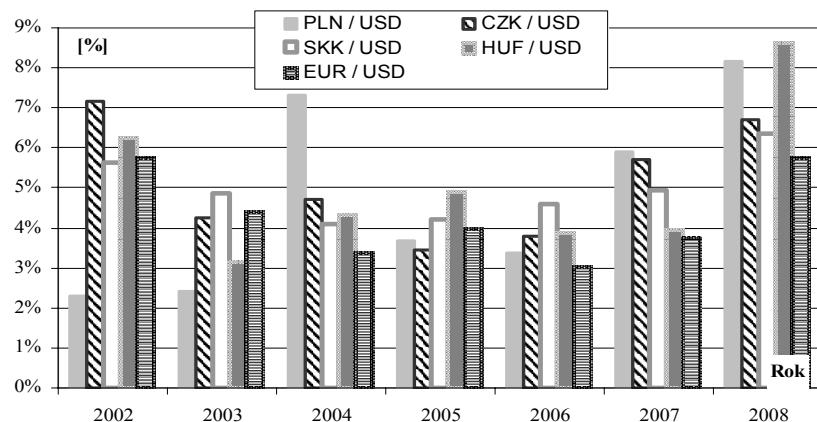
Source: own studies

The behavior of the exchange rates to 1 EUR was different (Fig 3). A common trend is hard to observe as the PLN showed a weakening trend till 2004 and then grew, CZK showed side-way movement till 2004 and then strengthened; SKK showed strengthening in the whole period while HUF showed side-way movement in the whole period; the USD had a weak trend for nearly the whole period and its strengthening has been noted in the last few months. Hence, the situation is different than in the formation of exchange rates to 1 USD.

Fig 4 presents a collation of the exchange rates volatility coefficient against 1 EUR on annual basis. In comparison with volatility level against 1 USD, we may note a greater relative stability of the analyzed exchange rates against 1 EUR - coefficients show lower values, but on the other hand the observed trends are not that clear. Most stable behavior against 1 EUR was observed in the CZK and SKK exchange rates, while the HUF exchange rate movement was more volatile, the greatest volatility among currencies of the region was noted for the PLN rate which until 2005 had always been at higher level and only since that year maintained the volatility level similar to that of other currencies.

Figure 3. Currency quotations 2002-2008 -1 EUR exchange rate 1 EUR exchange rate;
       Sty – January
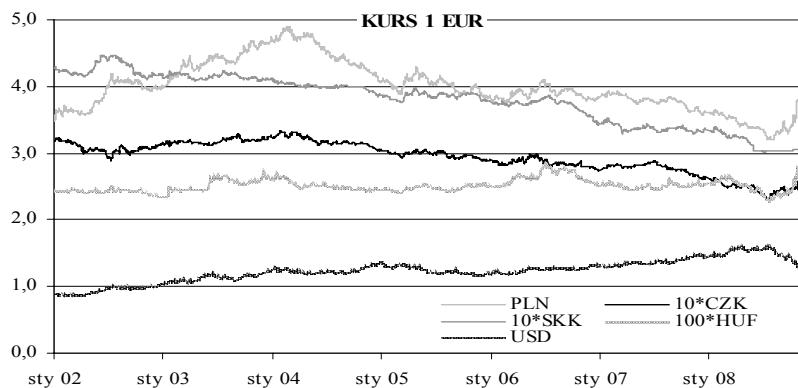
Source: FOREX quotations

Figure 4. Values of the exchange rate variance coefficient - 1 EUR quotations
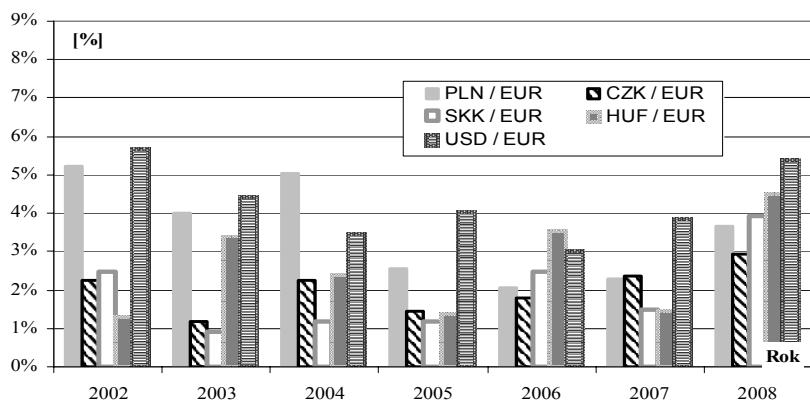
Source: own studies

   In the analysis of interrelations between time series values two methods the analysis of correlation and regression and the analysis of stationarity and co-integration, have been used in our study.

   In the first part of our research the interrelations were measured between the rates of exchange in individual years and in the whole period under study using the Pearson's linear coefficient. However, as indicated by the research practice, these results may not be reliable due to such features as non-stationarity and heteroscedasticity of certain time series. Hence, analysis of correlation was supplemented by the study of stationarity and co-integration. In this respect, the first stage was an analysis of integration degree for individual time series of

exchange rates. To this end we used the Phillips-Perron test as well as the ADF (because of similarity results of the first test are presented here). A second stage of the analysis of stationarity and co-integration consisted in a construction of co-integrating equation and verification of the residual stationarity hypothesis. The following form of the co-integrating equation was used[6]:

$$y_t = a_1 x_t + a_2 t + c , \qquad (1)$$

where:
y_t - dependent variable;
x_t - independent variable;
t - time;
c - constant.

The last stage of stationarity and co-integration analysis is construction of a model with an error correction mechanism of the form depending on the results of the first two steps[7]. In the case of level one stationary variables (stationary first differences of variables) co-integrated with each other, the auto-regressive model based on the first differences with the error correction should look like:

$$\Delta y_t = c + \sum_{i=1}^{k-1} \theta_i \Delta y_{t-i} + \sum_{i=0}^{k-1} \gamma_i \Delta x_{t-i} + \alpha ECM_{t-1} + \varepsilon_t , \qquad (2)$$

where:
ECM_{t-1} - residuals from the co-integrating equation.

In the case of non-cointegrated variables an application of the autoregressive model without error correction would seem more appropriate. Before the analysis of stationarity and co-integration the logarithms of time series values were taken and thus a percentage interpretation of the equation 2 parameters made possible.

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6 A.W. Gregory, B.E. Hansen, Residual - based tests for cointegration model with retime shifts, Journal of Econometrics, 1996, Vol. 70, p. 99 - 126.
7 R.F. Engle, C.W. Granger, Cointegration and error correction: representation, estimation and testing, 1987, Econometrica, Vol. 55, p. 251 - 276.

RESULTS

When comparing values of linear correlation coefficient of the exchange rates of regional currencies to 1 euro and the exchange rate of 1 euro to 1 USD (table 1) substantial similarities may be observed. All values of the correlation coefficient are positive i.e. EUR appreciation against USD results in the appreciation of other currencies, while depreciation involves the depreciation of these currencies. The above phenomenon is of different strength and for PLN in the period 2002-2003 it was weak, but grew strong in subsequent years, for CZK and SKK was strong in the whole period while for the HUF slightly weaker in the years 2003 and 2006, but remained always strong in other years.

Table 1. Correlation coefficients of exchange rates to 1 USD and the rate of exchange of USD to EUR

Pearson's correlation	EUR / USD							
	2002	2003	2004	2005	2006	2007	2008	from 1.05.04
PLN / USD	0.372	0.431	0.824	0.778	0.793	0.988	0.913	0.926
CZK / USD	0.961	0.963	0.907	0.929	0.987	0.969	0.901	0.946
SKK / USD	0.908	0.987	0.973	0.960	0.876	0.973	0.782	0.952
HUF / USD	0.983	0.634	0.838	0.972	0.488	0.929	0.870	0.921

Source: own calculations

Interrelations between the exchange rates of the currencies of the region vis-à-vis euro differ from those between the USD/EUR exchange rate. In the years 2002-2003 PLN is strongly linked with the USD, however already since 2004 most coefficients are negative. Exchange rates of other currencies in the region: CZK, SKK and HUF did not show strong links with the USD/EUR exchange rate in that period.

Table 2. Correlation coefficients of exchange rates to 1 Euro and the rate of exchange
        of 1 Euro to 1 USD

Pearson's correlation	USD / EUR							
	2002	2003	2004	2005	2006	2007	2008	from 1.05.04
PLN / EUR	0.906	0.836	-0.564	0.427	0.145	-0.918	-0.448	-0.757
CZK / EUR	-0.471	0.256	-0.453	0.504	-0.199	-0.800	-0.159	-0.808
SKK / EUR	0.207	-0.488	-0.581	-0.075	-0.434	-0.678	0.134	-0.831
HUF / EUR	-0.338	0.696	-0.144	-0.557	0.277	0.118	-0.364	-0.236

Source: own calculations

        In our analysis of correlation (table 1 and 2) the period since May 1, 2004
was additionally singled out as the time when inter alia Poland, the Czech
Republic, Slovakia and Hungary joined the European Union. In that period
currencies of these countries linked themselves strongly to the EUR, as indicated
by positive and close to unity values of the linear correlation coefficient in table
1 and negative values in table 2. If however, we discuss the strength of links with
EUR, the strongest are those with SKK exchange rate, then with CZK, PLN and
HUF.
        In table 3 we see interrelations between the rates of exchange for the
countries of the analyzed region to the USD and EUR. It turns out that the rates
to the USD show a very strong positive correlation indicating thus a strong impact
of the EUR/USD exchange rate on those relations, while those to the EUR are also
positive but the HUF/EUR exchange rate has weak ties to the exchange rates
of remaining currencies to 1 EUR, which may evidence a substantial autonomy
of the Hungarian currency.

Table 3. Correlation coefficients of the CE region countries

Currency	PLN	CZK	SKK	HUF
PLN	-	0.983	0.965	0.834
CZK	0.947	-	0.979	0.827
SKK	0.892	0.934	-	0.867
HUF	0.114	0.040	0.170	-

Explanations: values above the diagonal - 1 USD exchange rate, values below - 1 EUR
exchange rate.

Source: own calculations

The correlation study was supplemented by an analysis of stationarity and cointegration. Due to certain changes in the formation of the exchange rates after May 1, 2005 our study deals with time series since then. In general, all the analyses showed that the exchange rates were variables at the first integration level i.e. their first differences were stationary.

In the co-integrating equations presenting the exchange rate of regional currency to 1 USD with regard to EUR/USD, time and constant are very well aligned to empirical data as indicated by a high value of the determination coefficient, while residuals from these equations are stationary, as shown by the significance level of Phillips - Perron test being below 0.05 (table 4). It is then possible in accordance with the Engle-Granger's theorem to describe data with a model with an error correction mechanism.

Table 4. Co-integrating equation and the residual cointegration test-1 USD exchange rate

Currency	Independent variables			R2	Co-integration	
	EUR	trend	c		t	level
PLN	1.1372	-0.0001	1.5047	0.9584	-3.5309	0.0004
	0.0000	0.0000	0.0000			
CZK	1.1605	-0.0001	1.1912	0.9753	-2.4921	0.0124
	0.0000	0.0000	0.0000			
SKK	1.2497	-0.0001	1.4644	0.9796	-2.1437	0.0309
	0.0000	0.0000	0.0000			
HUF	1.3138	0.0000	0.9741	0.8818	-2.7963	0.0051
	0.0000	0.0000	0.0000			

Source: own calculations

Error correction models (Table 5) are marked by the relatively high, for increments models, values of determination coefficient and an absence of residuals autocorrelation. Coefficients values at the d(EUR) variable equal approximately 1 i.e. the exchange rates of the Central European countries adjust overnight to the long-term equilibrium with the EUR to 1 USD exchange rate.

Table 5. Error correction model -1 USD exchange rate

Dependent variables	Independent variables				R2	level p (B-G)
d(X)	d(X(-1))	d(EUR)	d(EUR(-1))	ECM(-1)		
PLN	-0.0816	0.9690	0.3429	-0.0202	0.4866	0.0158
	0.0047	0.0000	0.0000	0.0006		
CZK	-0.0573	1,0762	0.0226	-0.0178	0.6514	0.0000
	0.0588	0.0000	0.5749	0.0021		
SKK	-0.2118	0.9352	0.3698	-0.0123	0.6606	0.0020
	0.0000	0.0000	0.0000	0.0192		
HUF	-0.1076	1,0192	0.2363	-0.0167	0.4834	0.0000
	0.0003	0.0000	0.0000	0.0027		

Source: own calculations

Analogous research was carried out for link-ups between foreign exchange rates of regional currencies to 1 EUR with USD/EUR. And similarly to the earlier analysis the exchange rates show themselves to be non-stationary but their first differences are of stationary nature (Table 6). The co-integrating equations, except for HUF are well adjusted to empirical data (Table 7). Residuals from each equation are stationary.

Table 6. Co-integrating equation and residual-based co-integration test - exchange rates to euro

Currency	Independent variables			R2	Cointegration	
	USD	trend	c		t	level p
PLN	-0.1380	-0.0001	1,5046	0.8509	-3,3411	0.0008
	0.0000	0.0000	0.0000			
CZK	-0.1829	-0.0001	1,1970	0.9046	-2,8478	0.0043
	0.0000	0.0000	0.0000			
SKK	-0.2512	-0.0001	1,4645	0.9315	-1,9998	0.0437
	0.0000	0.0000	0.0000			
HUF	-0.3144	0.0000	0.9741	0.1910	-2,6270	0.0084
	0.0000	0.0000	0.0000			

Source: own calculations

Models with an error correction mechanism for linking regional exchange rates to 1 EUR with the USD/EUR exchange rate do not have as good statistical properties as in the previous research. Here we observe low values of the determination coefficient and in the case of models d(PLN) and d(SKK) - autocorrelation of residuals. Taking into account the coefficient values with

independent variables only, we may notice low values with the d(USD) variable and low values with the ECM(-1), which means that the regional exchange rates to 1 EUR adjust very slowly to a long term equilibrium with the USD/EUR exchange rate, or do not remain there.

Table 7. Error correction model - exchange rate to euro

Dependent variables d(X)	Independent variables				R2	Level p (B-G)
	d(X(-1))	d(USD)	d(USD(-1))	ECM(-1)		
PLN	0.0536	-0.1480	-0.0701	-0.0168	0.0459	0.9495
	0.0725	0.0000	0.0089	0.0007		
CZK	-0.0277	0.0402	0.0565	-0.0221	0.0163	0.0000
	0.3595	0.1132	0.0252	0.0003		
SKK	-0.0128	-0.0539	-0.0251	-0.0100	0.0116	0.1171
	0.6690	0.0011	0.1299	0.0174		
HUF	0.0175	-0.1422	0.0241	-0.0155	0.0313	0.0000
	0.5612	0.0000	0.4065	0.0021		

Source: own calculations

SUMMARY

Our studies confirm in full the assumptions put forward in the introduction and indicate in particular that:

1. The PLN rate of exchange was until 2004 strongly tied to the USD, and since 2004 links with the EUR exchange rate have been stronger.
2. The exchange rate of other currencies in the region were tied to the EUR earlier than the PLN exchange rate, as in the whole period under study these ties were strong.
3. The Euro impact on regional currencies is very strong. Our study shows that any changes in the EUR/USD rate result in straightaway changes in exchange rates of other currencies of the region versus the USD.
4. The exchange rates of regional currencies versus euro are to a large extent shaped by other factors than USD / EUR exchange rate.

Notwithstanding the fact that formally the Central European countries are outside the euro zone, currency integration is very strong and formation of these countries exchange rates should be perceived through formation of the euro exchange rate towards other currencies, in particular to the USD.

REFERENCES

Engle R.F., Granger C.W. (1987) Cointegration and error correction: representation, estimation and testing, Econometrica, 55, pp. 251-276

Flood R., Rose A. (1995) Fixing the Exchange Rate Regime: A Virtual Quest for Fundamentals, Journal of Monetary Economics, Vol. 36, pp. 33.

Frenkel J. (1978) A monetary approach to the exchange rate: doctrinal aspects and empirical evidence, Scandinavian Journal of Economics, , Vol. 78, No. 2, pp. 219 - 220.

Goodhart C. (1985) Money, information and uncertainty, Macmillan Press, London, p. 430.

Gregory A.W., Hansen B.E. (1996) Residual - based tests for cointegration model with retime shifts, Journal of Econometrics, 70, pp. 99-126

Ravenna F. (2005) The European Monetary Union as a Commitment Device for New EU Member States, European Central Bank, Working Paper No. 516, Frankfurt/Main, p. 26.

COMPARISON OF CONFIDENCE INTERVALS
FOR FRACTION IN FINITE POPULATIONS

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Abstract. Consider a finite population. Let $\theta \in (0,1)$ denotes the fraction of units with a given property. The problem is in interval estimation of θ on the basis of a sample drawn due to the simple random sampling without replacement. In the paper three confidence intervals are compared: exact based on hypergeometric distribution and two other based on approximations to hypergeometric distribution: Binomial and Normal. It appeared that Binomial based confidence interval is too conservative while the Normal based one does not keep the prescribed confidence level.

Keywords: confidence interval, approximate confidence interval, fraction, finite population

INTRODUCTION

Consider a population $\{u_1, \ldots, u_N\}$ containing the finite number N units. Let M denotes an unknown number of objects in population which has an interesting property. We are interested in an interval estimation of M, or equivalently, the fraction $\theta = \dfrac{M}{N}$. The sample of size n is drawn due to the simple random sampling without replacement (*lpbz* to be short). Let ξ_{bz} be a random variable describing a number of objects with the property in the sample. Its distribution is hypergeometric (Bracha 1996, Zieliński 2010)

$$P_\theta\{\xi_{bz} = x\} = \frac{\dbinom{\theta N}{x}\dbinom{(1-\theta)N}{n-x}}{\dbinom{N}{n}},$$

for integer x from the interval $\langle \max\{0, n-(1-\theta)N\}, \min\{n, \theta N\}\rangle$. Denote by $f_\theta(\cdot)$ and $F_\theta(\cdot)$ the probability distribution function and cumulative distribution function of ξ_{bz}, respectively.

Note that

$$E_\theta \xi_{bz} = n\theta, \qquad D_\theta^2 \xi_{bz} = \frac{N-n}{N-1} n\theta(1-\theta).$$

A construction of the confidence interval at a confidence level δ for θ is based on the cumulative distribution function of ξ_{bz}. If $\xi_{bz} = x$ is observed then the ends θ_L and θ_U of the confidence interval are the solutions of the two following equations

$$F_{\theta_L}(x) = \delta_1, \qquad F_{\theta_U}(x) = \delta_2.$$

The numbers δ_1 and δ_2 are such that $\delta_2 - \delta_1 = \delta$. In what follows we take $\delta_1 = (1-\delta)/2$ and $\delta_2 = (1+\delta)/2$. Analytic solution is unavailable. However, for given x, n and N, the confidence interval may be found numerically. In the Table 1 there are given exemplary confidence intervals for $N = 1000$ units, sample size $n = 20$, confidence level $\delta = 0.95$ and $\delta_1 = 0.025$.

Table 1. Confidence intervals for θ

x	θ_L	θ_U	x	θ_L	θ_U	x	θ_L	θ_U
0	0.000	0.167	7	0.155	0.591	14	0.459	0.880
1	0.001	0.247	8	0.192	0.638	15	0.511	0.913
2	0.012	0.316	9	0.232	0.683	16	0.565	0.942
3	0.032	0.377	10	0.273	0.727	17	0.623	0.968
4	0.058	0.435	11	0.317	0.768	18	0.685	0.988
5	0.087	0.489	12	0.362	0.808	19	0.753	0.999
6	0.120	0.541	13	0.409	0.845	20	0.833	1.000

Source: own calculations

The real confidence level equals

$$conf_H = F_\theta(x_g) - F_\theta(x_d) = \sum_{x=x_d}^{x_g} f_\theta(x),$$

where

$$x_d = \max\left\{x : F_\theta(x) \le \frac{1-\delta}{2}\right\} \text{ and } x_g = \min\left\{x : F_\theta(x) \ge \frac{1+\delta}{2}\right\}.$$

Equivalently,

$$conf_H = \sum_{x=0}^{n} f_\theta(x)\mathbf{1}_{(\theta_L,\theta_U)}(\theta),$$

where $\mathbf{1}_A(a) = 1$ if $a \in A$ and $= 0$ elsewhere.

Since the population is finite, the number of admissible values of θ is also finite. For example, for $x = 1$ admissible values of θ are $0.001, 0.002, \ldots, 0.247$. It means, that the number of units with the investigated property is one of 1, 2, $\ldots$ or 247.

The hypergeometric distribution is analytically and numerically untractable. Hence different approximations are applied. There are at least two approximations commonly used in applications: Binomial and Normal.

BINOMIAL APPROXIMATION

The distribution of ξ_{bz}, for relatively small values of θ and large values of N may be approximated by Binomial distribution $Bin(n,\theta)$. As a rule of thumb $\theta < 0.1$ and $N \ge 60$ is sometimes used (Johnson and Kotz 1969). The confidence interval for θ has the form (θ_L^B, θ_U^B), where

$$\theta_L^B = \beta^{-1}\left(\xi_{bz}, n - \xi_{bz} + 1; \frac{1-\delta}{2}\right), \quad \theta_U^B = \beta^{-1}\left(\xi_{bz} + 1, n - \xi_{bz}; \frac{1+\delta}{2}\right).$$

Here $\beta^{-1}(a,b;q)$ denotes the q-th quantile of the Beta distribution with parameters a,b (Clopper and Pearson 1934, Zieliński 2010). If $\xi_{bz} = 0$ then θ_L^B is taken 0. For $\xi_{bz} = n$, θ_U^B is taken 1. The true confidence level is equal to

$$conf_B = \sum_{x=0}^{n} f_\theta(x)\mathbf{1}_{(\theta_L^B,\theta_U^B)}(\theta).$$

NORMAL APPROXIMATION

The hypergeometric distribution may be approximated by a normal distribution with mean and variance equal to mean and variance of ξ_{bz}, i.e by the distribution $N(n\theta, \frac{N-n}{N-1}n\theta(1-\theta))$. To do so the rule of thumb $N\theta \geq 4$ is sometimes used (Johnson and Kotz 1969). The confidence interval for θ is obtained as a solution with respect to θ of the inequality

$$\left| \frac{\xi_{bz} - n\theta}{\sqrt{\frac{N-n}{N-1}n\theta(1-\theta)}} \right| \leq z_{\frac{1+\delta}{2}},$$

where z_q denotes the q-th quantile of the distribution $N(0,1)$. As a solution we obtain the confidence interval (θ_L^N, θ_U^N), where ($z = z_{\frac{1+\delta}{2}}\sqrt{\frac{n(N-n)}{N-1}}$)

$$\theta_L^N = \frac{z^2 + 2n\xi_{bz} - z\sqrt{z^2 + 4(n-\xi_{bz})\xi_{bz}}}{2(n^2 + z^2)}, \quad \theta_U^N = \frac{z^2 + 2n\xi_{bz} + z\sqrt{z^2 + 4(n-\xi_{bz})\xi_{bz}}}{2(n^2 + z^2)}.$$

The true confidence level is equal to

$$conf_N = \sum_{x=0}^{n} f_\theta(x)\mathbf{1}_{(\theta_L^N, \theta_U^N)}(\theta).$$

COMPARISON

Consider a population consisting $N = 1000$ units. From the population a sample of size $n = 20$ is drawn with respect to the *lpbz* scheme. The number ξ_{bz} of units with a given property is observed and confidence interval for the fraction θ of all such object in the population is constructed. The confidence level $\delta = 0.95$ is assumed.

In the Table 2 there are given confidence limits calculated on the basis of the exact distribution of ξ_{bz} as well as confidence limits based on two approximations.

Table 2. Comparison of confidence intervals

	Hypergeometric		Binomial		Normal	
x	θ_L	θ_U	θ_L^B	θ_U^B	θ_L^N	θ_U^N
0	0.000	0.167	0.000	0.169	0.000	0.159
1	0.001	0.247	0.001	0.249	0.009	0.234
2	0.012	0.315	0.012	0.317	0.028	0.299
3	0.032	0.377	0.032	0.379	0.052	0.359
4	0.058	0.435	0.057	0.437	0.081	0.414
5	0.087	0.489	0.086	0.492	0.112	0.467
6	0.120	0.541	0.118	0.543	0.146	0.517
7	0.155	0.591	0.153	0.593	0.182	0.566
8	0.192	0.638	0.191	0.640	0.220	0.612
9	0.232	0.683	0.230	0.685	0.259	0.657
10	0.273	0.727	0.271	0.729	0.300	0.700
11	0.317	0.768	0.315	0.770	0.343	0.741
12	0.362	0.808	0.360	0.809	0.388	0.780
13	0.409	0.845	0.407	0.847	0.434	0.818
14	0.459	0.880	0.457	0.882	0.483	0.854
15	0.511	0.913	0.508	0.914	0.533	0.888
16	0.565	0.942	0.563	0.943	0.586	0.919
17	0.623	0.968	0.621	0.968	0.641	0.948
18	0.685	0.988	0.683	0.988	0.701	0.972
19	0.753	0.999	0.751	0.999	0.766	0.991
20	0.833	1.000	0.831	1.000	0.841	1.000

Source: own calculations

In the Figure there are shown real confidence levels of above confidence intervals. It may be noted that the Binomial based confidence interval is too conservative: its length is greater than the hypergeometric one and its confidence level is higher. The Normal approximation based confidence interval does not keep the prescribed confidence level (here 0.95), so this interval should not be used in fraction estimation.

Figure 1.
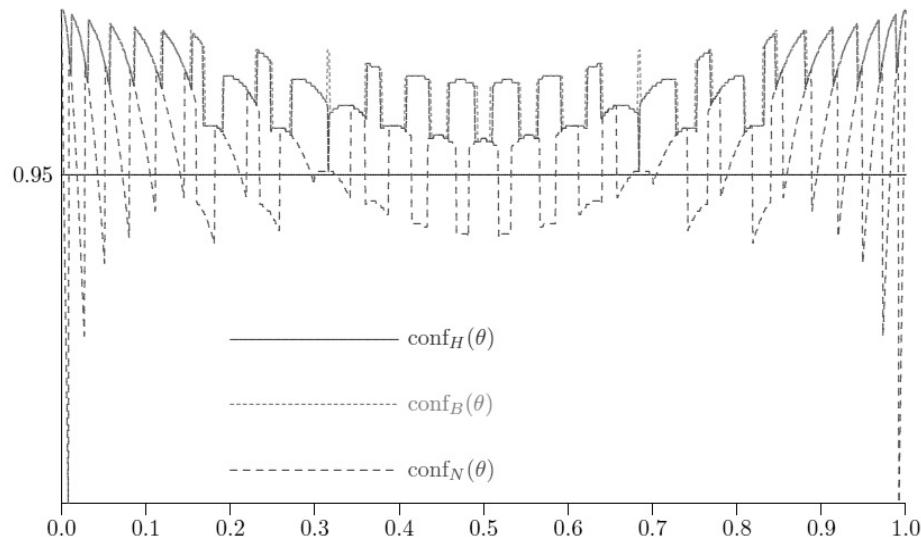
Source: own preparation

REFERENCES

Bracha Cz. (1996) Teoretyczne podstawy metody reprezentacyjnej, PWN , Warszawa.
Clopper C. J., Pearson E. S. (1934) The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial, Biometrika 26, 404-413.
Johnson N. L., Kotz S. (1969) Discrete distributions: distributions in statistics, Houghton Mifflin Company, Boston.
Zieliński W. (2010) Estymacja wskaźnika struktury, Wydawnictwo SGGW, Warszawa.