METODY ILOŚCIOWE
W BADANIACH EKONOMICZNYCH

QUANTITATIVE METHODS IN ECONOMICS

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BIAS REDUCTION IN KERNEL ESTIMATOR OF DENSITY FUNCTION IN BOUNDARY REGION

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Abstract: The properties of the classical kernel estimator of density function deteriorate when the support of density function is bounded. The use of classical form of kernel estimator causes the increase of the bias estimator, particularly in the so-called boundary region, close to end of support. It can also lead to undesirable situation where density function estimator has a different support than the density function. The paper presents selected bias reduction procedures, such as reflection method and its modification. An example is presented with an attempt to compare considered procedures.

Keywords: kernel estimator, density function, bias reduction, reflection method

INTRODUCTION

When the density function satisfies certain smoothness criteria (e.g. existing and being continuous of the density derivatives of appropriate orders over the entire real line), the kernel density estimator is characterized by some useful properties, such as: unnecessity of assuming that density belongs to a parametric family of distributions, its calculation is easy and it is asymptotically unbiased and is consistent estimator of unknown density function. The problems may arise for users when these smoothness conditions are not fulfilled, as in the case of some commonly known densities. E.g. when the density function of exponential distribution is being estimated, the kernel estimator is trying to estimate relatively high density for positive values of random variable, whereas for negative values the estimator is aiming to estimate zero. The discontinuity in the function results in the bias increasing of the estimator [Wand, Jones 1995].

The next situation when the properties of the kernel density estimator deteriorates is the bounded domain of definition of a density being estimated. In practical problems such a situation occurs often as many random variables considered in the problems of economic, technical or natural sciences are characterized by bounded support on one or both sides. In most situations left boundary equals zero when the data under consideration are measurements of positive quantities. In different analyses random variables with non-negative values are considered (duration of unemployment, the stock price, time of performing specific technical operation, the amount of inventory in the warehouse, time of growing plants, amount of atmospheric fall). The use of classical form of kernel estimator causes the increase of the bias estimator, particularly in the so-called boundary region, close to end of support. It is possible both when the kernel function is unbounded, and when the kernel function is bounded but partially is ejected out of the density function support. It can also lead to undesirable situation where density function estimator has a different support than the density function [c.f. Jones 1993]. Moreover, in presentation of the data for which the estimation is giving, the situation when any weight is assigned to the negative numbers is treated as unacceptable [Silverman 1986].

Modification of classical kernel estimator is needed to improve the estimator properties. It should be used especially in the situation when the integral of the kernel estimator is not 1 in appropriate support or estimator is not consistent for some observations.

Let density function f be continuous on interval $[0,\infty)$ and be 0 for $x<0$. For smoothing parameter h: interval $[0,h)$ is called boundary region and interval $[h,\infty)$ is interior region.

Note that for interior region it is possible to use the classical form of kernel density estimator. For boundary region information interval $[x-h,x+h]$ may locate outside the support what may cause that some of the observations are not used in construction of the density estimator [Albers 2012]. Estimation is based on reduced information, the bias is large resulting in poor estimation.

CLASSICAL KERNEL DENSITY ESTIMATOR

Function $K_{\nu,k}$ with support $[-1,1]$ is defined as kernel function of degree (ν,k), for $\nu \le k-1$ ($\nu,k \in N$), if it fulfils the following property [c.f. Horová et al. 2012]:

$$\int_{-1}^{1} x^j K_{\nu,k}(x)dx = \begin{cases} 0 \quad \text{for} \quad 0 \le j \le k-1,\ j \ne \nu, \\ (-1)^{\nu}\nu! \quad \text{for} \quad j = \nu, \\ \kappa_k \ne 0 \quad \text{for} \quad j = k, \end{cases} \tag{1}$$

where κ_k is kth moment of the kernel $K_{\nu,k}$.

For $\nu = 0$ and $k = 2$ kernel function $K_{0,2}(\cdot)$ is symmetric function around zero and $\int_{-1}^{1} K_{0,2}(x)dx = 1$. Any density function with support $[-1,1]$ with mean zero is kernel function of degree $(0,2)$ and in most cases they are used in construction of classical kernel density estimators.

Density kernel estimator based on sample $X_1, X_2, ..., X_n$ with kernel $K_{0,2}$ symmetric around zero with support $[-1,1]$ can be written as [Wand, Jones 1995], [Silverman 1996], [Domański et al. 2014]:

$$\hat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K_{0,2}\left(\frac{x - X_i}{h_n}\right), \tag{2}$$

where h_n is a smoothing parameter, such as $h_n > 0$, $h_n = h(n) : \{h(n)\}$, $h_n \underset{n\to\infty}{\to} 0$, $h_n n \underset{n\to\infty}{\to} \infty$.

Classical kernel density estimator is consistent for continuous $f(x)$ and for $h_n > 0$, $h_n \underset{n\to\infty}{\to} 0$ and $h_n n \underset{n\to\infty}{\to} \infty$. Moreover, it is a density function (is nonnegative and integrates to 1).

Kernel estimator of the ν-derivative of density function (assuming that derivatives exist and are continuous) is:

$$\hat{f}_n^{(\nu)}(x) = \frac{1}{nh_n^{\nu+1}}\sum_{i=1}^{n} K_{\nu,k}\left(\frac{x - X_i}{h_n}\right), \tag{3}$$

where $K_{\nu,k}$ is kernel function of degree (ν,k). For $\nu = 0$, $k = 2$ one can get (2).

Kernel estimator of the ν-derivative of density function for the appropriate kernel function is consistent in points of continuity of derivative.

BIAS REDUCTION OF KERNEL DENSITY ESTIMATOR

Let:

- $X_1, X_2, ..., X_n$ be a random sample drawn from a population with random variable X with density function f with support $[0,\infty)$ ($f(x)=0$ for $x<0$ and $f(x)>0$ for $x \geq 0$);
- $f^{(2)}$ be a second derivative of density function which is continuous away from $x=0$;
- function $K_{0,2}(\cdot)$ be symmetric and smooth kernel function of degree $(0,2)$ with support $[-1,1]$;
- $\hat{f}_n(x)$ be the kernel density function (2) with the smoothing parameter h_n.

Boundary behavior of the kernel estimator can be observed taking into regard its asymptotic properties at a sequence of points which is within one bandwidth of the boundary. Taking $x = ch_n$ for $c \in [0,1)$, kernel density estimator for point x, is defined as: $\hat{f}_n(ch_n) = \frac{1}{nh_n}\sum_{i=1}^{n} K_{0,2}\left(\frac{ch_n - X_i}{h_n}\right)$. For $x > h_n\,(c>1)$, kernel estimator of density function is asymptotically unbiased and consistent. Its expected value is the following:

$$E\left[\hat{f}_n(x)\right] = \frac{1}{nh_n}\sum_{i=1}^{n} E\left[K_{0,2}\left(\frac{X_i - x}{h_n}\right)\right] = f(x) + \frac{1}{2!} f^{(2)}(x) h^2 \kappa_2 + o\left(h^2\right), \qquad (4)$$

where κ_2 is defined in (1). For $0 \leq c \leq 1$ when $\int_{-1}^{c} K_{0,2}(u)du \neq 1$ in general, kernel estimator of density function is not consistent. Its expected value is:

$$E\left[\hat{f}_n(x)\right] = f(x)\int_{-1}^{c} K_{0,2}(u)du + o(1). \qquad (5)$$

It is possible to use an appropriate modification of the kernel estimator in the vicinity of the known boundary. It results in a family of boundary kernels $K_{0,2}^{L}(u,c)$ and the achieving $O\left(h^2\right)$ bias is possible. For different kernel functions and different values of c kernel density estimators based on kernel function from a family of boundary kernels improve the performance of estimator in the boundary region [Wand, Jones 1995].

Simple method used in bias reduction of kernel estimator is based on the estimator calculation only for positive values ignoring the boundary region and then setting kernel estimator to zero for negative values. It causes that the estimator

is zero for negative values but on the other hand the integral of the estimator is not 1 [Jones, Foster 1996].

Another approach uses the reflections of all the points in the boundary that results in a set $\{X_1,-X_1,X_2,-X_2,...\}$. Under the assumption that kernel function is symmetric and differentiable, the resulting estimator has zero derivative at the boundary.

This reflection method can be used directly in the kernel estimator by using appropriate modification of the kernel function outside the interval $[0,\infty)$, for example, symmetric reflection about zero, where parts of kernel function outside $[0,\infty)$ are deleted and next placed in the neighbour of zero in interval $[0,\infty)$.

Kernel estimator using reflection method is the following [Kulczycki 2005]:

$$\hat{f}_{nR}(x)=\frac{1}{nh_n}\sum_{i=1}^{n}\left[K_{0,2}\left(\frac{x-X_i}{h_n}\right)+K_{0,2}\left(\frac{x+X_i}{h_n}\right)\right]. \tag{6}$$

Estimator (6) is consistent estimator of function f but for x close to zero the bias is $O(h)$.

The Karunamuni and Alberts generalized reflection method improves the bias with low variance. The generalized reflection estimator is [Karunamuni, Alberts 2005]:

$$\hat{f}_{nGR}(x)=\frac{1}{nh_n}\sum_{i=1}^{n}\left[K\left(\frac{x-g_1(X_i)}{h_n}\right)+K\left(\frac{x+g_2(X_i)}{h_n}\right)\right], \tag{7}$$

where g_1 and g_2 are some transformation functions (e.g. cubic polynomials with coefficients ensuring criteria for the order of estimators $O(h^2)$).

SIMULATION STUDY

The simulation study was conducted to analyze the properties of chosen methods of the bias reduction of kernel density estimator.

The populations with density functions of bounded support $[0,\infty)$ were taken into consideration, particularly populations of two-parameters Weilbull distribution $W(0,\delta,\gamma)$, where δ is a scale parameter and γ is a shape parameter. The populations were regarded with the following parameters:

W1: $\delta=1,\gamma=0.1$,

W2: $\delta=1,\gamma=0.5$,

W3: $\delta=1,\gamma=1$,

W4: $\delta=1,\gamma=2$ (Rayleigh distribution),

W5: $\delta = 1, \gamma = 3.4$,
W6: $\delta = 1, \gamma = 5$,
W7: $\delta = 4, \gamma = 1$,
W8: $\delta = 4, \gamma = 2$.

The parameters of Weilbull distributions were chosen in such a way that it is possible to analyze the broad range of distributions with bounded supports. The populations are heterogeneous looking from e.g. measure of location, spread or asymmetry.

To extend the study and indicate the area of application of regarded methods, one more population was considered, the measure of agricultural productivity – agriculture value added per worker for countries in 2013. Data are in constant 2005 U.S. dollars. Source of the data is:
http://data.worldbank.org/indicator/EA.PRD.AGRI.KD [18.06.2015].

From each population the samples were chosen where $n = 10,20,...,100$. For each sample, the classical kernel density estimator and kernel density estimator with reflection were calculated using Gaussian kernel function and the reference rule or biased cross validation (in the case of W1) as the most popular methods of choosing the smoothing parameter. The chosen descriptive statistics calculated for samples ($n = 50$) from populations W1-W8 are presented in Table 1.

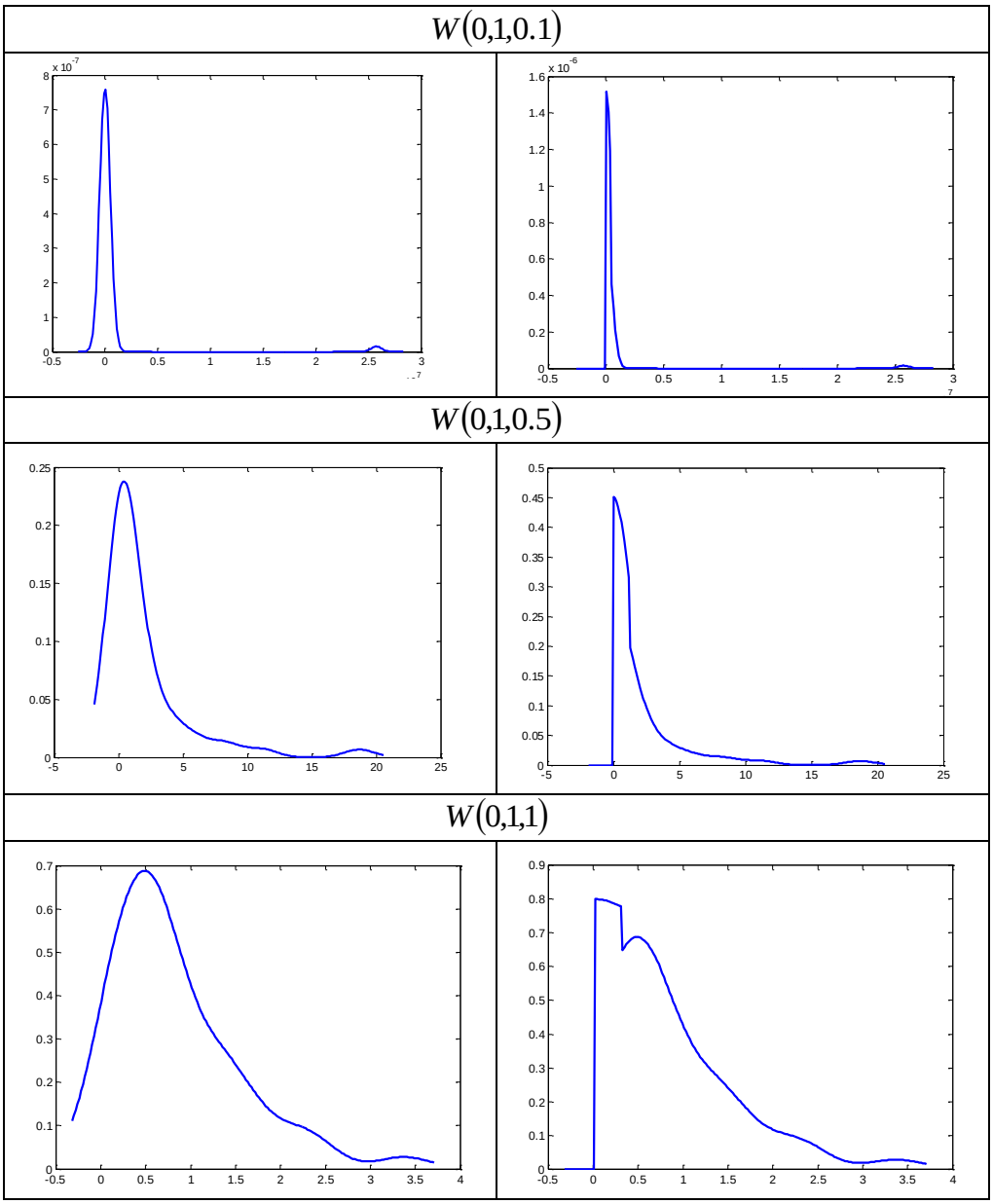
Table 1. Chosen descriptive statistics for samples from populations W1-W8 (n=50)

Sample from population $Wi(0,\delta,\gamma)$	Maximal value	Mean	Median	Standard deviation	Asymmetry	Kurtosis
$i = 1$ $\delta = 1,\ \gamma = 0.1$	2.5772	0.0518	0.0000	0.3644	0.0000	0.000
$i = 2$ $\delta = 1,\ \gamma = 0.5$	18.6900	1.9088	0.4418	3.4493	3.0234	13.3488
$i = 3$ $\delta = 1, \gamma = 1$	3.3774	0.8321	0.6525	0.7073	1.4203	5.1301
$i = 4$ $\delta = 1, \gamma = 2$	1.9582	0.8640	0.8397	0.3911	0.7166	3.2806
$i = 5$ $\delta = 1, \gamma = 3.4$	1.5101	0.8271	0.8312	0.2858	0.0493	2.5580
$i = 6$ $\delta = 1, \gamma = 5$	1.2919	0.9249	0.9549	0.2336	-0.3033	2.0451
$i = 7$ $\delta = 4, \gamma = 1$	17.4815	3.9782	2.5848	3.7848	1.4699	4.9171
$i = 8$ $\delta = 4, \gamma = 2$	7.9888	3.3012	2.7184	2.0423	0.5840	2.2218

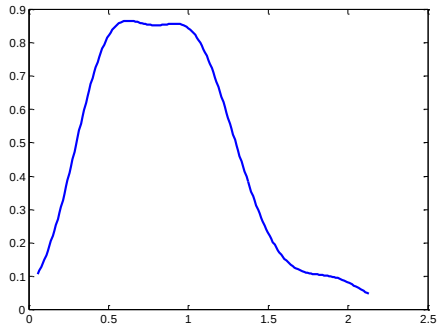
Source: own calculations

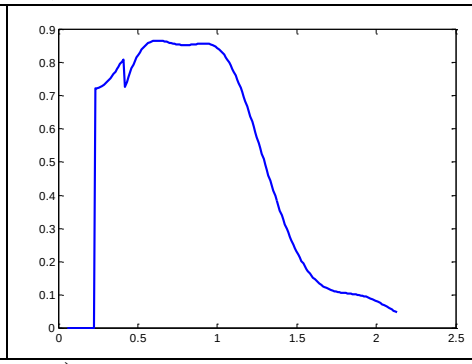
Exemplary results for sample size $n = 50$ are presented in Figures 1-2 with classical kernel density estimator (on the left) and kernel density estimator with reflection (on the right).

Figure 1. Classical kernel density estimator and kernel density estimator with reflection for populations W1-W8

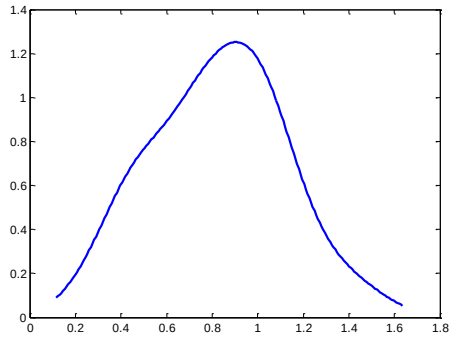


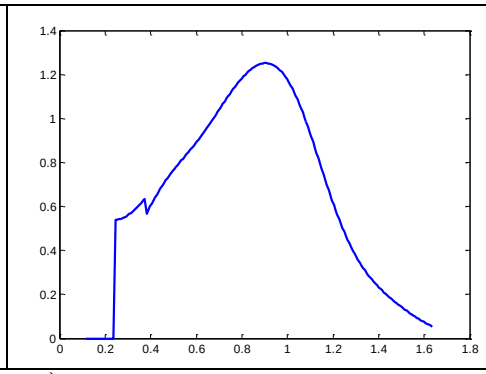
$W(0,1,2)$



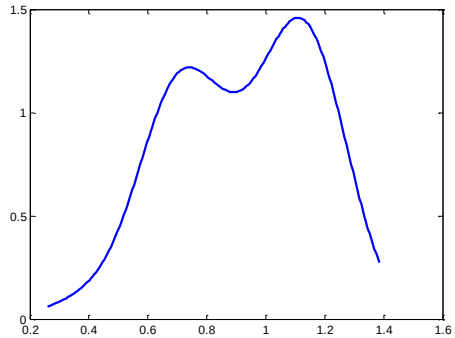


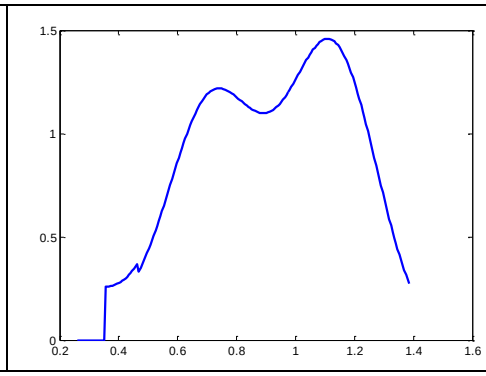
$W(0,1,3.4)$

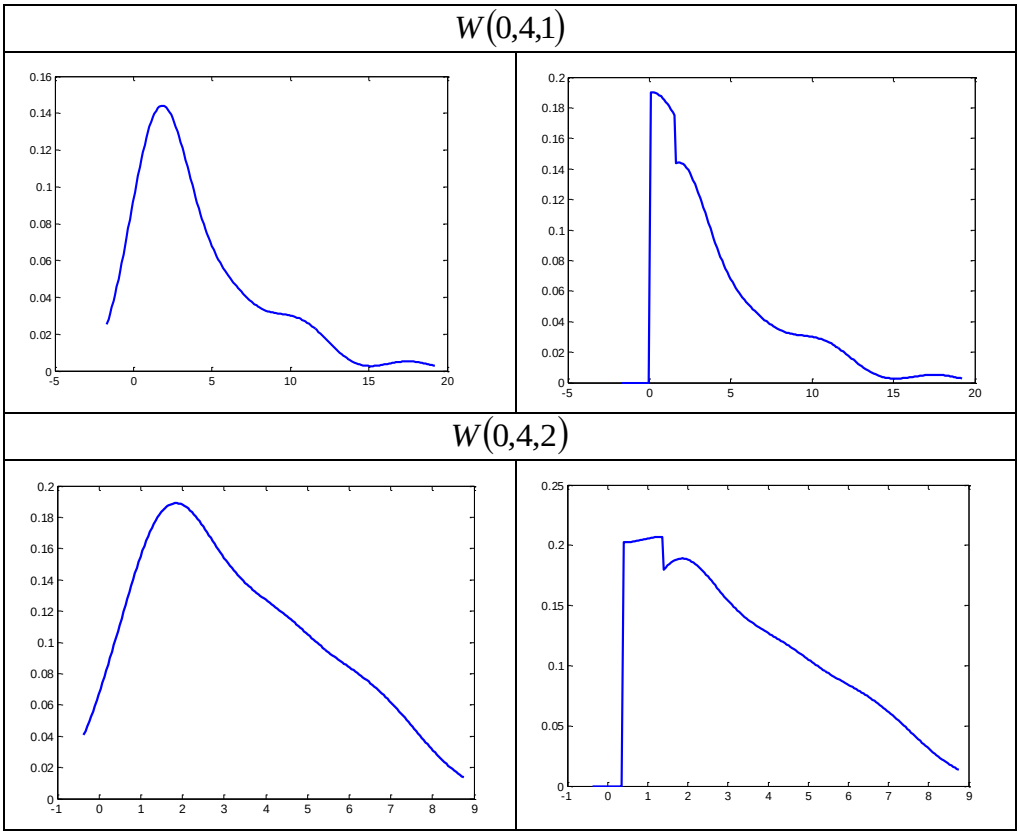




$W(0,1,5)$

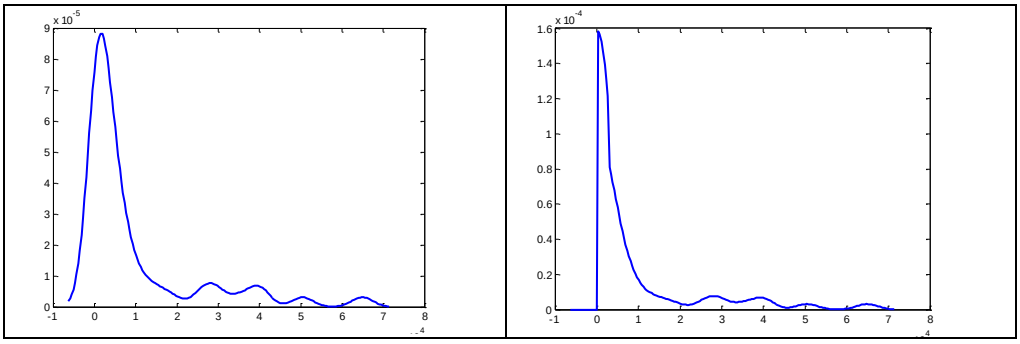






Source: own calculations

Figure 2. Classical kernel density estimator and kernel density estimator with reflection for agriculture value added per worker for countries in the world in 2013



Source: own calculations

SUMMARY

For many random variables considered in practical applications their density functions, by definition, are characterized by bounded support. Sometimes the estimator, e.g. kernel density estimator based on the samples from these populations, has different support than the density function. Such situation was observed for samples from Weilbull distribution, especially with small values of shape parameter. Modification of kernel estimator using the method of reflection ensure the users that the estimator is constructed only for non-negative values. But applying the reflection kernel estimator without initial stage of analyse the classical estimator may cause unnecessary limitation of the support (as it was in the case of W5). Further deeper analysis is needed for indicating such modifications of classical kernel estimator that the estimator will be with the same support as density function and there will be no lack of points of discontinuity. Such modification is necessary especially for practical implementations of regarded methods.

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GRANULAR CALCULATIONS IN THE REQUIREMENT ANALYSIS OF THE POLISH MARKET

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Abstract: The article presents the results of the analysis of the labour market requirements reported by Polish employers. Relations between the type of specialty, enhanced by the competence profile, and the requirement of proficiency in English were sought. The empirical material came from the research implemented within the "Human Capital Balance" project (V edition – 2014). The research procedure consisted of grouping objects with the method of *k*-means and the induction of decision rules based on the application of theories of rough sets. The analysis was performed using the RSES 2.1 system.

Keywords: competence, information granules, rough sets

INTRODUCTION

In the second half of the XX century L.A. Zadeh popularised the concept of the informative granule, also called the info-granule or information granule [Zadeh 1979, 1997]. The proposed term refers to certain sub-sets (classes) of the universe, which includes objects characterised by: indiscernibility, similarity or a similar behaviour. This term is associated with the formulation of granular calculations, which is defined by literature as the general theory of calculations. It aims

to effectively use the so-called grains, such as classes, clusters, sub-sets, groups and intervals. This will allow the construction of effective calculation models for purposes of complex applications with a significant amount of data, information and knowledge [Borowik et al. 2009, p. 154]. Granular calculations are perceived as the form of structural thinking or the procedure of structural problem solving. Depending on the assumed grain size (aggregation) in the tested set, one can observe the regularities (relations, dependencies) invisible with a different distribution, what allows a wider examination of the problem. The quality of results obtained from the application of the granular construction is affected by the structure of granules and methods of calculation implementation.

The aim of the article is to examine the competence needs of the Polish labour market. The main task is to analyse the competence demand (skills and qualifications) reported by the Polish employers. Studies were conducted in the paper using the RSES 2.1 system (Rough Set Exploration System 2.1), which is a computer tool, constructed by the team led by Z. Skowron. It enables the analysis of data in the table form using the rough set theory.

The studies used rough sets, used in the theory of granular calculations. The focus was on searching dependencies between the type of specialty, clarified by the competence profile, and the requirement of the English language proficiency. There were also attempts to answer three questions related to the analysed problem.

- Are there employers' requirements towards specialists concerning the knowledge of English?
- Which specialists (in particular) are required to know the English language?
- Is this regularity the same in the regions of Poland (information granules) characterised by a different level of economic development?

CONCEPT OF GRANULAR INFORMATION

Unlike traditional numerical calculations, data-oriented, granular calculations are directed to knowledge. That is why they are used in applications related to the discovery of knowledge and data exploration [Borowik et al. 2009, p. 155]. In 1982 Z. Pawlak proposed to use the granular information in the form of the rough set theory [Yao 2005].

The rough set theory helps to control large data sets and provides mathematical tools needed for a formal description of knowledge, especially the incomplete and inaccurate knowledge. Due to the fact that the granularity of the available information can cause inconsistency of the object description, this theory has been proposed as the tool of the granular information analysis [Nowak 2013]. The rough set theory allows, among others: searching dependencies between data, data reduction, determining the data weight, generating decision rules from data [Pawlak 1982]. In this method, the rule knowledge representation (in the form of IF… THEN) is the result of performing the decision rules induction process based on the set of data

representing the teaching models occurring in the object description. This description has the form of the decision table, where the rows represent next examples, and columns the selected features (describing attributes).

Among the methods of direct induction of decision rules it is possible to distinguish procedures based on the sequential coverage of the teaching cases set. This way we obtain the learning model, which has the form of a set of rules, which covers all cases from the learning set. An example of the algorithm of direct rule induction is LEM2 used in the studies [Grzymala-Busse 1992]. It is a popular procedure for creating a minimal set of rules, suitable for classification purposes. It creates a symbolic description discriminating the approximations of each decision class using the original proposal close to the rule of generating next covers. The LEM2 algorithm at the input gets the approximation of the analysed decision class. If this is the lower approximation, certain rules are generated, if the top one – the possible rules. In the study it is also possible to consider the marginal area, then the rough rules are generated [Skowron 1993].

CHARACTERISTICS OF WORK OFFERS FOR SPECIALSITS

The empirical material used in the conducted analyses came from studies implemented in 2014 within the "Human Capital Balance" project – HCB (V edition) [http://bkl.parp.gov.pl/dane]. It included the work offers (excluding internships and apprenticeships for students and apprentices) collected in the county labour offices (CLO) for all Polish provinces and at Careerjet.pl, a national web portal for job placement (job search engine). According to the research procedure, the offers were collected from the selected 160 CLO offices (10 offices per province). The sample was selected in an exhaustive manner, and the offers valid on the given day – on March 24, 2014. In case of the CLO the offer valid on the day of the conducted study was taken into account, while from the Careerjet.pl websites the first offers registered on this day were coded.

The article focused on information concerning offers directed to the following specialists:

- physical, mathematical and technical sciences (S1),
- for health matters (S2),
- teaching and education (S3),
- for economic and management matters (S4),
- for matters of the information and communication technologies (S5),
- from the field of law, social areas and culture (S6).

In the brackets there are designations of various groups of specialists adopted in the research.

Within the HCB studies the classification of competence included two groups of professional skills: formal (qualifications) and crucial. Formal competences related to: education – its level and profile (direction), experience (seniority) – the

course of the current employment, held certificates, permissions and diplomas, and additional resources (e.g. a car, Internet access, a computer). In contrast, crucial competences included skills: cognitive (searching and analysis of information and drawing conclusions), individual (psychological), cultural, physical (fitness), interpersonal (interpersonal contacts), management, disposition, office, technical (service, mounting and repairing devices), computer (computer skills and use of the Internet), mathematical (calculation), professional [Kocór et al. 2010].

In 2014 the most sought-after professional categories included: skilled workers, machine and equipment operators, specialists and sellers and service workers. In comparison to 2013 the labour demand has increased for: general practitioners, nurses, physiotherapists, professional subjects and language teachers, specialists for economy and programmers. In the case of skilled workers we sought: tillers, turners, mechanics, carpenters and seamstresses.

Among the requirements for candidates for future employees of utmost importance in 2014 was: experience (measured by the seniority at the similar offered position), level of the education and the skill to use a foreign language. The gender of candidates has slightly lost the significance. Employers recognised as the most importance competence: self-organisational, professional and interpersonal. It can be observed that there was a division of competence requirements related to the classification of professions into mental and physical. In the mental work the language, cognitive, computer and mathematical competence were more important. While in physical professions what counted was the physical fitness and technical abilities. On the basis of the owned information derived from candidates it was observed that they had insufficient professional experience and the inadequate motivation to work. Particularly clear was the gap regarding the competence: professional, self-organisational and interpersonal, that is the ones most valued by employers.

To regions with higher demand for specialists included: Lesser Poland and Silesian, Pomeranian, Opole, Świętokrzyskie, Warmia-Masuria and Podlasie. The specialists were most often sought to work by employers operating in the sectors of specialist services and services for the public (public and private education, health care and social care).

2014 was dominated by employers, who were searching for an employee, who could work without the long training. The more complex the obligations performed at work, the higher the expectations of employers within the professional experience. Also knowledge and formally certified skills, i.e. qualifications, gained on importance. Employers required formal documents confirming: practical abilities (e.g. a driving license, knowledge of resume writing, knowledge of specific computer programs), health care (e.g. current psycho-technical tests), taking part in obligatory training being the condition to undertake employment (e.g. safety training, fire training). For employers what also mattered was the fluent knowledge of English and German, most of all. It was important particularly for candidates for the so-called mental positions that is specialists, managers and office workers. While it was of no importance for physical professions.

ANALYSIS OF THE COMPETENCE NEEDS OF THE POLISH LABOUR MARKET

The study presented in the article concerning the search for dependencies between the type of specialty, clarified with the competence profile, and the requirement of the English language proficiency involved the use of the induction algorithm of the minimal set of decision rules (LEM2). The sample size used to perform calculations was 4636 offers. These offers came from particular regions of Poland. Data were included in the decision table, containing 12 conditional attributes, which took on the {yes/no} values and concerned the crucial competence: cognitive (k1), individual (k2), cultural (k3), physical (k4), interpersonal (k5), managerial (k6), disposition (k7), office (k8), technical (k9), computer (k10), mathematical (k11), professional (k12). The thirteenth attribute was related to the specialty –Spec{S1, S2, …, S6}. While the knowledge of English {yes/no} was defined as a decision attribute.

In the first stage of the study the focus was on searching for the correctness in the whole group of offers directed to specialists (the main granule – 4636 offers). While in the second one the group was divided into 4 sub-granules, which were obtained as a result of classification of regions of Poland in terms of the wealth degree. The division into granules at this stage was taken from the work of [Becker 2014]. The empirical material included the selected macroeconomic categories and came from the CSO [Gross domestic product…]. The division was made using the method of *k*-mean [Grabiński 1992, p. 124-127], using the Statistica 10 program. The obtained sub-granules created the next classes of wealth of individual regions of Poland:

- class 1: Masovia, (sub1),
- class 2: Lower Silesia, Silesia, Greater Poland, (sub2),
- class 3: Kuyavian-Pomeranian, Lublin, Lodz, Lesser Poland, Pomeranian, West Pomeranian, (sub3),
- class 4: Lublin, Opole, Podlasie, Subcarpathian, Świętokrzyskie, Warmia-Masuria, (sub4).

Performing calculations the coherence of calculations was not studied, because the LEM2 algorithm uses this type of information for generating possible rules (certain and approximate). The occurrence of reducers was tested in each set. One reducer was obtained in sub-granule 4.In this reducer attribute k11 has not appeared – mathematical competence. Due to the lack of reducers in individual sets the exhaustive sets of rules in the whole study were not generated.

The next step of the research procedure involved the formulation of the minimal sets of decision rules for each granule. Table 1 contains the number of rules generated for two adopted values of the cover parameter – 0,9 and 0,5. This factor defines the expected degree of coverage of the training set by derived rules. Results obtained for the main granule and individual sub-granules, with the coverage coefficient equal 0,9, indicated the greater number of rules than when it was the 0,5 level.

The more numerous sets had rules with a smaller number of conditional attributes (min 8). However, their individual coverage were small (about 2, 5% for the most reduced rules). In both sets the degree of rule reduction with the highest coverage has not been significantly different.

Table1**.** Summary of test results

Scope of research	Number of offers	Division of the offer number – English (yes/no)	Number of the minimal rule set			
			j. covers = 0,9		j. covers = 0,5	
			possible	certain	possible	certain
Main granule	4636	1930/2706	359	237	54	40
Sub-granule 1	747	423/324	160	115	47	34
Sub-granule 2	1445	614/831	218	152	54	43
Sub-granule 3	1554	622/932	214	143	41	30
Sub-granule 4	882	268/614	114	77	21	12

Source: own study

The support rate was calculated, which is the percentage expressed indicator of the number of the set of cases confirming the rule and the indicator of likelihood (accuracy, absolute support), expressing the percentage of the number of the non-empty sub-set of cases representing the given concept (“yes” or “no”) for the decision attribute. The highest values of the support rate were as follows: in the main granule – 4,1%; in sub-granules 1, 2, 3, 4 respectively: 5%, 3,3%, 5,4% and 7,6%. In the case of the likelihood ratio we obtained: in the main granule –7,6%; in sub-granules 1, 2, 3, 4 appropriately: 6,4%, 5,3%, 8,3% and 7,8%.

SUMMARY OF THE RESULTS

When analysing the studies involving both all offers directed to specialists, coming from the whole Poland (the main granule) and the offers obtained from regions grouped in particular sub-granules, the strongest regularities deserve attention.

In the offers grouped in the main granule, if the offer of work concerned specialists for the information and communication technology apart from the computer and professional competences what was expected was the knowledge of the English language. This rule had the strongest coverage also in sub-granule 1 and 3. In other sub-granules this trend had a much lower support. While if the offers (the main granule) were directed to the specialists in economy and management, with individual and psychological competences, or when no competence was required, the knowledge of the English language was not necessary. In offers derived from areas focused in sub-granules 2, 3, 4 the knowledge of the English language was also not expected. Sometimes additional competences were required from candidates for the employee, e.g., computer, disposition or managerial. Only in the Masovia region

(sub-granule 1) it was expected that the economists, with competences: computer, individual, interpersonal and disposition, will be able to use English. Offers directed to specialists in teaching and education (living in Poland – the main granule) did not include expectations concerning skills of the English use. This language was also not required if the demand was reported for the individual, physical, interpersonal and managerial competence. This trend was confirmed by certain rules obtained from the sub-granule 2, 3 and 4. The exception is the sub-granule 1 (Masovia region), where this relation has not been confirmed. In the case of health specialists, when the competence requirements were not defined, also the knowledge of English was not necessary. This rule was not confirmed in sub-granule 1 (Masovia region). For specialists of physical, mathematical and technical sciences only the approximate rules were obtained. Both in the main granule and in individual sub-granules, if no competence was required from the candidates, then the knowledge of English was not expected in about 80% cases.

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OVERLAPPING MULTIGRID METHODS AS AN EFFICIENT APPROACH FOR SOLVING THE BLACK-SCHOLES EQUATION

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Abstract: In this paper the modification of a two-level multigrid method by allowing an overlap between adjacent subdomains and its application to a one-dimensional Black-Scholes equation is described. The method is based on the finite-difference schema known as implicit Euler. Numerical experiments confirm the superiority of the proposed method in relation to the classic multigrid method in form of shortening computation time, memory savings and ease of parallelization. The comparison shows the advantages of overlapping grids vs method without them, mainly due to improved accuracy of the solution.

Keywords: option pricing, Black-Scholes model, multigrid method, finite-difference scheme

INTRODUCTION

One of the fast-evolving areas of financial mathematics is the modeling of dynamics of changes of the prices of financial instruments, in particular the problem of the option pricing. Many of proposed algorithms, which were developed over the years for the partial and stochastic differential equations [Sauer 2012] and are based on the finite-difference [Knabner and Angerman 2003] or the finite-element method [Zienkiewicz et al. 2005]. There exists also a class of methods that proved to be an effective alternative to those classic numerical algorithms that is class of multigrid methods [Shapira 2003], [Wesseling 2004]. Their effectiveness has been also noticed in option pricing tasks. In this paper the modification of a two-level multigrid method by allowing an overlap between subdomains is described. This approach improve the convergence of the iteration process and the approximation of the solution at the expense of less susceptibility to parallelization. Application of this idea to a one-dimensional Black-Scholes equation is

presented. To be more exact the European call option is considered (for details on option pricing theory see [Hull 2006] or [Haug 2007]):

$$\begin{cases} \dfrac{\partial V(s,t)}{\partial t} + \dfrac{\partial V(s,t)}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V(s,t)}{\partial s^2} - rV(s,t) = 0 \\ V(s,T) = \max\{s-K,\,0\} \\ \lim\limits_{s\to 0} V(s,t) = 0 \\ \lim\limits_{s\to\infty} \dfrac{V(s,t)}{s} = 1 \end{cases} \tag{1}$$

where *s* is a stock price of the underlying asset, *t* is a time, *r* is a risk-free interest rate, σ is a volatility of the stock and K is a strike price. The solution $V:(0,\infty)\times[0,T]\to R$ is the price of the option as a function of *s* and *t* and the analytic solution is known as a Black-Scholes formula.

Over the years many methods of solving the Black-Scholes model were developed, see [Fries 2007], in particular algorithms used to solve partial differential equations were successfully adapted to options pricing. The main dissimilarity between these two applications are the boundary conditions. For tasks from science subjects boundary conditions are exactly specified. In Black-Scholes model the domain is infinite and therefore only an approximation of boundary conditions is known. This issue influences the accuracy of the solution computed by the tested method and makes much harder to developed new, reliable algorithms.

In finite-difference schema the infinite domain $(0,\infty)\times[0,T]$ is change to the finite $(S_{\min}, S_{\max})\times[0,T]$ in such a way, that all probable values of *s* are within the domain. Let *M* be the number of parts, to which the space interval was divided and *N* the number of parts being the result of the partition of the time interval. Using the implicit Euler method, see [Hull 2006] or [Pascucci 2011], the continuous task (1) could be discretized as follows

$$\begin{cases} \dfrac{V_i^{\,j} - V_i^{\,j-1}}{\tau} + ih\dfrac{V_{i+1}^{\,j-1} - V_{i-1}^{\,j-1}}{2h} + \dfrac{1}{2}\sigma^2 (ih)^2 \dfrac{V_{i-1}^{\,j-1} - 2V_i^{\,j-1} + V_{i+1}^{\,j-1}}{h^2} - rV_i^{\,j-1} = 0 \\ V_0^{\,j} = 0 \\ V_M^{\,j} = S_{\max} - Ke^{-r\tau(N-j)} \end{cases} \tag{2}$$

for $i = 0,1,2,\ldots,M$ and $j = 0,1,2,\ldots,N$, where by $V_i^{\,j} = V(S_{\min} + ih,\, j\tau)$, *h* is a step of a space interval and τ a step of a time interval.

The approximate solution is calculated only in the nodes of the grid. The finer grid the better approximation you may expect. This however involves greater computational complexity. The implicit Euler schema on each time layer requires solving system of linear equations, which is usually done by some kind of iterative meth-

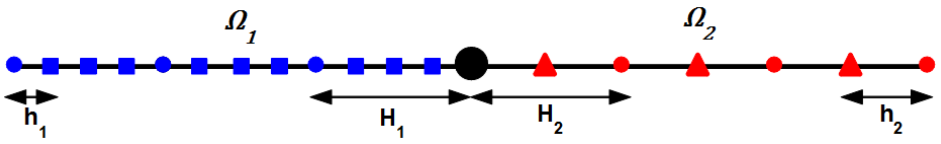
ods for sparse linear systems [Saad 2003] or dedicated methods using more sophisticated preconditioners [Bernardelli 2008], [Toselli & Widlund 2005], [Bhowmik & Stolk 2011]. In case of really big systems of linear equations, even when they are sparse, the memory and computational complexity of algorithms usually increases drastically. As an alternative, efficient approach, the multigrid methods may be considered, see [Shapira 2003], [Wesseling 2004].

THE MULTIGRID ALGORITHM FOR NON-OVERLAPPING GRIDS

The idea behind multigrid methods is to use a hierarchy of discretizations instead of one grid. The acceleration of the convergence of the iteration process is achieved by solving the task only on a coarse grid, which is faster due to the size of the problem, than solving the whole task on the fine grid. The definition of an interchange of information between coarser and finer layers is, besides the number and shape of the grids, at the same time a definition of the multigrid method. Multigrid methods are consistent with any discretization technique, starting from finite-difference [Strang 2007] and finite element [Zhu & Cangellaris 2006] methods. Also they are flexible if it comes to the regularity of the domain or boundary [Berridge & Schumacher 2002] and are known by their ease of parallelization, compare [Chiorean 2005]. In this paper the two-level multigrid method is described. For the comparison purpose two different splittings into grids are considered – the first with disjoint partitions and the second with overlaps.

Consider the decomposition of the space domain on each time layer into two disjoint subsets Ω_1 and Ω_2. Let Ω_H be the set of nodes on the coarse grid and by Ω_{H1} and Ω_{H2} denote Ω_H restricted to Ω_1 and Ω_2 respectively. We assume that these grids are uniform with the steps equal H_1 and H_2. Analogously let the fine grid Ω_h be decomposed into two subsets Ω_{h1} and Ω_{h2} with the steps denoted by h_1 and h_2 respectively. There is a simplifying assumption taken that the nodes of the coarse grid, are also nodes of the corresponding fine grid. The graphical representation of the two-level grid is presented on Figure 1.

Figure 1. Coarse grid and decomposition of the fine grid without overlap



Source: own elaboration

Let A_{H1}, A_{H1}, A_{h1} and A_{h2} be the matrices of the systems of linear equations related to the grids Ω_{H1}, Ω_{H2}, Ω_{h1} and Ω_{h2} respectively. Let V_{H1}^j, V_{H2}^j, V_{h1}^j and V_{h2}^j be the solutions of these systems of linear equations on the time layer $j = 0, 1, 2, ..., N-1$.

For the simplicity of notation let $V_H^j = V_{H1}^j \cup V_{H2}^j$ and $V_h^j = V_{h1}^j \cup V_{h2}^j$. To describe the multigrid method let introduce two operators:

- $R : \Omega_h \to \Omega_H$ – a restriction operator from the fine to the coarse grid,
- $I : \Omega_H \to \Omega_h$ – a prolongation operator from the coarse to the fine grid.

The algorithm for the non-overlapping decomposition of the grids could be describe in the following three steps on each time layer[1] $j = N, N-1, N-2, \ldots, 2, 1$:

STEP 1: Use the restriction operator R to calculate right side vectors of the matrix equations for the coarse grid

$$V_H^j = RV_h^j . \tag{3}$$

STEP 2: Use the newest calculated value at the common point of the two subdomains as the boundary condition for two systems of linear equations ($i = 1, 2$):

$$A_{Hi} V_{Hi}^{j+1} = V_{Hi}^j . \tag{4}$$

Solve these matrix equations in parallel.

STEP 3: Use the prolongation operator I to calculate the values of the solution in the nodes of the fine grid

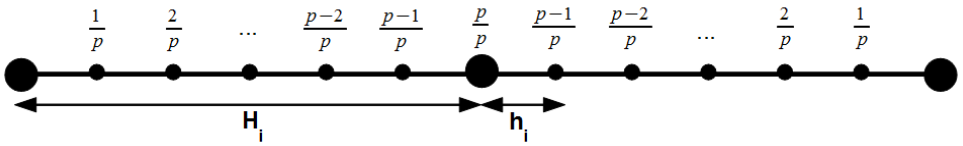
$$V_h^{j-1} = IV_H^j . \tag{5}$$

It can be shown [Strang 2007] that for the appropriately chosen operators R and I solutions of the equation defined on the fine and on the coarse grid are equivalent. In this paper we assume, that

$$R = \frac{1}{p} I^T \ , \tag{6}$$

if the step of the coarse grid is p times greater than the step of the fine grid. Interpolation operator I is defined on the stencil presented on Figure 2.

Figure 2. Graphical representation of the stencil for the interpolation operator I



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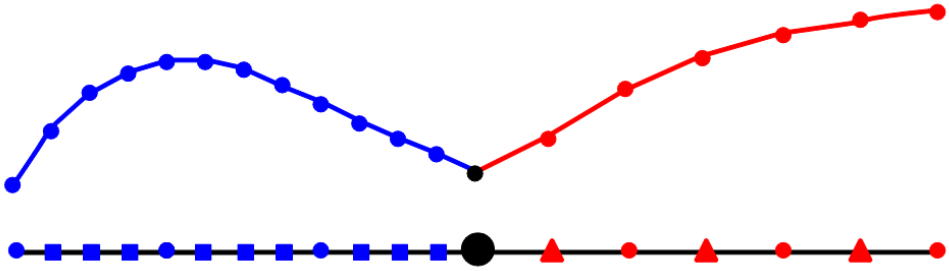
The solution limited to each subdomain should be smooth, but unfortunately, such an approach can result in the lack of smoothness in the whole domain. Demonstration of such a situation is given on Figure 3. The remedy could be an algorithm, which uses decomposition with overlapping grids.

[1] Black-Scholes equation is backward in time.

Figure 3. Example of non-smooth solution obtained by the multigrid algorithm with disjoint decomposition of the grids



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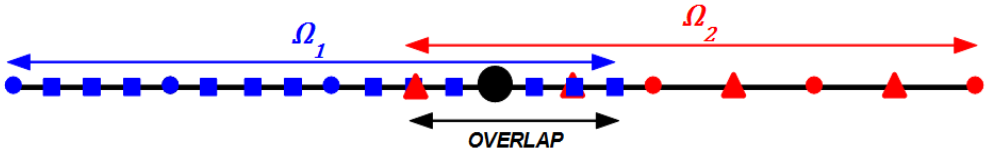
THE MULTIGRID ALGORITHM FOR OVERLAPPING GRIDS

A modification of the algorithm described in the previous subsection is made, by letting grids to overlap. Visualization of this idea is given on Figure 4. The concept of splitting the domain into smaller non-disjoint pieces is not new, [Toselli & Widlund 2005] or [Henshaw 2005], but explored rather only for partial differential equations, which have strict boundary conditions specified. Contrary, in the Black-Scholes equation only asymptotic character of the boundary conditions can be established. Therefore construction of algorithms for option pricing is much harder than it is for example for elliptic partial differential equations.

Figure 4. Coarse grid and decomposition of the fine grid with overlap



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To describe the algorithm presented already notation with minor changes can be used. For the clarity of the description we decompose the prolongation I and the restriction R operators into separate operators related to each subdomain, that is I_1, R_1 for Ω_1 and I_2, R_2 for Ω_2.

The algorithm for the overlapping decomposition of the grids could be summarized in the following steps on each time layer $j = N, N\text{-}1, N\text{-}2, \ldots, 2, 1$:

STEP 1: Use the restriction operator R_1 to calculate right side vector of the matrix equations for the coarse grid of the Ω_1:

$$V_{H1}^{j} = R_1 V_{h1}^{j}. \tag{7}$$

STEP 2: Based on values from the previous time layer use linear interpolation to calculate the value at the right end of the subdomain Ω_1 and use it as the boundary condition for the system of linear equations:

$$A_{H1}V_{H1}^{j+1} = V_{H1}^{j}. \tag{8}$$

STEP 3: Use the prolongation operator I_1 to calculate the values of the solution in the nodes of the fine grid of the Ω_1:

$$V_{h1}^{j-1} = I_1 V_{H1}^{j}. \tag{9}$$

STEP 4: Use the restriction operator R_2 to calculate right side vector of the matrix equations for the coarse grid of the Ω_2:

$$V_{H2}^{j} = R_2 V_{h2}^{j}. \tag{10}$$

STEP 5: Based on values from step 3 use linear interpolation to calculate the value at the left end of the subdomain Ω_2 and use it as the boundary condition for the system of linear equations:

$$A_{H2}V_{H2}^{j+1} = V_{H2}^{j}. \tag{11}$$

STEP 6: Use the prolongation operator I_2 to calculate the values of the solution in the nodes of the fine grid of the Ω_2:

$$V_{h2}^{j-1} = I_2 V_{H2}^{j}. \tag{12}$$

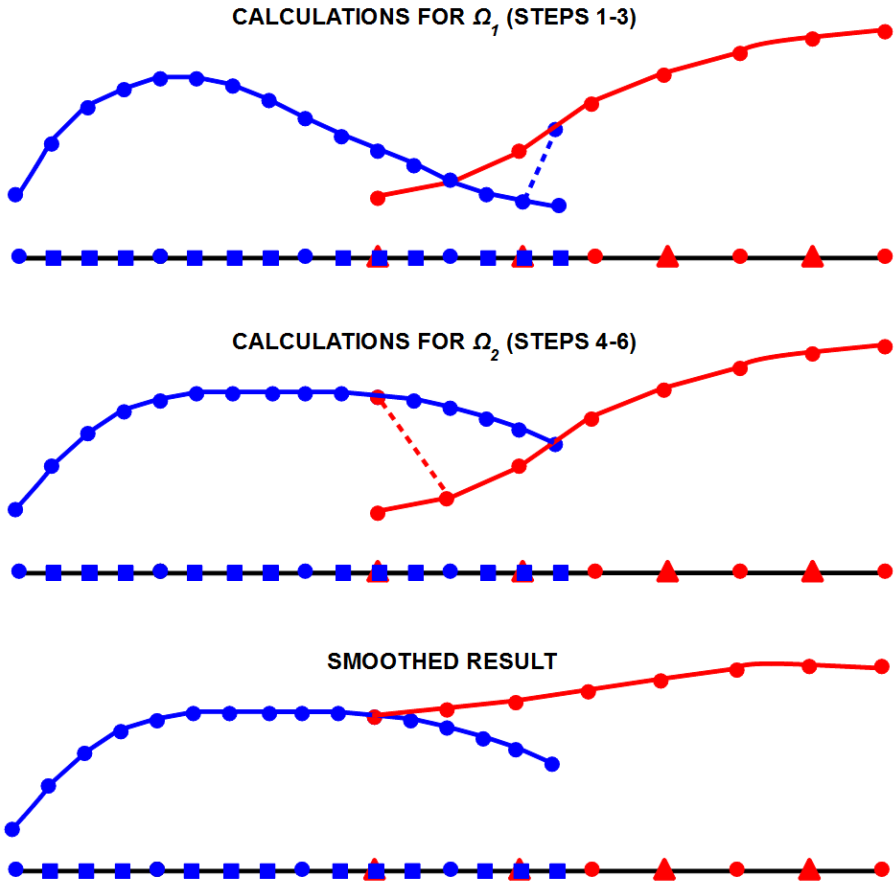
The exemplary difference between results of two algorithms on the given time layer is visualized on Figure 3 (non-overlapping grids) and Figure 5 (overlapping grids). Introducing overlaps firstly gives a fresh information about the boundary conditions for computations on each subdomain, and secondly makes the final solution smoother. Of course there are some disadvantages of this approach. Most of all computations for Ω_1 and Ω_2 can't be performed in parallel, steps 4-6 may be executed only after steps 1-3 are finished. The difference in performance however could be observed probably only on a computer cluster with many machines, not on the standard, even multi-processor computer.

NUMERICAL EXPERIMENTS

To verify the effectiveness of the proposed method with overlapping grids numerical experiments were conducted. As the test case the European call option problem (1) with the following parameters was chosen

$$T = 2,\ K = 10,\ r = 0.2,\ \sigma = 0.25,\ S_{min} = 0,\ S_{max} = 20. \tag{13}$$

Figure 5. The example of smoothed solution obtained by the multigrid algorithm with decomposition of the grids with overlaps



Source: own elaboration

Other parameters were changed during the experiments. The complete list of the parameters is as follows:
M – number of parts of the space interval (number of grid nodes on each time layer is equal $M+1$),
N – number of time layers used in the calculation, in experiments assumed as $N = round\left(M\left(2\max\{p_1, p_2\}\right)^{-1}\right)$
$overlap_1$, $overlap_2$ – size of the overlap measured as a number of fine grid points outside the corresponding subdomain,
p_1, p_2 – ratio of the step sizes between the coarse and fine grids on the subdomains, that is $p_i = H_i h_i^{-1}$ for $i = 1, 2$.

For every result there were measured the time of the computation and two types of errors:

- maximum residual error ($NORM_\infty$): $\left\| V - V^* \right\|_\infty = \max_{i=0,1,\ldots M} \left| V_i^0 - \tilde{V}_i^0 \right|$,
- absolute error at K (ABS_K): $\left| V(K) - \tilde{V}(K) \right|$,

where V is the computed solution and $\tilde{V}$ is the analytic solution from the Black-Scholes formula. The series of numerical experiments were divided into parts designed to verify dependence of errors on the ratio of number of grid nodes between subdomains, size of overlaps and ratio of the step sizes between the coarse and fine grids on the subdomains. Results of each part of the experiments and their brief descriptions are given in the remaining part of this section.

Selected results of the numerical experiments that verified the dependence between solution's accuracy and the number of grid nodes of the two subdomains are presented in Table 1.

Table 1. Dependence between errors and the ratio of number of grid nodes of two subdomains

Ratio	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
10% : 90%	12 000	0.030552	0.102905	0.009131	0.000673
30% : 70%		0.018038	0.011974	0.012346	0.000560
50% : 50%		0.015608	**0.011030**	0.011778	**0.000081**
70% : 30%		**0.014711**	0.017558	0.010776	0.002198
90% : 10%		0.033411	0.048296	**0.004385**	0.038145
10% : 90%	60 000	0.014191	0.021621	0.010784	0.000143
30% : 70%		0.014462	**0.002535**	0.011386	0.000120
50% : 50%		0.014556	0.003103	0.011268	**0.000024**
70% : 30%		0.014474	0.004428	0.011065	0.000431
90% : 10%		**0.013784**	0.010971	**0.009760**	0.007581

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $p_1 = p_2 = 6$, $overlap_1 = overlap_2 = 5$. In bold are the smallest values of errors for each M.

Source: own calculations

Maximum residual error ($NORM_\infty$) for the algorithm with overlaps seems to be worse or at most comparable to corresponding errors of the algorithm with disjoint decomposition of grids. In contrast, absolute error at K (ABS_K) is (except the extreme case of ratio 90% : 10%) much smaller for the algorithm with the overlap modification. In practice we are rather interested in the best possible value at the given point (or interval containing that point) not in the solution in the whole domain. This is why minimization of ABS_K error is desirable. Due to the computation complexity and given in Table 1 results in other two parts of the numerical experiment the ratio 50% : 50% was used.

The second part of the series of numerical experiments is related to the clue of the idea behind the described algorithm. It is intended to check how the size of an overlap influence the value of the errors. Selected results of these part of experiments are presented in Table 2.

Table 2. Dependence between errors and the size of overlaps, assuming $overlap_1 = overlap_2$

Overlap	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
1	12 000	**0.015608**	**0.008397**	**0.011778**	0.002106
2			0.009056		0.000317
3			0.009714		0.002023
5			0.011030		0.000081
6			0.011686		**0.000073**
7			0.012342		0.001103
1	60 000	**0.014556**	**0.002573**	**0.011268**	0.0004316
2			0.002705		0.0000719
3			0.002838		0.0004144
5			0.003103		0.0000244
6			0.003236		**0.0000065**
7			0.003368		0.0002299

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $p_1 = p_2 = 6$, ratio between number of fine grid nodes of two subdomains 50% : 50%. In bold are the smallest values of errors for each *M*.

Source: own calculations

In any considered case adding an overlap improved the accuracy of the solution. An observable fact resulting from comparison between different sizes of an overlap, is that increasing an overlap increases the $NORM_\infty$ error. Because of the approximation of the derivatives that was used in the discretization, the size of an overlap should be equal one step of the fine grid. Nevertheless concerning the second measured error (ABS_K) we can noticed, that it is rather effective to increase an overlap (taking into account the size *M* of the grid on the space interval). It seems to improve the accuracy (measured as a point error) of the solution for any reasonable set of parameters. In almost all numerical experiments parts of the experiments the size of an overlap was set to 5.

The last, third part of numerical experiments is directly related to the key concept of the multigrid method. There were measured errors while changing the parameters p_1 and p_2. Results are gathered in Table 3. As was expected, decreasing the size of the coarse grid makes errors greater. However ABS_K error of the solutions of the algorithm with overlapping grids may be and, depending on the size of the problem, usually is smaller for $p_1 = p_2 \in \{3,6\}$. Of course for greater values of considered in this part of numerical experiments parameters computations are noticeably faster.

Table 3. Dependence between errors and the ratio of the step sizes between the coarse and fine grids on the subdomains

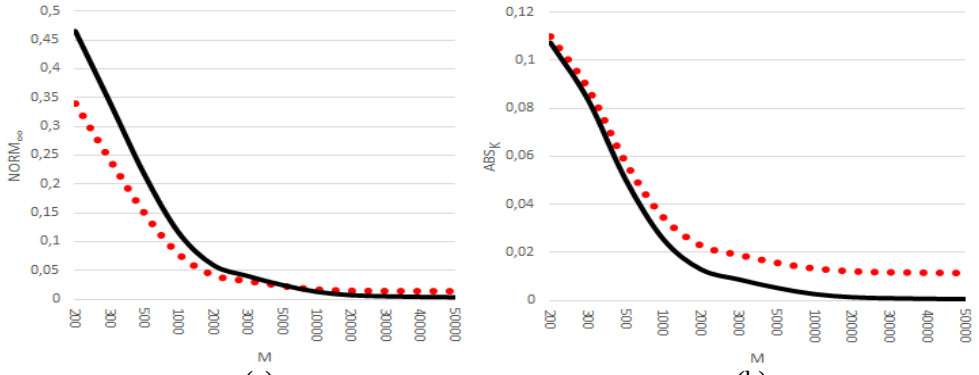
$p_1 = p_2$	M	$NORM_\infty$		ABS_K	
		NO	YES	NO	YES
2	10 000	**0.014518**	**0.003498**	**0.011335**	0.000352
3		0.014729	0.005880	0.011468	**0.000026**
4		0.015056	0.008259	0.011611	0.001133
6		0.016121	0.013006	0.011901	0.000091
10		0.019881	0.022429	0.012500	0.003678
2	60 000	**0.014462**	**0.001510**	**0.011173**	3.91531e-06
3		0.014478	0.001908	0.011195	**3.85165e-06**
4		0.014501	0.002307	0.011219	9.00507e-06
6		0.014556	0.003103	0.011268	2.44427e-05
10		0.014694	0.004694	0.011368	6.16543e-05

Source: own calculations

NO – algorithm without overlaps, YES – algorithm with overlapping grids. Used parameters: $overlap_1 = overlap_2 = 5$, ratio between number of fine grid nodes of two subdomains 50% : 50%. In bold are the smallest values of errors for each M.

In case of both measured errors algorithm with overlaps is performing considerably better than the algorithm with disjoint subdomains. The dependence of the $NORM_\infty$ and the ABS_K error on the value of M is given on Figure 6.

Figure 6. The value of the (a) $NORM_\infty$ error and (b) ABS_K error with increasing value of M. Used parameters: $overlap_1 = overlap_2 = 5$, $p_1 = p_2 = 6$, ratio between number of fine grid nodes of two subdomains 50% : 50%. Dotted line – no overlaps, continuous line – with overlaps



(a) (b)

Source: own elaboration

SUMMARY

In this paper the proposition of the multigrid method with overlapping grids is presented. The description includes the specification of the algorithm steps, its pros and cons as well as results of the numerical experiments. The method is designed to solve one-dimensional Black-Scholes problem for the European call option and was compared to the analogous method without overlaps. Based on the theory and numerical experiments one can formulate the following conclusions. Comparing to the method without overlapping grids presented method has

- improve convergence rate,
- better accuracy of the solution,
- comparable memory and computer power requirements,
- comparable time of computations,
- less parallelization possibilities.

Comparing to the multigrid method without decomposition of the domain proposed method has

- slightly worse accuracy of the solution,
- much faster time of computations,
- more parallelization possibilities,
- much less memory and computer power requirements.

Nevertheless the one-dimensional task is considered as reasonably easy to solve. The true advantage of the described method should become visible in more than one dimension. Such a generalization is one of the main goals of the further research. Also worth considering would be implementation for more than two subdomains and most of all checking the effectiveness of the method for other option pricing problems, especially those, for which the analytic solution is unknown. Generalization of the classic Black-Scholes model could include for example American options, model with variable coefficients or adding a dividend term to the model.

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SHORT TERM AGGREGATED SUPPLY CURVES FOR OECD COUNTRIES IN 1991 – 2013

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Abstract: Politicians used to determine such macroeconomic targets as GDP growth rate, inflation rate, unemployment rate. They are interested in relationships between: unemployment rate and GDP growth rate (the Okun's law), unemployment rate and inflation rate (the Philips curve) as well as between inflation rate and GDP growth rate (aggregate supply curve - SAS), put forth directly, eg. a straight line or a parabola, or indirectly, i.e. as a function Okun's with inserted Philips function. SAS derived from the Okun's and Philips curves estimated for OECD countries in the period 1991 to 2013 are analysed *vis a vis* curves reflecting direct relationships.

Keywords: Okun's law, short-term Philips curve, neutral inflation, short-term aggregate demand curve, B curve, OECD countries

INTRODUCTION

Economic policymakers determine usually a few strategic macroeconomic targets, for instance, the GDP growth rate, the inflation rate band, the range of the unemployment rate, the range of exchange rate, stability of public finances, improvement of infrastructure, increasing of living standards of citizens.

Majority (if not all) of these targets are expressed, directly or indirectly, in quantitative terms. Both macroeconomic policymakers and macroeconomic theoreticians are interested not only in the levels and dynamics of every of these targets but also in interdependencies between them, especially between GDP growth rate, inflation rate and unemployment rate. Relationships between two of the three are known as Okun's curve, Philips curve and the aggregate supply curve, respectively.

The aim of the paper is the show that any of the four short term aggregate supply curves, derived separately for every OECD country on the basis of two Okun's curves and two Philips curves, estimated on quarterly data 1991Q1 - 2013Q4, are inferior *vis*

a vis the two respective curves, reflecting the direct relationships, estimated on the same data sample.

RELATIONSHIPS BETWEEN MACROECONOMIC POLICY TARGETS

Let us assume that there are three macroeconomic policy targets: GDP growth rate, [r(GDP)], inflation rate, [r(p)], and unemployment rate, [UNR]. Between these targets there are, in particular, relationships between any two of them.

The relationship between UNR and r(GDP) is known as the Okun's law:

$$r(GDP) = f(UNR, \xi_{UG}) \quad (1)$$

where ξ_{UG} is the random variable (the letters U and G correspond to unemployment and r(GDP) thus indicating the independent variable and the dependent one.

The Okun's law is also expressed by a number of formulas[1]. The original formula adopted by A. Okun has the following form [Okun 1962]:

$$r(GDP) - r(pGDP) = f[UNR - t(UNR), \xi_{UG}] \quad (2)$$

where r(pGDP) is the potential r(GDP), t(UNR) is the long term UNR trend.

In turn, the relationship between UNR and r(p) is most often expressed by the short term Philips curve [Philips 1958][2]:

$$r(p) = f\,(UNR, \xi_{Up}) \quad (3)$$

where ξ_{Up} is the random variable (the letter p corresponds to r(p) thus indicating the independent variable. The concept of this curve has been widely criticized. Its critics have been awarded six times by the Nobel Prize Committee [Domitrovic 2011]. The subject of criticism, in particular, were proven by M. Friedman [1963] and E. S. Phelps [1968, pp. 678-711] the lack of applicability to long-term analyses and the assumption that the level of inflation is equal to nominal wage dynamics as well as the coexistence of the two phenomena instead of their time sequence [Machaj 2013] causing wage-inflation spiral that may be properly solved only by taking into account the proposals of T. Sargent and C. Sims concerning methods of analysis of causes and effects in the economy (the VAR model)
[http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2011/press.html].
Nevertheless Philips curve is still recognized in theory and applied in practice [Domitrovic 2011].

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[1] Some of them are given in main text and in references of: Shevchuk [2010, pp. 75-90]. A comprehensive list of empirical studies is given, i. a., in Durech et al. [2014, pp. 57-65] and in Elsby et al. [2010].

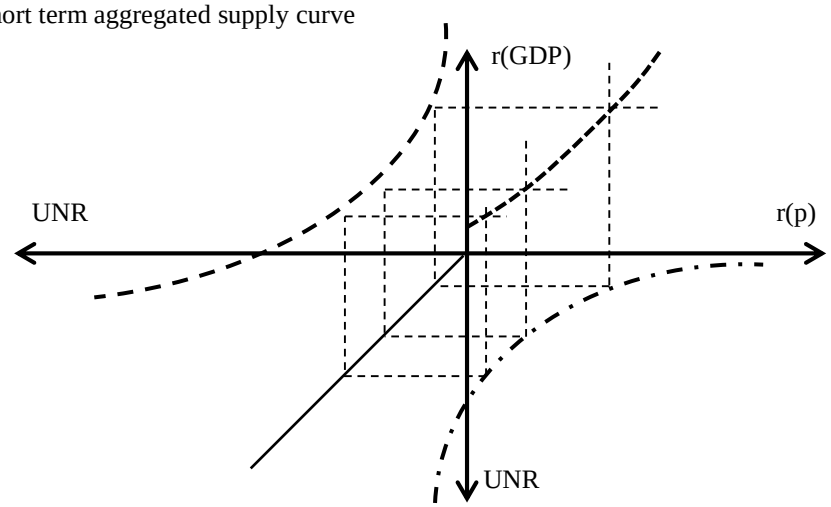
[2] There is a very vast literature on this subject including many papers comparing results of different investigations. See for instance, Carre [2010], Daniskova and Fidrmuc as well as Mavroeidis et al. [2014]. There are thousands of empirical analyses based on models with different specifications for the USA only.

Also, the relationship between r(p) and r(GDP) known as short term aggregated supply curve (SAS) is subject to both theoretical analyses and practical considerations [Błaszczuk 2014b][3]. It can be obtained analytically substituting the inverted Philips curve (3) into the Okun's curve (1):

$$r(GDP) = f\{G[r(p)]\} \quad (4)$$

or derived graphically (Figure 1).

Figure 1. Short term aggregated supply curve



Source: own elaboration based on [Burda and Wyplosz 1993, Chapter 11.5.3]

The shape of SAS depends, of course, on its analytical form, which, in turn, depends on the analytical forms of the respective Okun and Philips functions. In specific cases, it may be a linear function.

From the practical point of view, relationships between r(p) and r(GDP) should account for the J. Bednarczyk's [2011] concept of neutral inflation, which means the level of inflation in the economy ensuring its maximum r(GDP)[4]. In addition, J. Bednarczyk [2012] claims that the level of neutral inflation is different for different countries in different time points and that it does not need neither to be close to any level, decided in an arbitrary manner, as is the case in practice, for example 0% or 2%, nor be less than, for example, enlarged by 1.5 p.p. average level of inflation in the three EU countries with the lowest levels of inflation (excluding unusual situations).

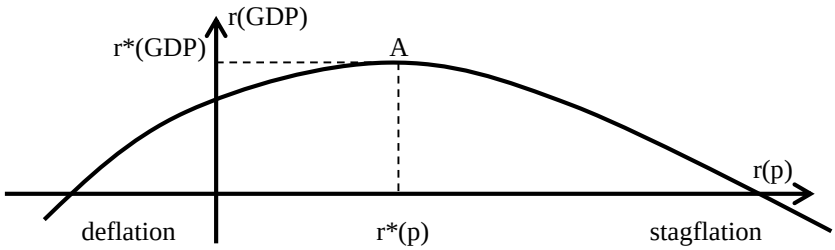
[3] Comprehensive presentations on the theoretical and empirical studies on this curve are given, for instance, in: Ball et al. [1988] and in: Mankiw and Reis [2010].

[4] In my opinion, the neutral inflation is the inflation that occurs when the GDP growth rate is maximum, so it is co-occurring with, not ensuring, the maximum GDP growth rate.

Moreover, in practice, the low pace of GDP growth rates, in particular close to 0, and even negative, coexist with levels of inflation, both much higher than the neutral inflation, referred to as stagflation, as well as the lower one - referred to as deflation that causes fear of monetary authorities[5].

In view of the above, a hypothesis was bet that the SAS takes the form, more or less, similar to the shape of the curve B shown in Figure 2 that reflects both the neutral inflation, r*(p), and the corresponding maximum GDP growth rate, r*(GDP), as well as the low rates of GDP going hand in hand, as is the case in practice, both with a negative inflation rates and its high levels.

Figure 2. The relationship between inflation rates and the GDP growth rates (B curve)



Source: own elaboration

Further on the SAS curve and the B curve are subject to empirical verification.

ASSUMPTIONS OF THE INVESTIGATION. SOURCES OF STATISTICAL DATA. METHOD OF ANALYSIS

Firstly, it has been assumed that the analysis cover all the OECD countries in the period of, more or less, the last two Juglar cycles, i.e. 1990Q1-2013Q4.

Secondly, it has been assumed that the OECD statistics would be used to ensure the comparability of the results. Therefore there were considered:

a) Harmonised Consumer Prices (HCPI) - all items, percentage changes from previous period, seasonally adjusted, http://stats.oecd.org/index.aspx?DatasetCode=MEI_PRICES [28 Apr 2014];
b) Harmonised Unemployment Rate (HUNR): all persons, seasonally adjusted, http://stats.oecd.org/index.aspx?DatasetCode=KEI [28 Apr 2014];
c) Growth Rate of Gross Domestic Product (r(GDP)) - expenditure approach, growth rate compared to previous quarter, seasonally adjusted, http://stats.oecd.org/WBOS/index.aspx [28 Apr 2014].

[5] This is because central bank cannot effectively counteract an increase of real interest rates as its interest rates may not be cut down below 0. Therefore, in order to fight deflation, it cuts interest rates in advance and, usually, increases the liquidity of commercial banks on a large scale, thus encouraging them to increase the supply of cheap loans to boost demand, which in turn will result in an increase in prices [Wojtyna 2004, pp. 252-277].

Unfortunately, these data do not correspond in full to the assumptions adopted. Namely, data on:

a) r(GDP) lack for Greece and for other countries start from different time points;
b) HCPI are lacking for 8 countries[6] and are available usually for periods shorter than for r(GDP) and HUNR, sometimes for relatively short time periods[7].

Moreover, in some cases estimated data (estimates) are given and in some other there are breaks (in collection and/or aggregation methods). In the absence of other information, estimates have been adopted, and the information about the breaks have been totally ignored.

Next it has been assumed for each country separately that r(GDP) dependence upon HUNR may be explained by the hyperbolic (h) and logarithmic (l) functions with intercept [Błaszczuk 2014a, p. 49]:

$$r^{hUG}{}_{kt}(GDP) = b^{hUG}{}_{0k} + b^{hUG}{}_{1k}\,(1/HUNR_{kt}) + \xi^{hUG}{}_{kt} \qquad (5.1)$$

$$r^{lUG}{}_{kt}(GDP) = b^{lUG}{}_{0k} + b^{lUG}{}_{1k}\,\ln(HUNR_{kt}) + \xi^{lUG}{}_{kt} \qquad (5.2)$$

where: k = 1, 2, … – country number and t = 1, 2, … – quarter number.

It has been adopted, therefore, that the increase in UNR causes slower and slower declines in r(GDP) and, moreover, it has been taken into account, appearing in practice, the possibility of a negative r(GDP).

The same functions have been applied for relationships between HUNR and HCPI [Błaszczuk, 2015, p. 17]:

$$HCPI_{kt} = b^{hUp}{}_{0k} + b^{hUp}{}_{1k}\,(1/HUNR_{kt}) + \xi^{hUp}{}_{kt} \qquad (6.1)$$

$$HCPI_{kt} = b^{lUp}{}_{0k} + b^{lUp}{}_{1k}\,\ln(HUNR_{kt}) + \xi^{lUp}{}_{kt} \qquad (6.2)$$

It has been adopted, therefore, that the increase in UNR causes slower and slower declines in r(p), and it has been taken into account, occurring in practice, possibility of deflation when HUNR are high.

Combining each of the Okun's functions with each of the Philips ones four SAS functions have been computed: SAS hh, SAS hl, SAS lh and SAS ll.

Then it has been assumed that r(GDP) may be explained by the polynomial function of the second degree of HCPI [Błaszczuk, 2014b, pp. 36-37]:

$$r_{kt}(GDP) = b^{pG}{}_{0k} + b^{pG}{}_{1k}(HCPI_{kt}) + b^{pG}{}_{2k}(HCPI_{kt})^2 + \xi^{pG}{}_{jt} \qquad (7)$$

Every of the five functions were estimated separately for each of the 25 countries, whereby the numbers of observations were not always the same. In all estimations it has been assumed that OLS may be used taking into account, inter alia, that the functions (7.1) trough (9) are linear after appropriate transformations.

[6] Australia, Canada, Chile, Israel, Japan, Korea, Mexico and New Zealand.

[7] Switzerland: 15 observations, Turkey: 35 observations; Iceland: 44 observations, in other cases: more than 60 observations.

RESULTS OF ANALYSIS

The results of the research are partially in accordance with expectations (for detailed results of estimation of the Okun's curve, the Philips curve and the B curve see Błaszczuk D. J. (2014a), Błaszczuk D. J. (2015) and Błaszczuk D. J. (2014b), respectively). Above all, do not dismay relatively low, and sometimes even very low, values of R^2 because dispersions of points on the vast majority of the 3 times 25 charts reflecting respective relationships do not allow to assume in advance any trends.

Okun's functions

It is worth noticing that, in general, the type of function does not affect the statistical significance of estimates of the structural parameters, and estimates of structural parameters for only nine countries are clearly statistically significant (t stat > 2,00)[8]. It is not exactly true in case of estimates of intercepts.

The estimates of the structural parameter in the logarithmic models of 8 countries[9] are clearly irrelevant statistically (t stat (b^L_{1j}) < 1.00)[10]. On the other hand estimates of intercept are clearly irrelevant statistically in 10 cases, however somewhat for different countries[11].

Slightly better results in this respect there were obtained in case of hyperbolic models. Namely, the United Kingdom has disappeared from the list of countries with clearly statistically irrelevant estimates of structural parameters, and as many as 6 countries[12] have disappeared from the list of countries with clearly statistically irrelevant estimates of intercept, but 4 new[13] popped up on it.

It is worth noting that, in general, type of function does not affect the statistical significance of the estimates of the structural parameters

Surprising is a significant convergence of both theoretical lines (due to a high negative correlation between estimates of the structural parameters of both functions) practically for all countries despite the fact that the values of the explaining variable are clearly (and sometimes by far) different from unity[14].

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[8] Belgium, Czech Republic, Finland, France, Hungary, Irland, Portugal, Spain and Switzerland.

[9] Denmark, Estonia, Germany, Poland, the Slovak R., Slovenia, Turkey and the UK.

[10] Expected and unexpected results have been also received by other authors, for instance, Klimczyk and Wronowska [2010, pp. 263-272].

[11] From the list disappeared: Poland and the United Kingdom and appeared on it: Austria, the Netherlands, Norway and Sweden.

[12] All except for Estonia and Turkey.

[13] Namely: Iceland, Luxembourg, the United Kingdom and the United States.

[14] In linear models with explanatory variables ln x and 1/x, respectively, a theoretical values for $x > 0$ but close to 1 are very similar because there is a dual weak inequality: $1 - 1/x = \ln x = x - 1$. Therefore in such a case the values of the coefficients do not play a significant role in determining theoretical values of the explained variable. In particular, for $1 \leq x \leq 2$, $1 - 1/x$ is a pretty good estimate of the bottom value of ln x. [Kuratowski [1971].

Finally, it should be noted that logarithmic lines are compatible with the expected ones in case of 11 countries[15] while in case of hyperbolic models such a compatibility takes place only in case of 9 countries[16]. The estimates of structural coefficients are statistically significant in each case for the same 4 countries[17] only.

Philips functions

The type of the function basically does not affect the statistical significance of the structural coefficients. The values of them are clearly irrelevant statistically in case of logarithmic models for 10 countries[18], and, in the case of hyperbolic ones Poland complements this list. Evaluations of structural coefficients are more important statistically in the case of the hyperbolic models, with the exception of four countries[19] for which they are only slightly lower from their counterparts.

Slightly different results have been obtained for the intercept. Only for three countries[20] the type of function does not affect the assessment of its statistical significance. Estimates of intercept are by far more important statistically in case of logarithmic models, with the exception of all six countries, for which values of the structural coefficients have opposite signs to the typical Philips curve[21], and Spain, Germany and Switzerland. In case of logarithmic models they were clearly insignificant statistically only for three countries[22], and, in case of hyperbolic models list of clearly irrelevant statistically intercepts covered 9 countries[23].

In this context, it should be noted that no matter what type of function in case of only six countries[24] theoretical lines are incompatible with the expected ones but only for two countries[25] the evaluations of the structural coefficients are statistically significant. For the remaining 19 countries evaluations of structural coefficients are statistically significant only in the case of seven countries[26].

[14] Expected and unexpected results have been also received by other authors, for instance, Klimczyk and Wronowska [2010, pp. 263-272].
[15] Hungary, Iceland, Ireland, Luxembourg, Poland, Portugal, Slovenia, Spain, Turkey, the United Kingdom and the United States.
[16] Poland and Slovenia have disappeared from the both lists above.
[17] Hungary, Ireland, Portugal and Spain. It should be noted that three of four remaining belong to the GIPSI group subject to huge problems during the last economic crisis.
[18] Germany, Hungary, Iceland, Italy, Luxembourg, Norway, the Slovak Republic, Spain, Turkey and the United States.
[19] Poland, the Slovak Republic, Slovenia and Turkey.
[20] Slovenia, Spain and Switzerland.
[21] Hungary, Iceland, Italy, Luxembourg, the Slovak Republic and the United Kingdom.
[22] Hungary, Italy and the United Kingdom.
[23] Austria, Belgium, the Czech R., Estonia, Finland, the Netherlands, Norway, Turkey and USA.
[24] Hungary, Iceland, Italy, Luxembourg, the Slovak Republic and the United Kingdom.
[25] The Slovak Republic and the United Kingdom.*
[26] Estonia, Finland, France, Ireland, the Netherlands, Portugal and Sweden.

SAS and B curves

Of the 25 countries under analysis, shapes of both the Philips curve and the Okun one were simultaneously in line with expectations (typical) for only 5 countries in case of both hyperbolic curves and only for two more in case of both logarithmic curves (see Table 1). Thus, typicality/non-typicality of the shape of the curve does not depend on the type of the function in 23 cases.

Table 1. Typicality of hyperbolic (h) and logarithmic (l) Okun (O) and Philips (Ph) curves

Country	O		Ph		Country	O		Ph		Country	O		Ph	
	h	l	h	l		h	l	h	l		h	l	h	l
Austria			+	+	Iceland	+	+			Slovenia		+	+	+
Belgium			+	+	Ireland	+	+	+	+	Spain	+	+	+	+
Czech R.			+	+	Italy					Sweden			+	+
Denmark			+	+	Luxembourg	+	+			Switzerland			+	+
Estonia			+	+	Netherlands			+	+	Turkey	+	+	+	+
Finland			+	+	Norway			+	+	United Kingdom	+	+		
France			+	+	Poland		+	+	+	United States	+	+	+	+
Germany			+	+	Portugal	+	+	+	+	X				
Hungary	+	+			Slovak R.					X				

Source: own elaboration

As a result, there are 5 different situations (see Table 2 and Annexes 1 through 5[27]). As expected, the SAS hh [long broken line] and SAS ll [starred line] are linear while the other two - SAS hl [exponential solid line] and SAS lh [logarithmic broken line] - have respective nonlinear shapes. Both linear functions practically overlap in all 23 cases when the typicality of the shape of the SAS curve does not depend on the type of O and Ph functions. Moreover, all four SAS curves match themselves in the proximity of the average values of (p) and r(GDP).

[27] HCPI values occur on the x-coordinate while r(GDP) on the y-coordinate. The degree of convexity/concavity of the respective curves is not fully comparable between the countries because of the different calibration of the axes.

Table 2. Combinations of typicality/non-typicality of the Okun and Philips curves

Curve		Okun (O)		Total
		typical	non-typical	
Philips (Ph)	typical	5[a)]	12[b)]	19
	non-typical	4	2	6
Total		9[a)]	14[b)]	25

Note: a) +2 in case of both logarithmic curves. (b) + 2 in case of both hyperbolic curves as well as in case of Okun hyperbolic curve and any Philips curve.

Source: own elaboration

In no case the shape of SAS curve reflects dependencies between r(p) and r(GDP) observed in practice described by the linear trend (solid straight line) or polynomial of the second order one (solid parabolic line). Moreover, the goodness of fit measured by R^2 in case of the polynomial functions is better from a few dozen percent to ten times than in case of the linear functions.

In all five cases with typical shapes of O and Ph curves and two with both non-typical ones, all the SAS curves have positive slopes. In first case SAS hl are increasing faster and faster while SAS lh are increasing slower and slower and opposite is true in the other case. The linear trends have negative slopes for Turkey in the first group and the Slovak Republic in the second while B curves have typical shapes for all countries.

There is a mirror reflection of the above in all twelve cases with non-typical shapes of O curves and typical Ph ones and all four cases with typical shapes of O curves and non-typical Ph ones. Namely, all the SAS curves have negative slopes. Moreover, in the first case SAS hl are decreasing faster and faster while SAS lh are decreasing slower and slower and opposite is true in the other case. In the first group of countries the linear trends have negative slopes for the Czech Republic, Finland and the Netherlands while B curves have untypical shapes for Denmark and Sweden. In the second group the linear trend has positive slope for Hungary only while B curve has untypical shapes for Iceland only.

Somewhat different pictures are in case of the two countries with non-typical hyperbolic Okun's functions. Negative slopes have SAS hh and SAS hl while positive have SAS lh, SAS ll and the linear trend that is not far from SAS ll. B curve has typical shape for Slovenia only.

Summing up, in cases when the (linear) SAS hh and SAS ll have positive slopes and (nonlinear) SAS hl and SAS lh with positive slopes grow faster and faster their shapes are similar to those discussed in macroeconomic literature.

CONCLUSIONS

Further research is required for countries with non-typical Okun's and/or Phlips and/or B curves. One of the directions is their estimation for different time periods, starting, for instance, from the beginning of previous business cycle (approximately

2001Q1[28]) and finishing on the recent available data. The other possibility is to substitute simultaneous relationships by the lagged ones thus allowing for time sequence of the respective variables (endogenous or exogenous)[29].

On top of that the attempt must be made to explain the reasons of differences across the analysed OECD countries.

Despite of the above, the SAS curves, widely discussed in the macroeconomic literature and very helpful in understanding of many macroeconomic issues, especially connected with macroeconomic policy measures, have no empirical justification whatever the shapes of Okun's and Philips curves. They are inferior, as regards the goodness of fit not only with respect to the B curve (polynomial of the second order) but even to the linear trend.

Nevertheless, when they are linear or nonlinear and growing faster and faster, they may reflect quite well behaviour of the respective economy in given period.

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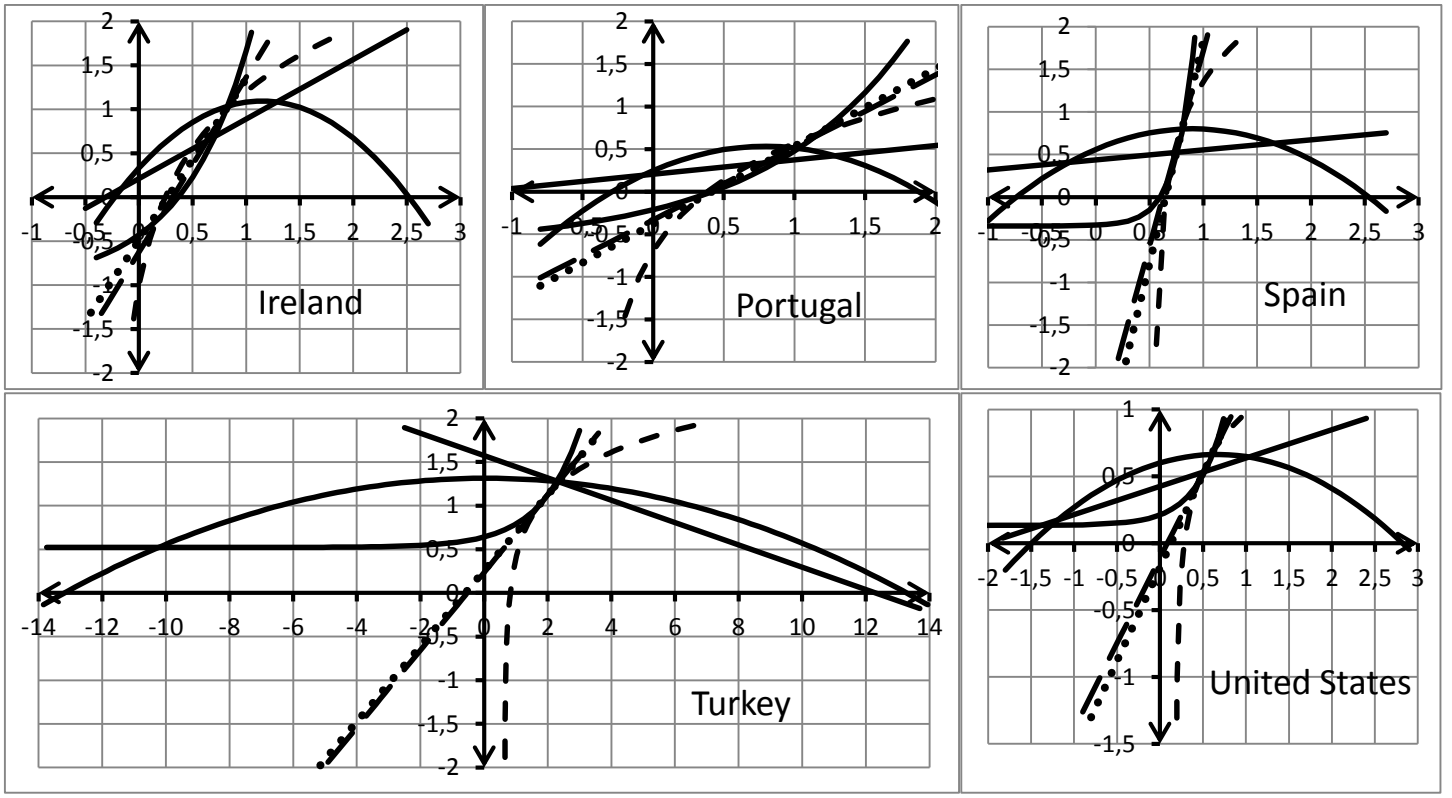
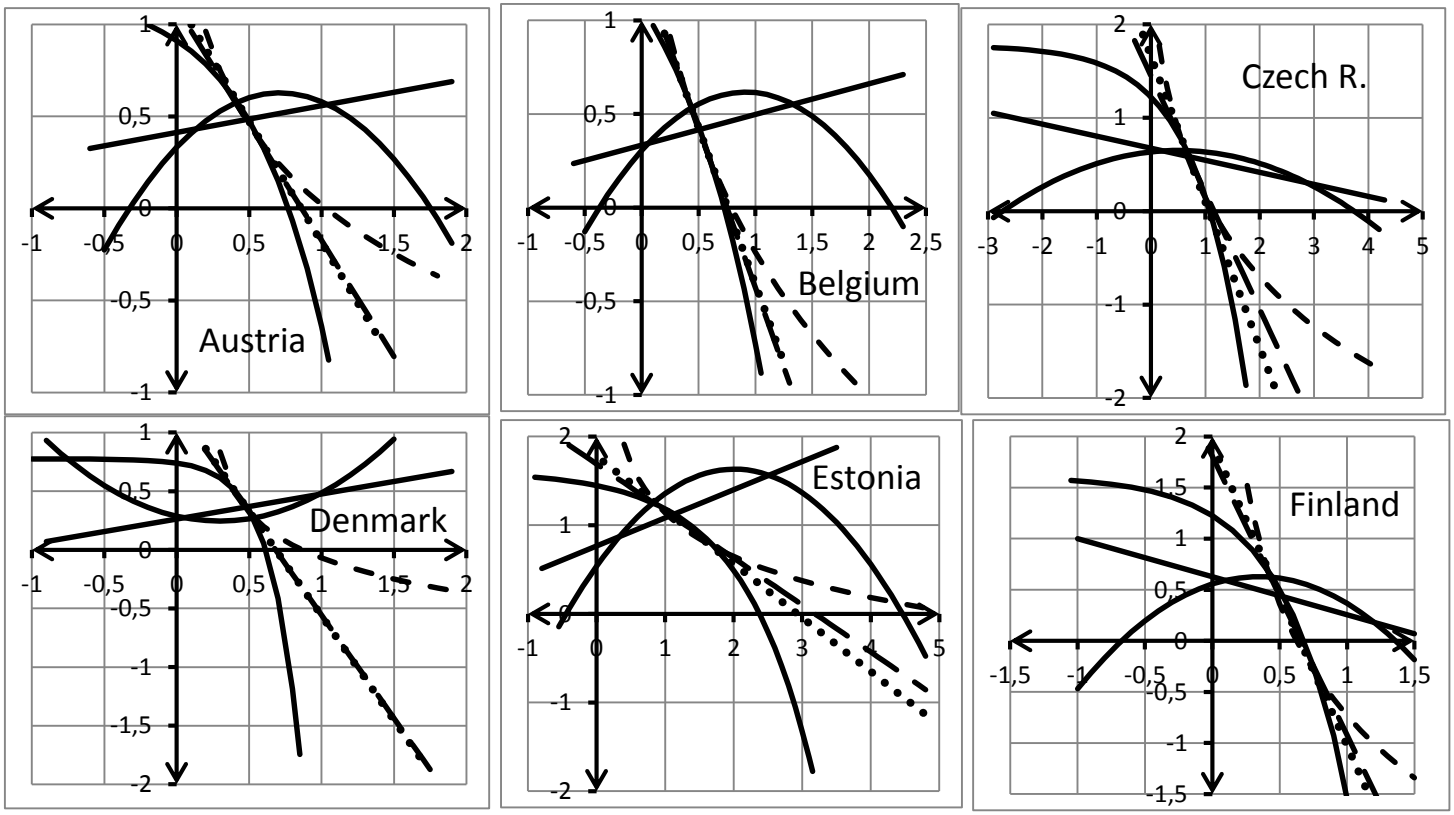
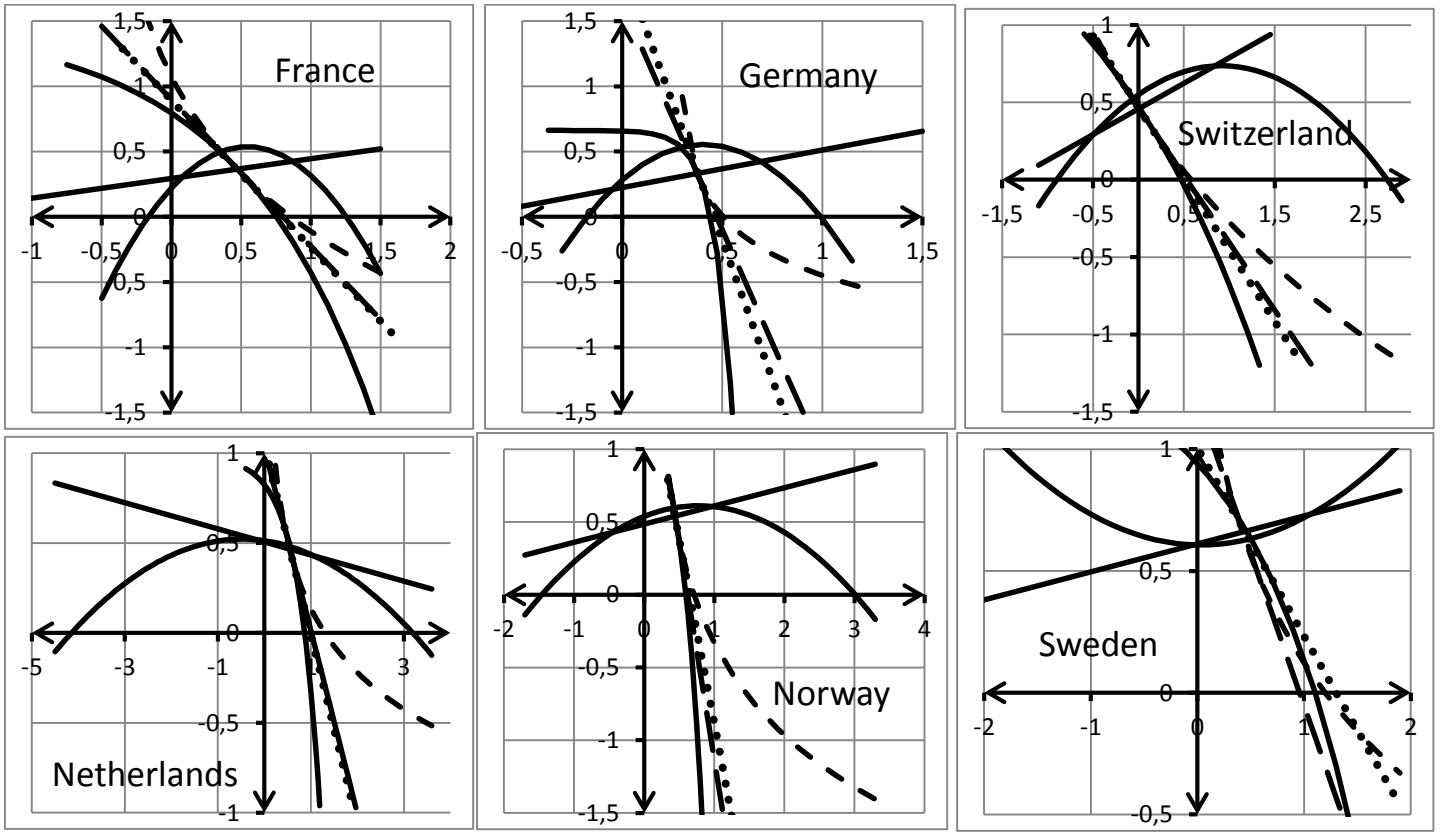
Annex 1. Six SAS curves for countries with typical Okun and Philips curves



Source: own computations

Annex 2. Six SAS curves for countries with non-typical Okun curves and typical Philips curves





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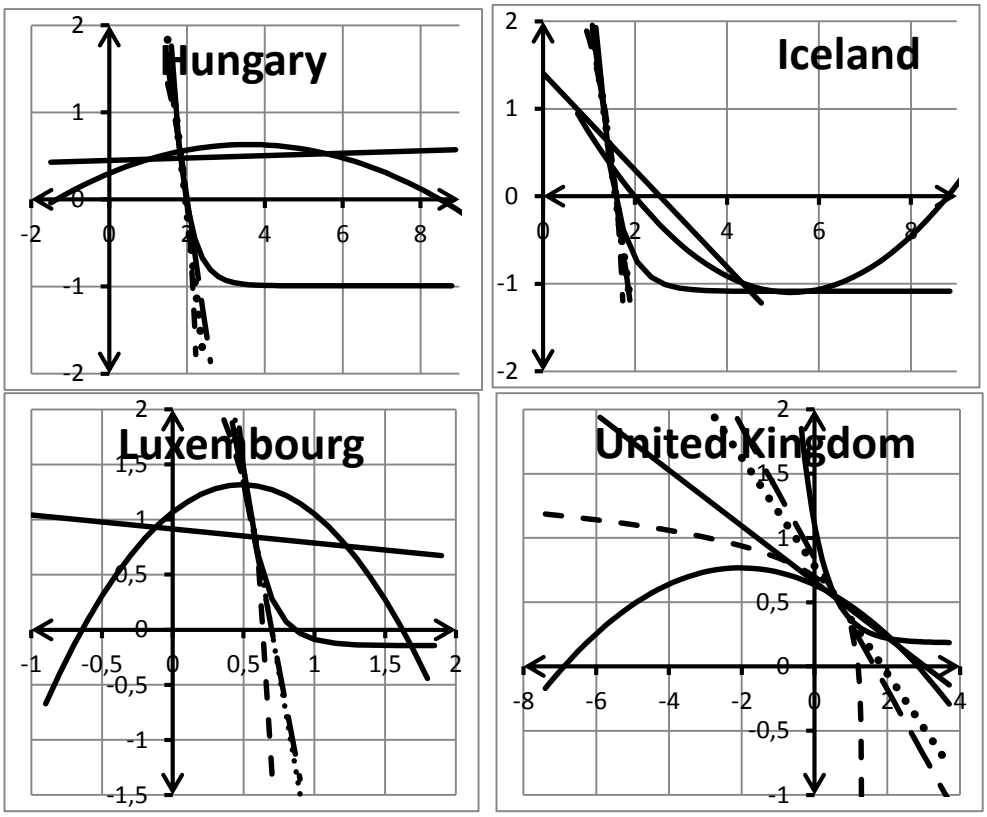
Annex 3. Six SAS curves for countries with typical Okun curves and non-typical Philips curves



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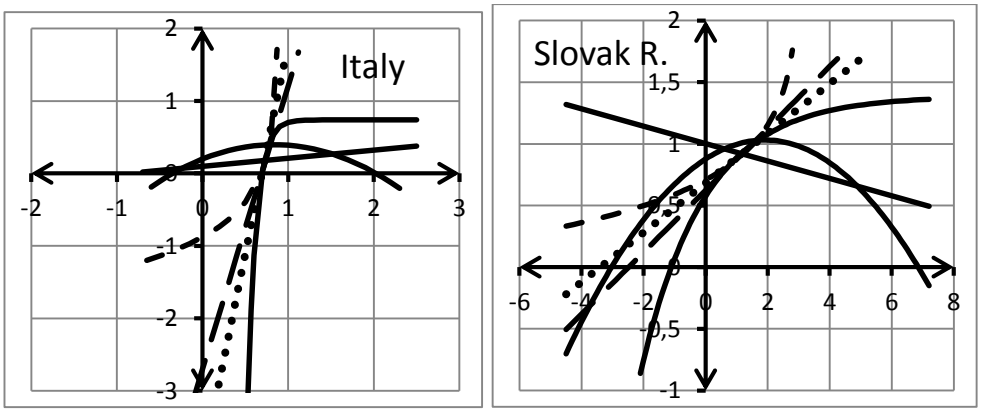
Annex 4. Six SAS curves for countries with non-typical Okun and Philips curves



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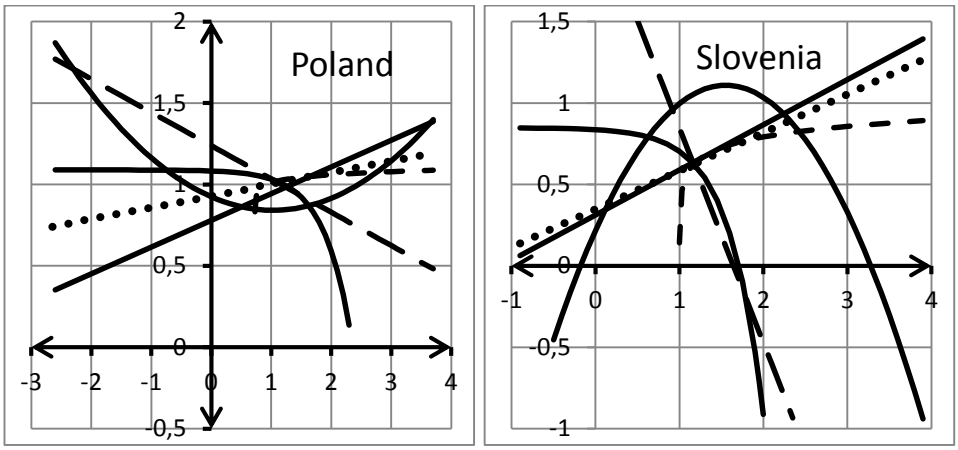
Annex 5. Special cases



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QUANTITATIVE METHODS IN ECONOMICS
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ROBUSTNESS OF TWEEDIE MODEL OF RESERVES WITH RESPECT TO DISTRIBUTION OF SEVERITY OF CLAIMS

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Abstract: The aim of the work is to discuss the robustness of estimation procedures and robustness of prediction in Tweedie's compound Poisson model. This model is applied to the claim reserving problem. The quality of parameter estimators and predictors is studied when the distribution of severity of claims is disturbed. The ε-contamination class of distributions is considered. The example, where errors of estimators are large is presented. The simulation methods, using the R programming environment, are applied.

Keywords: loss reserves, Tweedie model, Poisson and gamma distribution, ε-contamination, generalized linear model, mean square error, bias, prediction

INTRODUCTION

The claim reserving problem is one of the most important in the insurance mathematics and is the main task of insurance actuaries. There are a lot of models, methods and algorithms setting claim reserves. We can mention chain ladder method [Mack 1991, 1993, 1999], [Wüthrich et al. 2009], Bornhuetter-Ferguson method [Bornhuetter, Ferguson 1972], [Taylor 2000], [Mack 2008], bootstrap methods [England, Verrall 1999], [Liu, Verrall 2009], lognormal model [Han, Gau 2008], Bayesian and credibility methods [Ntzoufras, Dellaportas 2002], [Gisler, Wüthrich 2008], [Sánchez, Vilar 2011]. For excellent overview we refer to [England, Verrall 2002] and [Wüthrich, Merz 2008]. The main aim is to predict an appropriate random variable which describes future payments or estimate parameters of that random variable distribution like mean, variance. It is also interesting to estimate prediction and estimation errors.

To do this we select a statistical model, methods of estimation and prediction. But the observed random variables sometimes do not satisfy the

assumptions of a chosen model. It is interesting to verify how our estimators or predictors, their errors and parameters of their distributions can change when we have some deviation from the model. In mathematical statistics such a problem is called the problem of examine the robustness of a statistical procedure. The concept of robustness is presented in [Huber 1981], [Hampel 1986] and [Zieliński 1983]. We will use the concept of Zieliński and will measure the difference in bias and mean square error (*MSE*) of estimators and predictors when model assumptions are not satisfied and the distribution of number of claims and severity of claims are different from the distributions in the choosen model. We assume the Tweedie model as the main (basic) model. The numbers of payments are Poisson random variables and severity of payments is Gamma distributed. These assumptions lead to Tweedie's compound Poisson model (see [Jørgensen, de Souza 1994], [Jørgensen 1997], [Smyth, Jørgensen 2002] and [Ohlsson, Johansson 2006]). The model belongs to the exponential dispersion family and we use GLM for the parameter estimation. The model considered in this paper was described in [Wüthrich 2003], [Peterset al. 2009] and [Boucher, Davidov 2011]. In the first paper the method of parameter estimation is presented and the model is applied to the motor insurance data. It is compared with the chain ladder and lognormal models. The dispersion parameter is assumed to be constant. There is not presented a method of verifying the assumptions of the model. In the second paper the model is compared with a Bayes model and the last paper presents the extension and the dispersion parameter is not assumed to be constant. We use the model and methods of estimation described in [Wüthrich 2003] and measure the variation of *MSE* of parameter estimation and prediction when the distribution of severity of claims is not gamma distribution.

So far there have not been papers which present results about robustness of the Tweedie model for prediction reserves and the verification of model assumptions may be difficult or even impossible. We consider the ε-contamination distribution of the value of claim, where $\varepsilon \in [0,1]$ is the power of disturbance. If $\varepsilon = 0$ we have the model distribution, if $\varepsilon = 1$ we have total departure from this distribution and we replace it with other, differing in shape, variance, coefficient of asymmetry. By introducing this type of disturbances we move away from the exponential family of distributions. This may affect the quality of the GLM estimators. As a basic model we consider the model with parameters estimated in [Wüthrich 2003]. We will consider a model with constant Tweedie's dispersion coefficient, the work associated with its variation we plan to in the future.

Applying simulation methods we generate run-off triangles of numbers and severities of claims using disturbed distributions, and next compute *MSE* and bias of estimation and prediction. For all simulations and GLM procedures the R language environment was used.

DEFINITION OF THE BASIC MODEL

Let C_{ij} be incremental claim payments, where i denotes the accident year and j development year and $i \in \{0,1,\ldots,I\}$, $j \in \{0,1,\ldots,J\}$. Let R_{ij} be numbers of claims corresponding to these payments. At time J we observe the following sets

$$D_J = \{C_{ij} : i = 0,1,\ldots,I;\ j = 0,1,\ldots,J-i\}$$

and

$$DR_J = \{R_{ij} : i = 0,1,\ldots,I;\ j = 0,1,\ldots,J-i\}.$$

If $I = J$ the sets are upper triangles. In this paper $J > I$, but despite that, these sets are called upper triangles. The main aim is prediction of the lower triangle i.e. variables C_{ij}, where $i + j > J$ and $i \le I$, $j \le J$, or estimation of parameters of their distributions. It is also interesting to estimate the expected value of a random variable $S_i = \sum_{j=J-i+1}^{J} C_{ij}$, $i = 1,2,\ldots,I$, called the reserves for the year i, and a random variable $S = \sum_{i=1}^{I} S_i$, called the total reserve, as well as the prediction of these variables.

In the basic model we assume that R_{ij} are independent and have Poisson distributions $Poiss(\lambda_{ij} w_i)$, where $\lambda_{ij} w_i > 0$ are expected values and w_i numbers of policies. The individual payments $X_{ij}^{(k)}$ for $k = 1,2,\ldots$ are independent and $Gamma(\gamma, s_{ij})$ distributed with shape parameter $\gamma > 0$, expected value $EX_{ij}^{k} = \tau_{ij} = \gamma s_{ij}$ and variance $VarX_{ij}^{(k)} = \gamma s_{ij}^2 = \tau_{ij}^2 / \gamma$. Variables R_{ij} i $X_{lm}^{(k)}$ are independent for all indices. Then

$$C_{ij} = \sum_{k=1}^{R_{ij}} X_{ij}^{(k)},$$

if $R_{ij} > 0$, and $C_{ij} = 0$ otherwise, hence $P(C_{ij} = 0) = \exp(-\lambda_{ij} w_i)$. Conditionally, given $R_{ij} \neq 0$, the variable C_{ij} has $Gamma(R_{ij}\gamma, s_{ij})$ distribution.

Let $Y_{ij} = C_{ij} / w_i$. Random variables Y_{ij} have Tweedie distribution with the density function (for $y > 0$)

$$f_{Y_{ij}}(y; \mu_{ij}, \phi, w_i, p) = \sum_{r=1}^{\infty} \left(\frac{(w_i/\phi)^{\gamma+1} y^{\gamma}}{(p-1)^{\gamma}(2-p)} \right)^{r} \frac{1}{r!\Gamma(r\gamma) y} \exp\left(\frac{w_i}{\phi} \left(y \frac{\mu_{ij}^{1-p}}{1-p} - \frac{\mu_{ij}^{2-p}}{2-p} \right) \right),$$

where

$$p = \frac{\gamma+2}{\gamma+1} \in (1,2), \mu_{ij} = \lambda_{ij}\tau_{ij}, \phi = \lambda_{ij}^{1-p}\tau_{ij}^{2-p}(2-p)^{-1}$$

(for details see [Wüthrich 2003]).
We have

$$EY_{ij} = \mu_{ij}, \qquad VarY_{ij} = \frac{\phi}{w_i}V(p) = \frac{\phi}{w_i}\mu_{ij}^{p}, \qquad \frac{E(Y_{ij} - EY_{ij})^3}{Var^{1,5}Y_{ij}} = \frac{p}{\sqrt{\lambda w_i (2-p)}}.$$

In this paper, as in [Wüthrich 2003], the parameter p is estimated. Additionally we assume a multiplicative model, i.e. there exist parameters $\alpha(i)$, $\beta(j)$, $i = 0,1,\ldots,I$, $j = 0,1,\ldots,J$ such that

$$\mu_{ij} = \alpha(i)\beta(j).$$

Hence choosing the logarithmic link function we have

$$\eta_{ij} = \ln(\mu_{ij}) = \mathbf{x}_{ij}b$$

where $b^T = [b_{0,0}, b_{1,0}, \ldots, b_{I,0}, b_{0,1}, \ldots, b_{0,J}]$ and a matrix $\mathbf{x}$ of rows $\mathbf{x}_{ij}$ is defined so that $\ln(\mu_{ij}) = \ln\alpha(i) + \ln\beta(j) = b_{0,0} + b_{i,0} + b_{0,j}$. The vector of unknown parameters is equal to $B = [b_{0,0}, b_{1,0}, \ldots, b_{I,0}, b_{0,1}, \ldots, b_{0,J}, \phi, p]$, and the logarithm of the likelihood function (for $R_{ij} = r_{ij}$ and $Y_{ij} = y_{ij}$, $i = 0,1,\ldots,I$, $j = 0,1,\ldots,J-i$) is equal to

$$L(\mu, p, \phi) = \sum_{\{i,j: r_{ij} \neq 0\}} \left[r_{ij} \ln\left(\frac{(w_i/\phi)^{\gamma+1} y_{ij}^{\gamma}}{(p-1)^{\gamma}(2-p)} \right) - \ln(r_{ij}!\Gamma(r_{ij}\gamma)y_{ij}) \right] + \sum_{i,j} \frac{w_i}{\phi} \left(y_{ij} \frac{\mu_{ij}^{1-p}}{1-p} - \frac{\mu_{ij}^{2-p}}{2-p} \right),$$

where $\gamma = (p-2)(1-p)^{-1}$ and $\mu = [\mu_{ij}]_{i=0,1,..,I,\ j=0,1,..,J}$ is a matrix.

The estimation in this model is described in [Wüthrich 2003]. We remind the most important steps.

1. Choose an initial $\hat{p}$.
2. For $\hat{p}$ known, use GLM and find estimates for $b_{0,0}, b_{1,0}, \ldots, b_{I,0}, b_{0,1}, \ldots, b_{0,J}$ and estimates for μ_{ij} equal to $\hat{\mu}_{ij} = \exp(\hat{b}_{0,0} + \hat{b}_{i,0} + \hat{b}_{0,j})$.
3. For given $p = \hat{p}$ and $\mu = \hat{\mu}$ find MLE for ϕ equal to

$$\hat{\phi} = -\left((1+\gamma) \sum_{i,j} r_{ij} \right)^{-1} \sum_{i,j} w_i \left(y_{ij} \frac{\mu_{ij}^{1-p}}{1-p} - \frac{\mu_{ij}^{2-p}}{2-p} \right).$$

4. Given $\hat{\phi}$ and $\hat{\mu}$, find maximum of the likelihood function and choose the next $\hat{p}$.
5. Repeat steps 2, 3, 4, 5 until convergence of $\hat{p}, \hat{\mu}$.

The vector

$$\hat{B} = [\hat{b}_{0,0}, \hat{b}_{1,0}, \ldots, \hat{b}_{I,0}, \hat{b}_{0,1}, \ldots, \hat{b}_{0,J}, \hat{\phi}, \hat{p}]$$

is the estimator for *B*, and applying equalities

$$\gamma = \frac{p-2}{1-p}, \qquad \tau_{ij} = (2-p)\phi\mu_{ij}^{p-1}, \qquad \lambda_{ij} = \frac{\mu_{ij}}{\tau_{ij}}$$

we obtain the estimators for γ, τ_{ij} and λ_{ij}. The predictor for C_{ij}, $i + j > J$, is defined as $\hat{C}_{ij} = w_i\hat{\mu}_{ij}$ and it is an estimator for EC_{ij}. Hence, the predictor for S_i (S) and the estimator for ES_i (ES) is defined as $\hat{S}_i = \sum_{j=J-i+1}^{J}\hat{C}_{ij}$ ($\hat{S} = \sum_{i=1}^{I}\hat{S}_i$).

DISTURBED MODELS

We use simulation methods to obtain data from assumed distributions. Values of the parameters of the basic model are equal to estimated values presented in [Wüthrich 2003] and [Boucher, Davidov 2011], (see Appendix 1). Assume that $I = 8$, $J = 10$. Table 1.3 (Appendix 1) provides values of w_i and p, γ, ϕ. Tables 1.1 and 1.2 (Appendix 1) provide values of the parameters λ_{ij} and τ_{ij}. The parameters μ_{ij} satisfy $\mu_{ij} = \lambda_{ij}\tau_{ij}$. Given these values the basic model and parameters of distributions of numbers and severity of claims for all periods are uniquely determined. Entries in the upper and lower triangle in Tables 1.1 and 1.2 are parameters of observed and predicted distribution respectively.

Disturbances in the model concern distributions of severity of claims. Distributions of severity of claims incurred in accident year i and reported in development year j are in the form of ε-contamination distributions

$$F(x) = (1-\varepsilon)F_0(x) + \varepsilon G(x),$$

where F_0 is a distribution function of severity of claims in the basic model and *G* is a distribution function of contamination. The parameter $\varepsilon \in [0,1]$ determines the power of disturbance. In particular, $\varepsilon = 0$ means that there is no disturbance, for $\varepsilon = 1$ distribution F_0 is replaced by *G*.

Table 1 provides the basic and contaminating distributions and their parameters. A disturbance by means of a given distribution is called a type. Thus there are three types of disturbances according to type of distribution (I – lognormal, II –Weibull, III – Pareto). In all cases the contaminating distribution has the same mean as the basic distribution, but different variance and coefficient of asymmetry. Chosen distributions have variance the same as the basic distribution (type Ia and IIa) or $z\gamma$ times greater than the one in the basic model (then its value is such that square of the coefficient of variation is equal to 1.5 or 2,

while in the basic distribution it equals $1/\gamma = 0.211$). Thus we assume that the parameter z is equal to 1.5 and 2. An asymmetry of contaminating distribution differs from that in the basic model (see Table 1). Only in type IIa it is less than asymmetry in the basic model.

Table 1. Contaminating distributions (a_z denotes coefficient of asymmetry of distribution with parameter z)

Type	Distribution	Parameters	Coefficient of asymmetry a	value of a
Ia	Lognormal $L(m_{ij}, \sigma^2)$	$\sigma^2 = \ln(1+\gamma^{-1})$ $m_{ij} = \ln(\tau_{ij}) - 0{,}5\sigma^2$	$3(\sqrt{\gamma})^{-1} + (\sqrt{\gamma})^{-3}$	1.47
Ib	Lognormal $L_z(m_{z,ij}, \sigma_z^2)$	$\sigma_z^2 = \ln(1+z)$ $m_{z,ij} = \ln(\tau_{ij}) - 0{,}5\sigma_z^2$	$3\sqrt{z} + (\sqrt{z})^3$	$a_{1,5} = 5.51$ $a_2 = 7.07$
IIa	Weibull $W(c_{ij}, t)$	t satisfies $\frac{\Gamma^2(1+t^{-1})}{\Gamma(1+2t^{-1}) - \Gamma^2(1+t^{-1})} = \gamma$ $c_{ij} = (\Gamma(1+t^{-1})\tau_{ij}^{-1})^t$	$\frac{\Gamma(\frac{t+3}{t}) - 3\Gamma(\frac{t+2}{t})\Gamma(\frac{t+1}{t}) + 2\Gamma^3(\frac{t+1}{t})}{[\Gamma(\frac{t+2}{t}) - \Gamma^2(\frac{t+1}{t})]^{1,5}}$	0,45
IIb	Weibull $W_z(c_{z,ij}, t_z)$	t_z satisfies $\frac{\Gamma^2(1+t_z^{-1})}{\Gamma(1+2t_z^{-1}) - \Gamma^2(1+t_z^{-1})} = \frac{1}{z}$ $c_{z,ij} = (\Gamma(1+t_z^{-1})\tau_{ij}^{-1})^{t_z}$	$\frac{\Gamma(\frac{t_z+3}{t_z}) - 3\Gamma(\frac{t_z+2}{t_z})\Gamma(\frac{t_z+1}{t_z}) + 2\Gamma^3(\frac{t_z+1}{t_z})}{[\Gamma(\frac{t_z+2}{t_z}) - \Gamma^2(\frac{t_z+1}{t_z})]^{1,5}}$	$a_{1,5} = 2.70$ $a_2 = 3.33$
IIIb	Pareto $Pa_z(\theta_z, \varphi_{z,ij})$	$\theta_z = 2z(z-1)^{-1}$ $\varphi_{z,ij} = \tau_{ij}(\theta_z - 1)$	$\frac{6z-2}{(3-z)\sqrt{z}}$	$a_{1,5} = 3.81$ $a_2 = 7.07$
IV	gamma $G(\gamma, s_{ij})$	$s_{ij} = \tau_{ij}\gamma^{-1}$ or $s_{ij} = \tilde{\tau}_{ij}\gamma^{-1}$	$2\gamma^{-0,5}$	0.92

Type IV consists in using gamma distribution as a contamination. Its mean is equal to average value of payments for data given in [Wüthrich 2003] (see Table 2.2, Appendix 2). The average payments computed on the basis of that paper differ much from τ_{ij} (cf. Tables 1.2 and 2.2) and are almost the same in successive development years, while parameters τ_{ij} decrease with increasing development period. It might indicate the difference between distributions. So it is the type worth carrying out. In this case two types of distributions of number of claims are considered. In the first case (type IVa) it is Poisson distribution with parameters given in Table 1.1 (Appendix 1). Then (similar to types I, II, III) distribution

of number of claims is the same as the one in the basic model and distribution of severity of claims is of the form of ε-contamination. In the second case (type IVb) numbers of claims are generated from ε-contamination distributions where Poisson distributions with parameters given in Table 2.1 (Appendix 2) are contaminating distributions. The parameters are equal to average numbers of claims per policy obtained on the basis of paper [Wüthrich 2003]. It means that for every i, j we generate numbers of claims according to distributions $(1-\varepsilon)Poiss(\lambda_{ij}) + \varepsilon Poiss(\tilde{\lambda}_{ij})$ where parameters λ_{ij} and $\tilde{\lambda}_{ij}$ are given in Table 1.1 and 2.1 respectively. If the number of claims is generated from $Poiss(\lambda_{ij})$ distribution then severity of claims is generated from $Gamma(\gamma, \tau_{ij}/\gamma)$ distribution, otherwise from $Gamma(\gamma, \tilde{\tau}_{ij}/\gamma)$ distribution. Parameters τ_{ij} and $\tilde{\tau}_{ij}$ are given in Table 1.2 and 2.2 respectively. The data presented in the paper [Wüthrich 2003] allows us to determine parameters (in particular expected values) of distributions of number and severity of claims for the upper triangle. Parameters for the lower triangle of contaminating distributions are calculated by applying chain ladder method to the run-off triangles of the paper.

The goal is to check how the choice of different distributions influences the bias of a parameter estimator, its mean square error and the mean square error of prediction.

A study for a chosen contaminating distribution and the value of ε is called a scenario. A case of generating random variables from the basic model is called the zero scenario. For every scenario we generate independently 10000 tables of numbers of claims and the corresponding ones of sums of payments, that is the variables R_{ij} and C_{ij}. Based on data from the upper triangle we estimate elements of the vector B according to the steps described in the previous section and then we estimate the other ones. We also predict the lower triangle, i.e., quantities C_{ij} for $i + j > J$.

Let $\hat{\theta}_l$ be the value of an estimator $\hat{\theta}$ of a parameter θ based on the *l*-th simulation, $l = 1,2,\ldots,10000$. As the parameter estimates of distribution of variable $\hat{\theta}$ we take:

- expectation: $E\hat{\theta} = \frac{1}{10000}\sum_{l=1}^{10000}\hat{\theta}_l$ and variance: $Var\,\hat{\theta} = \frac{1}{10000}\sum_{l=1}^{10000}(\hat{\theta}_l - E\hat{\theta})^2$;
- mean square error: $MSE(\hat{\theta}) = Var\,\hat{\theta} + bias^2(\hat{\theta})$, where $bias(\hat{\theta}) = E\hat{\theta} - \theta$;
- standard error: $SE(\hat{\theta}) = \sqrt{MSE(\hat{\theta})}$;
- relative bias: $Rbias(\hat{\theta}) = \frac{bias(\hat{\theta})}{\theta} \cdot 100\%$;

- variation coefficient: $\upsilon(\hat{\theta}) = \frac{(Var\hat{\theta})^{1/2}}{E\hat{\theta}} \cdot 100\%$;
- standard percentage error: $SPE(\hat{\theta}) = \frac{SE(\hat{\theta})}{\theta} \cdot 100\%$ which measures percentage of the standard error of the estimate in actual value of the parameter.

Analogously, let $\hat{Z}_l$ be the value of a predictor $\hat{Z}$ of some random variable Z based on the *l*-th simulation, $l = 1,2,\ldots,10000$, and let Z_l be an observed value of the variable. We take the following measures:

- mean square error of prediction: $MSEP(\hat{Z}) = \frac{1}{10000}\sum_{l=1}^{10000}(\hat{Z}_l - Z_l)^2$;
- standard error of prediction: $SEP(\hat{Z}) = \sqrt{MSEP(\hat{Z})}$;
- bias of predictor: $bias(\hat{Z}) = \frac{1}{10000}\sum_{l=1}^{10000}(\hat{Z}_l - Z_l)$;
- relative bias of predictor: $RbiasP(\hat{Z}) = \frac{bias(\hat{Z})}{\bar{Z}} \cdot 100\%$, where $\bar{Z} = \frac{1}{10000}\sum_{l=1}^{10000} Z_l$;
- standard percentage error of prediction: $SPEP(\hat{Z}) = \frac{SEP(\hat{Z})}{\bar{Z}} \cdot 100\%$.

PROPERTIES OF ESTIMATORS AND PREDICTORS IN THE BASIC MODEL

As a result of simulations for the zero scenario the bias and mean square error of estimators of elements of vector *B*, matrices λ, τ, μ and vector of expected reserves for successive accident years and expected value of the total reserve are obtained. Relative bias of estimators of parameters p, γ, ϕ is of the order of 10^{-3}% and standard error is less than 0.9% of actual value of the parameter (see Table 5, the third row).

Table 2. Relative bias and *SPE* of estimators $\hat{\lambda}_{ij}$, $\hat{\tau}_{ij}$, $\hat{\mu}_{ij}$

	j										
	0	1	2	3	4	5	6	7	8	9	10
Rbias for $\hat{\lambda}_{ij}$											
min	-0.003	-0.010	-0.020	-0.004	-0.025	-0.052	0.003	-0.008	-0.038	-0.12	-14.27
max	0.021	0.015	0.005	0.020	-0.001	-0.027	0.027	0.019	-0.013	-0.10	-14.23
Rbias for $\hat{\tau}_{ij}$											
min	-0.003	-0.006	-0.013	-0.013	-0.020	-0.037	-0.037	-0.046	-0.123	-0.19	-42.37
max	0.002	-0.001	-0.007	-0.008	-0.015	-0.032	-0.032	-0.041	-0.117	-0.18	-42.36
Rbias for $\hat{\mu}_{ij}$											
min	-0.004	-0.014	-0.028	-0.005	-0.028	-0.048	0.035	0.033	0.100	0.07	-1.50
max	0.025	0.016	0.003	0.023	0.000	-0.018	0.063	0.065	0.131	0.10	-1.45

SPE for $\hat{\lambda}_{ij}$											
min	0.94	1.05	1.73	2.51	3.02	4.63	5.87	6.58	11.19	13.51	104.28
max	1.04	1.16	1.79	2.60	3.10	4.70	5.94	6.64	11.24	13.55	104.36
SPE for $\hat{\tau}_{ij}$											
min	0.25	0.27	0.50	0.70	0.81	1.14	1.39	1.54	2.51	2.98	68.220
max	0.27	0.29	0.51	0.72	0.82	1.15	1.41	1.55	2.52	2.99	68.221
SPE for $\hat{\mu}_{ij}$											
min	1.12	1.26	2.06	2.99	3.61	5.54	7.06	7.92	13.54	16.32	128.32
max	1.25	1.39	2.12	3.10	3.71	5.63	7.13	8.00	13.59	16.38	128.47

Source: Authors' calculations

Table 2 provides the minimum and maximum value (over i) of the relative bias and *SPE* of estimators of parameters λ_{ij}, τ_{ij}, μ_{ij} for all development years. Variation of *Rbias* and *SPE* with respect to *i*, for fixed *j*, is small. *SPE* is increasing function of development year *j*. Absolute value of relative bias of estimators of parameters λ_{ij}, τ_{ij}, μ_{ij} for $j<10$ is not greater than 0.2%, for $j=10$ it grows tens times. In the last development year bias of all these estimates is negative. Thus even in the basic model it can be observed large average underestimate of both expected values (severity and number of claims) in the last development year, compared to the others. There is an even bigger jump for *SPE* in the last development year.

Table 3. Estimation of expected value and prediction of reserves

			Estimation of ES_i				Prediction of S_i		
i	ES_i	$\hat{ES}_i$	*Rbias*	*V*	*SE*	*SPE*	*RbiasP*	*SEP*	*SPEP*
1	326	321	-1.45	130.33	418	128.44	3.58	600	180.41
2	21575	21588	0.06	16.24	3506	16.25	0.29	6042	27.91
3	40746	40779	0.08	10.61	4325	10.62	-0.31	7966	19.60
4	89306	89371	0.07	6.55	5855	6.56	-0.18	11935	13.38
5	138537	138630	0.07	4.96	6872	4.96	-0.01	14888	10.74
6	204637	204723	0.04	3.88	7939	3.88	-0.06	18138	8.86
7	361456	361481	0.01	2.83	10245	2.83	-0.06	25456	7.05
8	598005	597967	-0.01	2.29	13678	2.29	-0.03	33644	5.63
total	1454587	1454860	0.02	2.79	40528	2.79	-0.05	60912	4.19

Source: Authors' calculations

The results for estimation of expected values of reserves and prediction of reserves are shown in Table 3. The last row shows the errors for the total reserve.

Table 4. Errors: *SPE* and *SPEP* (in parentheses) for reserves C_{ij}

	j							
i	3	4	5	6	7	8	9	10
1								128.4(180.4)
2							16.4(28.2)	128.5(182.3)
3						13.6(27.1)	16.4(28.7)	128.4(187.8)
4					8.0(18.1)	13.6(27.6)	16.4(29.5)	128.4(185.8)
5				7.1(17.5)	8.0(18.0)	13.6(27.7)	16.4(28.8)	128.5(185.3)
6			5.6(15.1)	7.1(18.0)	8.0(18.7)	13.6(28.2)	16.4(29.5)	128.4(185.5)
7		3.7(10.6)	5.6(15.5)	7.1(18.4)	8.0(19.1)	13.6(28.6)	16.4(30.0)	128.3(194.2)
8	3.1(9.2)	3.7(10.5)	5.6(15.5)	7.1(18.2)	8.0(19.2)	13.6(28.9)	16.4(30.1)	128.4(186.7)

Source: Authors' calculations

Absolute relative bias of estimates, υ, *SPE* and *SPEP* are decreasing functions of accident year *i*. Large values of υ, *SPE* and *SPEP* for $i = 1$ (compared to the other *i*) are related to the consideration of the reserve only for development year $j = 10$. Table 4 presents *SPE* for estimates of EC_{ij}, $i + j > J$, and *SPEP* for prediction of C_{ij}, $i + j > J$. For all *i* both the errors are increasing functions of *j*, however for $j = 10$ are much greater than the others.

PROPERTIES OF ESTIMATORS AND PREDICTORS IN THE DISTURBED MODEL

We analyze behavior of estimators and predictors in contaminated models with respect to the type and the power of disturbance.

First consider disturbances of types I, II, III. It turns out that the type of disturbance does not influence on results. The power of disturbance and the variance of contaminating distribution are much more important. In all tables below we denote:

m_a, M_a – the minimum and maximum value of a concerned error in all disturbances of the type "a",

$m_{b;z}$, $M_{b;z}$ – the minimum and maximum value of a concerned error in all disturbances of the type "b" given *z*,

$r_a = M_a - m_a$ and $r_{b,z} = M_{b,z} - m_{b,z}$.

The minima and maxima are taken also over $\varepsilon \in \{0; 0.01; 0.1; 0.2; 0.5; 1\}$.

Let us remind that the type "a" of disturbance denotes contamination by lognormal or Weibull distribution with the variance as the one in the basic model and type "b" denotes contamination by lognormal, Weibull or Pareto distribution with the variance $z\gamma$ times greater than the one in the basic model, $z \in \{1.5; 2\}$ (cf. Table 1).

Table 5. Behavior of errors of estimators $\hat{p}$, $\hat{\gamma}$, $\hat{\phi}$ for disturbances of types I, II, III

	Rbias			*SPE*		
	$\hat{p}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{p}$	$\hat{\gamma}$	$\hat{\phi}$
$\varepsilon = 0$	0.0010	-0.0024	-0.0050	0.1054	0.8608	0.4404
m_a	-0.0002	-0.0144	-0.0117	0.1046	0.8546	0.4356
M_a	0.0025	0.0080	0.0018	0.1071	0.8745	0.4462
$m_{b;1,5}$	0.0010	-1.3545	-0.6749	0.1054	0.8608	0.4404
$M_{b;1,5}$	0.1722	-0.0024	-0.0050	0.3105	2.4678	1.2611
$m_{b;2}$	0.0010	-1.8251	-0.9076	0.1054	0.8608	0.4404
$M_{b;2}$	0.2328	-0.0024	-0.0050	0.3757	2.9667	1.5124

Source: Authors' calculations

Table 5 presents behavior of errors of estimators of parameters p, γ, ϕ. For each type of contamination $|Rbias|$ and *SPE* are increasing functions of variance of contaminating distribution. For disturbances of the type "b" these errors are also increasing functions of the power of disturbance. For $\varepsilon = 0{,}01$ values of the errors are close to those in the basic model regardless of variance of contaminating distribution. But for $\varepsilon = 1$ and for contaminating distribution of the type "b" with parameter $z = 2$, $|Rbias|$ increases more than one hundred times and *SPE* more than three times, compared to the results for the basic model. However, in all the types of disturbances the errors are less than 3%.

Tables 6, 7, 8 provide these errors for estimators $\hat{\lambda}_{ij}$, $\hat{\tau}_{ij}$, $\hat{\mu}_{ij}$. Here the minimum and maximum for a given type is taken also over all accident years *i*. The reason is that for every estimator, for every $\varepsilon \in \{0; 0.01; 0.1; 0.2; 0.5; 1\}$ and for every type I, II, III the oscillation of the errors with respect to *i* is similar to the one in the basic model (see Table 2) and it does not exceed 1%.

Table 6. Behavior of errors of estimators $\hat{\lambda}_{ij}$ for disturbances of types I, II, III

	j										
	0	1	2	3	4	5	6	7	8	9	10
	Rbias										
M_a	0.03	0.04	0.05	0.07	0.04	0.07	0.07	0.16	0.08	0.18	-11.09
r_a	0.05	0.07	0.07	0.12	0.10	0.16	0.21	0.19	0.35	0.63	3.22
$M_{b,1,5}$	0.02	0.13	0.57	0.76	0.80	1.01	1.13	1.05	1.07	0.75	-11.76
$r_{b,1,5}$	0.19	0.15	0.60	0.78	0.81	1.06	1.31	1.21	1.18	0.97	4.63
$M_{b,2}$	0.02	0.17	0.69	1.02	1.12	1.29	1.35	1.38	1.62	1.54	-12.20
$r_{b,2}$	0.24	0.18	0.71	1.07	1.15	1.29	1.52	1.52	1.84	1.85	5.05

SPE											
M_a	1.07	1.18	1.81	2.64	3.12	4.73	5.97	6.77	11.14	13.71	107.04
r_a	0.14	0.13	0.10	0.14	0.11	0.18	0.14	0.22	0.19	0.27	3.42
$M_{b,1,5}$	1.48	1.64	2.52	3.68	4.36	6.69	8.56	9.67	15.89	19.64	139.05
$r_{b,1,5}$	0.54	0.59	0.80	1.17	1.38	2.04	2.76	3.02	4.85	6.13	34.15
$M_{b,2}$	1.59	1.78	2.76	4.07	4.81	7.34	9.32	10.53	17.67	22.22	145.14
$r_{b,2}$	0.65	0.73	1.03	1.55	1.78	2.71	3.43	3.88	6.55	8.72	40.27

Source: Authors' calculations

The oscillation of *Rbias* of every estimator is very small for all the types of disturbance (not greater than 6.4%). For estimators $\hat{\lambda}_{ij}$ and $\hat{\mu}_{ij}$ the standard percentage error and its oscillation are increasing functions of variance of contaminating distribution. The power of disturbance also affects these values (the greatest ones are achieved for $\varepsilon = 0.5$ or $\varepsilon = 1$). For estimators $\hat{\lambda}_{i,10}$ and for disturbances of the type "a" the oscillation of *SPE* does not exceed 4% and for disturbances of the type "b" with parameter $z = 2$ exceeds 40%. For estimators $\hat{\mu}_{i,10}$ the oscillation is even greater. However, *SPE* is stable for estimators $\hat{\tau}_{ij}$ and for every i it is increasing function of development year j.

Table 7. Behavior of errors of estimators $\hat{\tau}_{ij}$ for disturbances of types I, II, III

	j										
	0	1	2	3	4	5	6	7	8	9	10
Rbias											
M_a	0.005	0.004	0.003	0.002	0.002	-0.016	-0.026	-0.014	-0.11	-0.13	-41.04
r_a	0.015	0.014	0.018	0.030	0.035	0.031	0.049	0.050	0.08	0.15	1.27
$M_{b,1,5}$	0.137	0.001	-0.010	-0.011	-0.015	-0.022	-0.040	-0.045	-0.10	-0.19	-41.33
$r_{b,1,5}$	0.142	0.123	0.530	0.741	0.808	1.005	1.086	1.135	1.40	1.50	5.47
$M_{b,2}$	0.186	0.000	-0.007	-0.013	-0.020	-0.023	-0.037	-0.059	-0.14	-0.19	-41.40
$r_{b,2}$	0.192	0.154	0.729	0.983	1.075	1.356	1.511	1.490	1.86	1.98	6.32
SPE											
M_a	0.28	0.29	0.52	0.72	0.83	1.17	1.42	1.57	2.49	3.03	68.13
r_a	0.03	0.03	0.02	0.03	0.03	0.04	0.03	0.04	0.02	0.07	0.80
$M_{b,1,5}$	0.59	0.58	1.26	1.71	1.89	2.47	2.82	3.01	4.27	5.01	68.92
$r_{b,1,5}$	0.33	0.31	0.75	1.00	1.08	1.32	1.42	1.45	1.80	2.01	1.40
$M_{b,2}$	0.67	0.66	1.49	2.00	2.22	2.87	3.29	3.44	4.88	5.65	68.99
$r_{b,2}$	0.41	0.39	0.98	1.29	1.41	1.71	1.88	1.87	2.39	2.64	1.76

Source: Authors' calculations

Table 8. Behavior of errors of estimators $\hat{\mu}_{ij}$ for disturbances of types I, II, III

	j										
	0	1	2	3	4	5	6	7	8	9	10
Rbias											
M_a	0.04	0.05	0.05	0.08	0.05	0.09	0.10	0.23	0.23	0.43	2.70
r_a	0.06	0.08	0.08	0.15	0.13	0.18	0.24	0.24	0.43	0.77	4.30
$M_{b,1,5}$	0.04	0.03	0.06	0.10	0.11	0.14	0.22	0.15	0.25	0.45	2.38
$r_{b,1,5}$	0.08	0.08	0.11	0.16	0.19	0.22	0.41	0.45	0.57	0.73	4.00
$M_{b,2}$	0.03	0.06	0.04	0.10	0.09	0.18	0.15	0.17	0.31	0.49	2.00
$r_{b,2}$	0.08	0.10	0.13	0.18	0.17	0.27	0.36	0.33	0.56	0.79	3.90
SPE											
M_a	1.28	1.41	2.15	3.15	3.72	5.67	7.17	8.16	13.45	16.56	133.06
r_a	0.17	0.16	0.12	0.17	0.13	0.21	0.15	0.27	0.24	0.31	5.45
$M_{b,1,5}$	1.83	2.03	3.10	4.50	5.31	8.11	10.32	11.68	19.15	23.70	189.12
$r_{b,1,5}$	0.71	0.77	1.05	1.50	1.74	2.53	3.33	3.67	5.81	7.37	59.45
$M_{b,2}$	1.98	2.20	3.39	4.96	5.84	8.89	11.25	12.71	21.36	27.27	207.28
$r_{b,2}$	0.86	0.94	1.33	1.95	2.24	3.33	4.16	4.71	7.93	10.94	77.56

Source: Authors' calculations

Table 9. Behavior of errors of estimation of expected value of reserve ES_i and the total reserve *ES* and behavior of errors of prediction of reserve S_i and the total reserve *S* for disturbances of types I, II, III

	Estimation						Prediction					
i	M_a	r_a	$M_{b,1,5}$	$r_{b,1,5}$	$M_{b,2}$	$r_{b,2}$	M_a	r_a	$M_{b,1,5}$	$r_{b,1,5}$	$M_{b,2}$	$r_{b,2}$
	Rbias						*RbiasP*					
1	2,67	4,24	2,35	3,90	1,93	3,80	3,58	5,14	4,29	7,00	3,58	6,84
2	0,43	0,75	0,42	0,71	0,43	0,69	0,29	0,84	0,72	1,46	0,56	0,86
3	0,15	0,27	0,19	0,35	0,29	0,55	0,40	0,71	0,30	0,83	0,38	1,01
4	0,15	0,18	0,11	0,32	0,20	0,30	0,13	0,44	0,26	0,57	0,28	0,59
5	0,07	0,12	0,07	0,17	0,18	0,26	0,05	0,23	0,17	0,41	0,27	0,58
6	0,05	0,08	0,09	0,15	0,09	0,14	0,10	0,25	0,13	0,37	0,19	0,35
7	0,05	0,09	0,05	0,11	0,08	0,11	0,18	0,33	0,11	0,23	0,11	0,27
8	0,03	0,08	0,08	0,11	0,08	0,15	0,06	0,12	0,12	0,17	0,13	0,27
total	0,06	0,09	0,06	0,09	0,10	0,16	0,07	0,18	0,09	0,14	0,05	0,12
	SPE						*SPEP*					
1	132,98	5,30	189,12	60,68	206,54	78,10	183,91	6,12	264,49	84,25	284,70	104,29
2	16,39	0,21	23,52	7,36	26,96	10,71	28,21	0,40	40,20	12,30	44,20	16,29
3	10,73	0,33	15,19	4,66	17,25	6,63	19,92	0,57	28,57	8,99	31,84	12,31
4	6,61	0,15	9,40	2,89	10,50	3,98	13,56	0,33	19,31	6,21	21,30	7,92
5	5,04	0,14	7,13	2,18	7,97	3,01	10,78	0,29	15,51	4,77	17,12	6,39
6	3,91	0,09	5,59	1,71	6,15	2,27	8,97	0,20	12,98	4,11	14,08	5,21
7	2,88	0,06	4,11	1,28	4,49	1,65	7,05	0,16	10,05	3,09	10,99	4,06
8	2,33	0,06	3,30	1,02	3,63	1,34	5,69	0,15	8,07	2,44	8,91	3,28
total	2,83	0,08	4,00	1,22	4,47	1,68	4,27	0,10	6,10	1,91	6,66	2,47

Source: Authors' calculations

Consider behavior of errors of estimators of expected value of reserves and predictors of reserves. In all types of disturbances the errors behave similarly (see Table 9). The oscillation of relative bias is less than the one of standard percentage error. *SPE* is increasing function of variance of contaminating distribution and increasing function of power of disturbance. Owing to oscillation of errors and their maxima it may be concluded that estimators of expected reserves are robust with respect to the relative bias. However, on account to the other measures of quality, the variance of contaminating distribution and the power of disturbance influence behavior of estimators. For disturbances of the type "b" even for $\varepsilon = 0{,}1$ *SPE* is larger than the one in disturbances of the type "a", moreover they are increasing functions of ε .The errors of prediction are larger than errors of estimation of ES_i , but exhibit analogous trend.

Let us consider disturbances of type IV. Here all distributions that disturb distributions of severity of claims have γ and *p* equal to ones in zero scenario but, compared to this, *Rbias* for $\hat{\gamma}$ is more than 100 times larger, even for $\varepsilon = 0.01$ (cf. Table 10). If power of disturbance increases then *Rbias* also increases significantly, for $\varepsilon = 0.2$ is greater than 23, while for $\varepsilon = 0$ equals -0.0024. For $\varepsilon = 1$, for all i, j ,

there is no mixture of gamma distributions but, similar to the case of $\varepsilon = 0$, we have a gamma distribution. It may explain lower values of errors than that for $\varepsilon \in \{0.1; 0.2; 0.5\}$. In value of *SPE* the bias plays major role, variance of estimator does not change so much (cf. Table 10). Errors of estimation of *p* do not rise as much as errors of $\hat{\gamma}$, they are about 10 times smaller. Comparing Tables 5 and 10 it may be concluded that type IV has larger influence on behavior of estimators than changing of the shape of distribution in the types I, II, III. Moreover, regardless of the power of disturbance, errors for type IVa are greater than that for type IVb.

Table 10. Behavior of errors of estimators $\hat{\boldsymbol{p}}$, $\hat{\gamma}$, $\hat{\phi}$ for disturbance of type IV

	ε									
	0.01		0.1		0.20		0.50		1.00	
	IVa	IVb	IVa	IVb	IVa	IVb	IVa	IVb	Iva	IVb
	Rbias									
$\hat{\boldsymbol{p}}$	-0.2014	-0.0645	-1.5552	-0.6086	-2.3717	-1.1207	-2.3714	-1.8847	0.8831	0.0396
$\hat{\gamma}$	1.6748	0.5356	14.1979	5.1915	23.0648	9.9105	23.0759	17.6414	-6.7869	-0.3153
$\hat{\phi}$	0.8613	0.2543	6.9897	2.4319	11.1164	4.5339	12.7848	7.7715	2.0335	-0.1608
	SPE									
$\hat{\boldsymbol{p}}$	0.2340	0.1264	1.5609	0.6218	2.3754	1.1291	2.3784	1.8893	0.9046	0.1214
$\hat{\gamma}$	1.9512	1.0437	14.2619	5.3132	23.1158	9.9976	23.1726	17.6968	6.9348	0.9839
$\hat{\phi}$	0.9988	0.5202	7.0168	2.4945	11.1336	4.5765	12.8064	7.7962	2.1280	0.4980

Source: Authors' calculations

Analogously to the basic distribution and disturbances of types I, II, III, errors *Rbias* and *SPE* of estimators $\hat{\lambda}_{ij}$, $\hat{\tau}_{ij}$, $\hat{\mu}_{ij}$ oscillate little with accident year *i* and the largest are for development year $j = 10$ (for disturbance of type IVb). Therefore tables presenting these errors for chosen ε include minima (min) and maxima (max) taken over $i \in \{0,1,\ldots,8\}$ or maxima and the upper bound for the oscillation *r*=max-min (see Tables 11, 12, 13).

For disturbances of type IVa distributions of number of claims are the same as that in the zero scenario. Thus it is suprising that values of the errors of $\hat{\lambda}_{ij}$ are very large in comparison to the ones for disturbances of the other types (cf. Tables 6 and 11) and are increasing functions of ε for $j < 10$. Even for $\varepsilon = 0.1$ the minimum of absolute values (over $i \in \{0,1,\ldots,8\}$) of errors is larger than the maximum of the corresponding ones for disturbances of types I, II, III. For disturbances of type IVb distributions of number of claims are mixture of Poisson distributions with different expectations. Thus expected values of generated

variables differ from parameters λ_{ij} of the basic model. Here values of errors are larger than for disturbances of the type I, II, III but not as large as in type IVa (see Table 11). The maximum of *Rbias* is reached for $\varepsilon = 0.5$, while the maximum of *SPE* for $\varepsilon = 0.5$ if $j \le 5$, for $\varepsilon = 1$ if $5 < j < 10$ and for $\varepsilon = 0.01$ if $j = 10$.

Table 11. Behavior of errors of estimators $\hat{\lambda}_{ij}$ for type IV

j	0	1	2	3	4	5	6	7	8	9	10
type IVa, *Rbias*											
	$\varepsilon = 0.1$, $r < 2.5$										
max	1.32	-2.12	-2.21	1.07	12.40	8.46	31.90	41.50	18.90	18.50	-28.24
	$\varepsilon = 1$										
min	-11.79	-22.87	11.49	56.76	153.5	140.1	330.3	409.3	255.4	260.7	-31.60
max	-0.38	-12.89	25.92	77.04	186.3	171.2	386.2	475.3	301.5	307.4	-22.76t
type IVa, *SPE*											
	$\varepsilon = 0.01$, $r \le 0.15$										
max	1.07	1.21	1.85	2.65	3.42	4.84	7.96	10.27	12.51	14.89	103.1
	$\varepsilon = 1$, r similar to the corresponding r for *Rbias*										
max	11.83	22.89	26.03	77.18	186.6	171.6	388.1	478.3	306.2	312.3	95.98
type IVb, *Rbias*											
	$\varepsilon = 0.01$, $r \le 0.04$										
max	0.07	-0.03	-0.18	-0.29	-0.33	-0.46	-0.39	-0.39	-0.56	-0.72	-14.19
	$\varepsilon = 0.5$, $r \le 0.3$										
max	1.49	-0.97	-5.48	-7.61	-8.29	-10.07	-10.84	-11.56	-13.19	-12.85	-28.34
	$\varepsilon = 1$, $r \le 0.1$										
max	-0.01	0.03	0.13	0.21	0.12	0.22	0.07	-0.09	-0.45	-0.69	-8.05
type IVb, *SPE*											
	$\varepsilon = 0.01$, $r \le 0.2$										
max	1.04	1.15	1.80	2.64	3.17	4.77	6.12	7.01	11.55	13.92	103.8
	$\varepsilon = 0.5$, $r \le 0.3$										
max	1.81	1.65	6.00	8.36	9.62	11.98	16.51	19.61	23.96	24.65	84.80
	$\varepsilon = 1$, $r \le 0.3$										
max	1.07	1.18	1.99	3.45	6.42	8.43	18.36	24.21	29.76	30.98	91.21

Source: Authors' calculations

Table 12 presents errors of estimator $\hat{\tau}_{ij}$. For disturbance of type IVb, minima (*m*) and maxima (*M*) are taken over $i \in \{0,1,\ldots,8\}$ and $\varepsilon \in \{0; 0.01; 0.1; 0.2; 0.5; 1\}$. For type IVa, if $j < 10$, the errors reach the largest values for $\varepsilon = 0{,}5$, but they are less than maximum values for errors of $\hat{\lambda}_{ij}$. In this type only for $\varepsilon = 0{,}01$ or $j = 10$ the errors are close to the ones in the other types. For $\varepsilon > 0{,}01$ (except $j = 10$) the errors in type IVb are smaller than in type IVa. It

is worth noting that for $j = 10$ the errors *SPE* are close to each other and are in the interval $(56;68)$ for the type IVb, and interval $(65;69,5)$ for the other types.

Table 12. Behavior of errors of estimators $\hat{\tau}_{ij}$ for type IV

j	0	1	2	3	4	5	6	7	8	9	10
type IVa, *Rbias*											
	$\varepsilon = 1.01, r \le 0.03$										
max	-0.08	0.16	0.74	1.07	1.42	1.59	2.19	2.41	2.05	2.01	-41.11
	$\varepsilon = 0.1, r \le 0.2$										
max	-0.48	1.31	6.01	8.99	11.94	13.47	18.55	20.15	18.59	18.50	-27.88
	$\varepsilon = 0.5, r \le 1.5$										
max	1.31	3.20	12.97	20.44	29.31	31.51	44.72	48.55	43.82	43.92	-17.12t
type IVa, *SPE*											
	$\varepsilon = 0.01, r \le 0.03$										
max	0.29	0.33	0.92	1.32	1.69	2.01	2.84	3.26	3.47	3.83	68.40
	$\varepsilon = 0.1, r \le 0.2$										
max	0.66	1.34	6.04	9.03	12.00	13.56	18.76	20.44	19.06	19.00	67.85
	$\varepsilon = 0.5, r \le 1.5$										
max	1.37	3.21	12.99	20.46	29.33	31.55	44.78	48.62	44.03	44.16	67.18
type IVb, *Rbias*											
m	-1.47	-0.04	-0.14	-0.18	-0.24	-0.30	-0.57	-0.82	-1.26	-1.40	-41.33
M	0.03	1.22	6.05	8.40	9.19	11.48	12.32	12.20	14.34	14.38	-18.31
type IVb, *SPE*											
m	0.26	0.27	0.54	0.77	0.88	1.23	1.52	1.70	2.60	3.09	56.53
M	1.49	1.25	6.08	8.46	9.28	11.62	12.71	12.79	15.23	15.34	67.81

Source: Authors' calculations

Consider the behavior of errors of estimators $\hat{\mu}_{ij}$ (cf. Table 13). For type IVb and *Rbias*, because of very small oscillation, we give minima (*m*) and maxima (*M*) over *i* and ε. Both *Rbias* and *SPE* are increasing functions of power of disturbance for $j < 10$ (excluding *Rbias* for type IVb). The errors are very large for type IVa even for $\varepsilon = 0,1$, compared to that in types I, II, III and IVb. Notice that parameters μ_{ij} in type IVb differ less from parameters μ_{ij} of the basic model than in type IVa. It may explain such large difference in errors between these types of disturbance.

Table 14 presents behavior of estimators of expected reserves and predictors of reserves. It gives maxima (*M*) and the oscillation taken over ε. For type IVb values of *Rbias* and *RbiasP* are close to that for types I, II, III. *SPE* and *SPEP* are decreasing functions of ε for $i = 1$ and increasing ones for $i > 1$. The oscillation of these errors is smaller for $i = 1$ and larger for $i > 1$ than the oscillation for the types I, II, III. The errors of estimation (for $\varepsilon \ge 0,1$) are lower (even more than 10 times) than for type IVa. For the case IVa only for $i = 1$ or

$\varepsilon = 0{,}01$ the errors have values comparable to those in the basic model. For type IVa the errors of prediction are smaller than the ones of estimation of expected value of reserve and they are more stable when ε is changing. It may be caused by moving away from the values of μ_{ij} in the basic model. As it was mentioned, the errors of estimators of these parameters are large, they estimate completely different values, better corresponding to the future reserves.

Table 13. Behavior of errors of estimators $\hat{\mu}_{ij}$ for type IV

j	0	1	2	3	4	5	6	7	8	9	10
type IVa, *Rbias*											
	$\varepsilon = 0.01$, $r \leq 0.25$										
max	0.07	-0.09	0.39	0.97	2.57	2.27	5.52	6.94	3.91	4.06	-1.27
	$\varepsilon = 0.1$, $r < 3$										
max	0.83	-0.83	3.67	10.17	25.86	23.13	56.73	70.67	41.81	41.32	-1.41
	$\varepsilon = 1$										
min	-7.62	-22.75	18.18	77.22	218.2	194.2	498.9	636.1	368.8	377.4	-29.27
max	7.37	-10.22	37.35	106.0	269.8	241.9	596.4	755.8	445.0	454.9	-17.81
type IVa, *SPE*											
	$\varepsilon = 0.01$, $r \leq 0.4$										
max	1.27	1.41	2.17	3.30	4.70	6.26	10.69	13.68	15.88	18.75	129.6
	$\varepsilon = 0.1$, $r < 3$										
max	1.47	2.83	4.27	10.75	26.47	24.47	60.23	76.42	50.92	51.43	127.6
	$\varepsilon = 1$										
min	1.26	10.30	18.36	77.42	218.6	194.9	501.8	641.0	376.4	385.42	97.24
max	7.70	22.77	37.47	106.2	270.3	242.7	600.1	761.6	453.7	464.0	109.2
type IVb, *Rbias*											
m	-0.09	-0.07	-0.07	-0.14	-0.12	-0.11	-0.01	-0.49	-0.18	-0.28	-1.49
M	0.06	0.08	0.08	0.07	0.06	0.19	0.34	0.31	0.29	0.60	1.81
type IVb, *SPE*											
	$\varepsilon = 0.01$, $r \leq 0.15$										
max	1.25	1.37	2.12	3.15	3.78	5.73	7.41	8.52	14.01	16.88	128.8
	$\varepsilon = 1$, $r \leq 0.5$										
max	1.29	1.40	2.37	4.21	7.66	10.27	22.17	29.31	36.03	37.50	110.5

Source: Authors' calculations

CONCLUSION

Except the disturbance of type IVa, in behavior of the errors the power of disturbance and the variance of contaminating distribution play a key role, i.e., the errors are robust to considered disturbances of shape of distribution. The relative bias oscillates less with increasing variance of contaminating distribution than standard percentage error. If variance of contaminating distribution equals the variance of the basic model then oscillation of *SPE* is several times smaller than for contaminating distribution with larger variance. Moreover, *SPE* is increasing

function of the power of disturbance. However, for type IVa both these errors are larger. Here products $\lambda_{ij}\tau_{ij}$ of contaminating distributions differ more from μ_{ij} in the basic model than for the other types. This departure from the assumptions causes the largest differences in the errors of estimation. The difference of expected value of claims does not have significant influence if this is accompanied by the appropriate difference of the expectation of number of claims (see Type IVb). Notice that in the other types of disturbances values μ_{ij} are preserved. The conclusion from analysis is that however the use of the Tweedie model and presented method of estimation allows us to predict reserves, the inference about expected value of payments and of number of payments and of reserves may lead to significant errors. For type IVb and $\varepsilon = 1$ estimates of expected values of number and severity of claims are closer, in the sense of *Rbias* and *SPE*, to parameters in the basic model than to values of actual distributions. The contamination by distributions with parameters μ_{ij} different from that in the basic model (even for small ε) leads also to larger errors of estimation of expected values of reserves than contamination of shape of distributions of claim severities without changing parameters μ_{ij}.

Table 14. Behavior of errors of estimation of ES_i and *ES* and bahavior of errors of prediction of reserve S_i and total reserve *S* for type IV

	type IVa				type IVb			
	Estimation		Prediction		Estimation		Prediction	
i	*M*	*r*	*M*	*r*	*M*	*r*	*M*	*r*
	Rbias		*RbiasP*		*Rbias*		*RbiasP*	
1	-1.31	22.20	3.58	7.90	1.80	3.25	3.58	4.02
2	423.33	423.27	2.66	2.37	0.54	0.82	0.54	1.04
3	409.08	409.00	-0.01	2.06	0.29	0.44	0.36	0.67
4	617.81	617.74	-0.18	21.54	0.22	0.41	0.02	0.97
5	575.38	575.31	-0.01	17.96	0.20	0.26	0.68	0.86
6	472.18	472.14	-0.06	19.30	0.14	0.22	0.13	0.61
7	348.97	348.96	-0.06	7.33	0.06	0.10	0.57	0.63
8	260.79	260.79	-0.03	4.79	0.07	0.09	0.82	0.86
Total	370.82	370.80	-0.05	10.64	0.10	0.14	0.42	0.48
	SPE		*SPEP*		*SPE*		*SPEP*	
1	129.5	26.5	190.30	9.89	128.7	18.23	184.19	29.38
2	432.0	415.7	37.86	9.96	37.0	20.72	64.15	36.25
3	413.3	402.6	26.86	7.27	26.0	15.34	45.38	25.79
4	620.7	614.1	26.83	13.45	19.8	13.26	35.25	21.87
5	576.9	572.0	21.62	10.88	15.0	10.01	27.15	16.41
6	473.1	469.3	21.97	13.11	10.5	6.57	20.46	11.59
7	349.4	346.5	10.61	3.56	6.7	3.82	14.58	7.54
8	261.0	258.7	7.93	2.30	4.5	2.18	10.32	4.69
Total	371.4	368.6	12.14	7.95	7.5	4.73	9.94	5.75

Source: Authors' calculations

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APPENDIX 1

Table 1.1. Values of parameters $10000\lambda_{ij}$ in the basic model

	j										
i	0	1	2	3	4	5	6	7	8	9	10
0	571.74	227.45	39.944	17.485	13.291	6.1327	4.4623	4.2884	2.0140	2.0033	0.0645
1	589.64	234.57	41.194	18.032	13.707	6.3246	4.6019	4.4226	2.0770	2.0660	0.0665
2	621.31	247.17	43.407	19.001	14.443	6.6644	4.8492	4.6602	2.1886	2.1770	0.0701
3	610.80	242.99	42.673	18.679	14.199	6.5516	4.7671	4.5813	2.1515	2.1401	0.0689
4	613.92	244.23	42.891	18.775	14.271	6.5850	4.7915	4.6047	2.1625	2.1510	0.0692
5	592.70	235.79	41.408	18.126	13.778	6.3574	4.6258	4.4455	2.0878	2.0767	0.0668
6	605.79	241.00	42.323	18.526	14.083	6.4979	4.7281	4.5438	2.1339	2.1226	0.0683
7	584.01	232.33	40.801	17.860	13.576	6.2642	4.5580	4.3804	2.0572	2.0463	0.0659
8	597.80	237.81	41.764	18.282	13.897	6.4121	4.6656	4.4838	2.1057	2.0946	0.0674

Table 1.2. Values of parameters τ_{ij} in the basic model

	j										
i	0	1	2	3	4	5	6	7	8	9	10
0	2997.0	2467.6	1710.0	1436.6	1355.9	1151.8	1077.1	1068.1	910.77	909.75	440.79
1	3016.6	2483.7	1721.1	1445.9	1364.7	1159.3	1084.1	1075.1	916.71	915.68	443.67
2	3050.0	2511.3	1740.2	1462.0	1379.8	1172.2	1096.2	1087.0	926.88	925.84	448.59
3	3039.1	2502.2	1733.9	1456.7	1374.9	1168.0	1092.2	1083.1	923.55	922.51	446.98
4	3042.3	2504.9	1735.8	1458.3	1376.4	1169.2	1093.4	1084.3	924.54	923.51	447.46
5	3019.8	2486.4	1723.0	1447.5	1366.2	1160.6	1085.3	1076.3	917.71	916.68	444.15
6	3033.8	2497.9	1730.9	1454.2	1372.5	1165.9	1090.3	1081.2	921.95	920.92	446.20
7	3010.5	2478.7	1717.6	1443.0	1361.9	1157.0	1082.0	1072.9	914.86	913.83	442.77
8	3025.3	2490.9	1726.1	1450.1	1368.7	1162.7	1087.3	1078.2	919.37	918.34	444.96

Table 1.3. Number of policies and values of parameters p, γ, ϕ

I	0	1	2	3	4	5	6	7	8
w_i	112953	110364	105400	102067	99124	101460	94753	92326	89545
p = 1.1741431		γ = 4.7424055		ϕ = 1481.7243					

Source: Authors' calculations based on data from Wüthrich [2003]

APPENDIX 2

Table 2.1. Values of parameters $10000\widetilde{\lambda}_{ij}$

	j										
i	0	1	2	3	4	5	6	7	8	9	10
0	551.47	309.86	37.626	11.863	4.5152	2.1248	1.1509	1.0624	0.5312	0.3541	0.0885
1	579.45	302.82	36.425	9.786	2.8089	1.2685	1.0873	0.4530	0.5437	0.4530	0.0899
2	607.78	278.94	38.046	9.298	3.9848	1.7078	0.4744	0.2846	0.2846	0.4089	0.0905
3	602.35	283.93	29.490	9.014	4.0170	2.2534	1.1757	0.9797	0.4573	0.4058	0.0898
4	600.46	272.29	30.669	9.483	4.9433	2.2194	0.7062	0.6888	0.4515	0.4007	0.0887
5	583.88	265.33	29.568	8.969	3.1540	2.2669	0.8898	0.6688	0.4384	0.3890	0.0861
6	585.21	290.65	30.817	8.126	3.6938	1.9596	0.9170	0.6892	0.4518	0.4009	0.0888
7	597.88	266.34	28.919	8.773	3.8086	1.9323	0.9043	0.6796	0.4456	0.3953	0.0875
8	601.93	248.37	24.904	9.154	3.7345	1.8947	0.8867	0.6664	0.4369	0.3876	0.0858

Table 2.2. Values of parameters $\widetilde{\tau}_{ij}$

	j										
i	0	1	2	3	4	5	6	7	8	9	10
0	2864.2	2126.4	2106.9	3042.9	4061.4	2565.4	1229.1	2077.0	206.0	3910.8	321.0
1	3052.2	1991.8	2341.9	1438.8	2240.6	2697.8	4486.0	22278.2	7043.8	5166.6	331.8
2	3120.7	2152.2	2743.6	2853.6	3872.7	3888.9	11375.6	3293.7	6552.0	4949.2	350.5
3	3140.2	2032.4	2634.6	3359.2	3559.0	4237.6	2293.6	6192.0	4370.8	4898.6	346.9
4	3073.2	2146.5	2267.9	3070.5	7051.5	5026.6	16549.0	7178.1	4395.9	4926.8	348.9
5	3179.0	2132.7	1939.3	2731.5	3339.8	4096.2	5706.1	7228.6	4426.9	4961.4	351.4
6	3093.4	2145.3	1797.3	2992.9	9911.5	3851.2	5611.5	7108.7	4353.5	4879.2	345.6
7	3010.0	2078.6	2072.2	3121.9	4870.7	3766.0	5487.4	6951.6	4257.2	4771.3	337.9
8	3123.5	2249.1	2194.4	2921.9	5129.4	3966.0	5778.8	7320.8	4483.3	5024.7	355.9

Source: Authors' calculations based on data from Wüthrich [2003]

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AN EXAMPLE OF NETWORK DEA – ASSESSMENT OF OPERATING EFFICIENCY OF UNIVERSITIES

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Abstract: The purpose of the article is to present the use of network DEA for evaluating the efficiency of Polish universities. Network DEA assessment of operating efficiency of universities was compared with the single stage DEA results that focus on teaching efficiency, research efficiency and aggregate performance. The study presented in the paper – though limited in scope – shows that biggest Polish universities are diversified in regard to the efficiency of their performance.

Keywords: DEA, network DEA, efficiency, universities

INTRODUCTION

Since the initial work by Charnes et al. (1978), Data Envelopment Analysis (DEA) has been methodological developed and widely successfully applied to assess the relative efficiency of entities – Decision Making Units (DMUs) in many areas. Education is an area of application that attracted the most attention in the early days of the development of the DEA and is still very popular. According to [Liu et al. 2013] there are 184 papers that concern DEA applications in education in journals indexed by the Web of Science database from 1978 through 2010. It provides a fifth place of the most common field of study in DEA literature. Looking for the reasons it is worth noting that DEA allows multi input and output variables, it can take into account many different fields of the performance [Chodakowska 2015]. Outcomes of DEA may provide valuable information supporting the management of educational institutions. It allows identifying strengths and weaknesses, the mode of fund allocation among institutions, or the optimal size of them [Nazarko, Šaparauskas 2014]. Brief references of the newest efficiency studies in the higher educational institutions is given below.

Among others, DEA was explicitly used to asses research productivity ([Johnes, Yu 2008], [Kao, Hung 2008]), teaching quality ([Celik, Ecer 2009], [Johnes 2006]) or aggregate performance ([Johnes 2014], [Leitner et al. 2007], [Worthington, Lee 2008]) of universities or theirs departments. Such a distinction represent the two most important activities of a university: teaching and researching with transferring the new knowledge. There were also attempts to assess the allocative efficiency ([Abbasi et al. 2010], [Tauer et al. 2007]).

Network DEA models are relatively recent innovations on methodology that gain popularity in various application areas [Liu et al. 2013]. The examples of network DEA application in the area of higher education proposed [Saniee Monfared, Safi 2013].

The aim of this paper is to use a network DEA approach to perform an efficiency analysis of the Polish universities.

METHODS

The proposed methodology

Network DEA to measure the relative efficiency of a system, taking into account its internal structure, was popularized by Färe and Grosskopf ([Färe, Grosskopf 1996], [Färe, Grosskopf 2000], [Färe et al. 2007]).

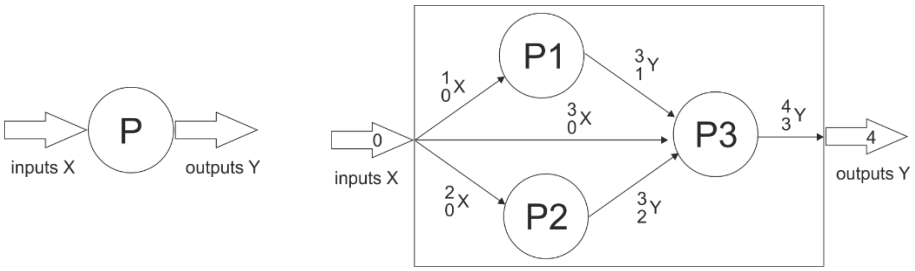
The main difference between network DEA and conventional non-network DEA is that while conventional DEA considers a single process that consumes all the inputs and produces all the outputs, network DEA considers the existence of several stages each of which consumes its owns set of inputs and produce its own set of outputs, in addition to consuming and producing intermediate products. These intermediate products are defined as inputs for some stages are outputs for others [Saniee Monfared, Safi 2013].

The differences between non-network and networks models are illustrated in Fig. 1. Inputs are denoted by $X=(X_1, \ldots, X_m)$ and outputs by $Y=(Y_1, \ldots, Y_s)$. In non-network technology inputs are transformed in the process P into outputs [Färe, Grosskopf 2000]. In network DEA model available inputs are attached to sub-technologies. Assuming that there are three production processes P1, P2 and P3 that use the same source of inputs, ${}^{1}_{0}X$ is employed by P1, ${}^{2}_{0}X$ by P2, and ${}^{3}_{0}X$ by P3. The total amounts used in three activities cannot exceed the total amount available [Färe et al. 2007]:

$$X \geq \sum_{i=1}^{3} X_0^i \tag{1}$$

P3 uses ${}^{3}_{0}X$ as exogenous input and ${}^{3}_{1}Y$, ${}^{3}_{2}Y$ as intermediate inputs. The final product from P3 is output vector ${}^{4}_{3}Y$.

Figure 1. A non-network versus network technology



Source: adopted from [Färe et al. 2007]

Since its introduction various network models have been developed and proposed to measure the efficiency [Kao 2014]. In this case, in order to evaluate the efficiency of Polish universities, two-stage process was considered where X are inputs to the first stage and outputs from the first stage at the same time are inputs to the second stage. The implemented DEA model inspired by (Saniee Monfared, Safi 2013) and can be written as follows [Chiu et al. 2011], [Zhu 2003]:

$$\min\left(\theta_{1j_0} + \theta_{2j_0}\right)/2$$

(stage 1)

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_{1j_0} x_{ij_0} \quad i = 1, 2, \ldots, I \\ &\textstyle\sum_{j=1}^{n} \lambda_j q_{ej} \ge q_{ej_0} \quad e = 1, 2, \ldots, E \\ &\textstyle\sum_{j=1}^{n} \lambda_j z_{dj} \ge \tilde{z}_{dj_0} \quad d = 1, 2, \ldots, D \end{aligned} \tag{2}$$

(stage 2)

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \mu_j p_{sj} \le \theta_{2j_0} p_{sj_0} \quad s = 1, 2, \ldots, S \\ &\textstyle\sum_{j=1}^{n} \mu_j z_{dj} \le \tilde{z}_{dj_0} \quad d = 1, 2, \ldots, D \\ &\textstyle\sum_{j=1}^{n} \mu_j y_{rj} \ge y_{rj_0} \quad r = 1, 2, \ldots, R \\ &\lambda_j, \mu_j \ge 0 \quad j = 1, 2, \ldots, n \end{aligned}$$

where:

$X_j = \left(x_{1j}, x_{2j}, x_{3j}, \ldots, x_{Ij},\right)$ – input vector of stage 1,

$Q_j = \left(q_{1j}, q_{2j}, q_{3j}, \ldots, q_{Ej},\right)$ – output vector of stage 1,

$P_j = \left(p_{1j}, p_{2j}, p_{3j}, \ldots, p_{Sj},\right)$ – input vector of stage 2,

$Y_j = \left(y_{1j}, y_{2j}, y_{3j}, \ldots, y_{Rj},\right)$ – output vector of stage 2,

z_{dj} – the *d*th intermediate variables of the *j*th DMU,

$\tilde{z}_{ij_0}$ – decision variables,

$j = 1, 2, \ldots, n$ – number of DMUs,

θ_{j_0} – efficiency ratio taking values in the range <0.1>; 1 for fully effective entities. The larger θ_{j_0} is, the better efficiency DMU_{j0} has.

A Case Study

In Poland in the 2013/2014 academic year there were nearly 1 550 thousand students in 133 public and 312 non-public higher education institutions [Jagielski, Żebrowska 2013]. To show an example of network DEA assessment of operating efficiency of universities was considered a data set that consists of 12 Polish universities with h-index over 80. They are also the largest Polish universities taking into account number of undergraduate and graduate students (Table 1).

Table 1. Performance of selected universities – data from 2013

University	Abbrev.	h Index	Grants [PLN]	The number of students	The number of PhD students
University of Warsaw	Univ Warsaw	197	108 016 954	46 125	3 167
Jagiellonian University in Cracow	Jagiellonian Univ	163	115 497 162	42 374	3 220
Warsaw University of Technology	Warsaw Univ Technol	121	25 080 067	34 135	1 248
University of Wroclaw	Univ Wroclaw	101	33 121 948	26 239	1 554
University of Gdańsk	Univ Gdansk	98	20 554 024	27 640	1 431
Nicolaus Copernicus University in Toruń	Nicholas Copernicus Univ	95	21 058 072	28 034	881
AGH University of Science and Technology in Cracow	Agh Univ Sci Technol	93	23 629 991	33 244	987
Adam Mickiewicz University in Poznań	Adam Mickiewicz Univ Poznan	91	43 969 843	40 633	1 382
University of Łódź	Univ Lodz	88	17 360 475	36 734	3 220
Poznań University of Technology	Poznan Tech Univ	84	15 444 946	34 135	597
Wrocław University of Technology	Wroclaw Univ Technol	82	27 481 164	34 428	1 091
University of Silesia in Katowice	Silesian Univ	80	15 074 819	26 908	1 372

Source: [Jagielski, Żebrowska 2013], [Ranking 2014]

To assessment of operating efficiency of universities was done under the assumption that the most important resources are people. The number of academic staff contains Table 2. Universities differ in the number of research workers, especially in the number of employed assistants. For the purposes of this study, the following translation and grouping of academic positions was done: assistant – polish *asystent*, junior assistant professors – polish *adiunkt* and *docent*, associate professor and professor – polish *profesor nadzwyczajny* and *profesor zwyczajny*.

This division of academic staff was done due to the assumed difference in the duties usually expressed in the number of normal working hours of teaching, payments, and employers' expectations. It is noteworthy that at Polish universities the smallest group of employees are assistants. The largest group are junior assistant professors and associate professors.

Table 2. Employment at universities in 2013

University	The number of professors	The number of junior assistant professors	The number assistants	The number of other teachers
Univ Warsaw	865	1 721	152	587
Jagiellonian Univ	671	1 708	643	617
Warsaw Univ Technol	474	1 087	221	366
Univ Wroclaw	422	1 111	43	239
Univ Gdansk	463	749	162	285
Nicholas Copernicus Univ	481	885	319	315
Agh Univ Sci Technol	441	1 152	330	225
Adam Mickiewicz Univ Poznan	733	1 481	14	646
Univ Lodz	551	1 015	200	395
Poznan Tech Univ	198	560	254	234
Wroclaw Univ Technol	358	1 063	223	261
Silesian Univ	355	1 014	146	293

Source: [Jagielski, Żebrowska 2013], [Ranking 2014], [The National Science Centre]

Using the data simple single stage DEA CCR-I models that the focus on teaching efficiency, research efficiency or aggregate performance can be applied.

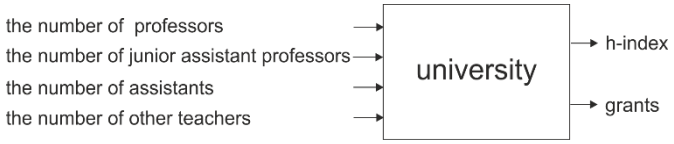
The teaching efficiency model (Fig. 2) assumes that the academic staff should above all teach, so the inputs in form of numbers of professors, assistant professors, assistants and lecturers should produce the graduates: bachelors, master and PhD students. In Poland, the number of graduates is practically equal to the number of students in their last year of studies. Author is aware that such assumption exclude the quality measurement of teaching but promotes cost minimizing. But the determination of the quality of teaching is not possible without the study of labour market situation of graduates, without taking into account localisation of the university and in particular its candidates among which recruits its students.

Figure 2. Teaching efficiency

Source: adopted on the basis of [Saniee Monfared, Safi 2013]

Using DEA scholarly productivity of academic staff can also be calculated (Figure 3.). Presented model takes into account the total value of grants founded by National Science Centre (NCN) and h-index.

Figure 3. Research efficiency



Source: adopted on the basis of [Saniee Monfared, Safi 2013]

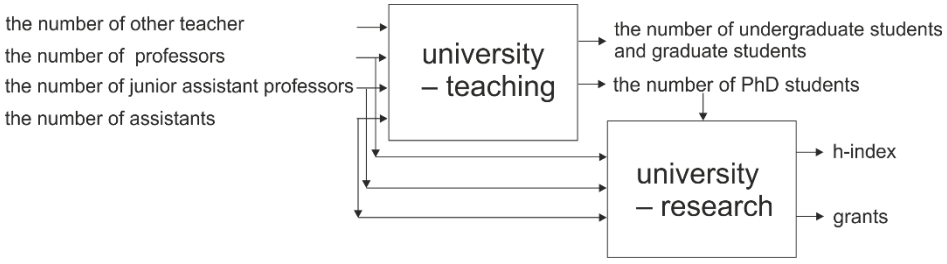
Aggregated model do not distinguishes between research and teaching activities. Inputs in form of number of academic staff produce graduates, high cited articles and win grants (Figure 4.).

Figure 4. Teaching and research efficiency model – aggregated model

Source: adopted on the basis of [Saniee Monfared, Safi 2013]

The network DEA model was built on the assessment that 1/3 working time is used for teaching by professors, assistant professors and assistant, while other teachers spend the whole time for teaching. In addition, PhD students assist with scholarly works.

Figure 5. Teaching and research efficiency model – a network DEA



Source: adopted on the basis of [Saniee Monfared, Safi 2013]

In the all models author assumed that the working time devoted to didactic and research work can be represented by the data of number of employees. Moreover, the division of 1/3 and 2/3 for teaching and scientific work, done in the network model work is simplistic and might be controversial. Author is aware that in practise, depending on university, even with the same group of workers, number of teaching hours spread unevenly.

The results of the network DEA model against three alternative single-stage models presented in Table 3.

RESULTS, DISCUSSION AND LIMITATIONS

The network DEA approach was tested and its results were compare with the single-stage DEA models.

Table 3. Results of the single-stage and network DEA models

University	Teaching efficiency	Research efficiency	Aggregated efficiency	Network DEA efficiency
Univ Warsaw	0.949	1	1	0.974
Jagiellonian Univ	0.851	1	1	0.925
Warsaw Univ Technol	0.804	0.918	0.918	0.860
Univ Wroclaw	1	1	1	1
Univ Gdansk	0.928	1	1	0.964
Nicholas Copernicus Univ	0.669	0.816	0.818	0.713
Agh Univ Sci Technol	1	0.978	1	0.814
Adam Mickiewicz Univ Poznan	1	1	1	1
Univ Lodz	1	0.683	1	0.794
Poznan Tech Univ	1	1	1	1.000
Wroclaw Univ Technol	0.984	0.833	1	0.854
Silesian Univ	0.900	0.783	0.907	0.807

Source: own calculation performed using Solver Microsoft Excel and VBA procedures

It is worth notice that the average efficiency is high, regardless the model. It is the effect of correspondence between the number of DMUs and the sum of the number of inputs and outputs. Among the compared models the network DEA has the highest discriminant power since they use more detailed data.

DEA is an effective technique for measuring the relative efficiency of a set of decision making units (DMUs) that apply multiple inputs to produce multiple outputs. However, it should be recalled that the efficiency scores do not measure the performance of Polish universities relative to their counterpart in the world. Polish universities do not occupy important places in global rankings. Academic Ranking of World Universities [http://www.shanghairanking.com] assesses two Polish high schools and in databases records three more.

Apart for that, author is of the opinion that the DEA implementation can be a complementary tool to rankings with fixed weight such as mentioned Academic Ranking of World Universities or The Ranking Web or Webometrics

[http://www.webometrics.info/] or local comparisons to see for example Perspektywy education institutions ranking [http://www.perspektywy.pl/portal/].

Network DEA approach that takes the internal structure of the system into account in measuring efficiency, can supersede conventional DEA models. However, author agrees that comparison of teaching and scholarly achievements of universities is complex and evokes a considerable amount of controversy [Nazarko, Šaparauskas 2014].

In terms to further research, a key limitation of this work is arbitrary selection of variables and the model. Definitely it would be reasonable to check other variables and the relationship (transformation processes) between the assets and the outcomes of the universities. Another issue, emphasized by researchers, as the most difficult area in academic performance evaluation, is measuring teaching quality. The number of students are limited by the regulations, students' achievements recorded on the diplomas, and their subsequent situation on the labour market as the output performance are in high degree the result of the students' initial ability which they have already acquired before enrolment at the university. Analogically, typical research performance indicators included number of publications or citation, and journal impact factors are discussed and criticised. Furthermore, teaching and research although considered as the two major tasks of the universities, are not the only outputs. Universities produce also some extremely difficult to measure social outputs.

CONCLUSIONS

The paper presented an example of network DEA method in the efficiency assessment of Polish universities. The study presented in the paper – though limited in scope – shows that the biggest Polish universities are diversified in regard to the efficiency of their performance. The results of the analysis show that selection of variables and models affect unit's efficiency. On the basis of selected Polish universities, author shows that the application of network DEA could be useful and provides additional knowledge about the efficiency of educational institutions, but proper interpretation of these results requires a high degree of caution when formulating conclusions.

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QUANTITATIVE METHODS IN ECONOMICS
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GRAPH METHODS IN AN ANALYSIS OF A LEVEL AND A STRUCTURE OF PUBLIC COSTS OF UNIVERSITIES

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Abstract: In this article a usage of graph methods as a tool for a classification of polish public universities due to a generated level and a structure of costs is shown. The research was conducted in a spatial interpretation and referring to 57 public universities of an academic function. The usage of graph methods constitutes a tool which can be used by decision makers in a process of searching for ineffective usage of public sources. The outcomes should be a basis for further cause and effect analysis.

Keywords: taxonomic methods, public universities, an analysis of a level and a structure of costs

INTRODUCTION

Managing finances of a public university depends on efficient management of economic information. Its presentation should show clear and open financial data. The lack of proper presentation of a situation may lead to incorrect financial politics in relation with a public university and inside the university. Spatial analysis of a level and a structure of costs should be used as a tool in a process of an effective usage of a public source. The outcomes should be fully used in creating a strategy of higher education development and in internal cause and effect financial analysis of every university.

An article constitutes a continuation of researches of the authors in order to show the usefulness of the usage of chosen taxonomic methods in a process of checking a level and a structure of costs of public universities. [1]

The aim of this article is to present the possibilities of the usage of graph methods in a process of public universities grouping due to a level and a structure of costs – in order to rationalize a politics of public universities financing.

THEORETICAL BASIS OF THE METHOD

A classification of public universities due to a level and a structure of costs in a spatial presentation was conducted with the usage of graph methods. By using graphs it is possible to show relations between researched objects [Nowak 1990].

One of the well known graph method is wroclaw taxonomy created by K. Florek, J. Łukasiewicz, J. Perkal, H. Steinhaus and S. Zubrzycki, a tree method by R.C. Prim and a graph method by W.Pluta .

In this article a taxonomic graph method was used (so called wrocławska taxonomy) which is also called a tree method. The rules of wroclaw taxonomy method were created by An Application Group of a Public Mathematical Institute in Wrocław [Florek and others 1951]. The basis for building a matrix of a distance between objects is calculated on the basis of Euklid certificate [Panek, Zwierzchowski 2013].

A base to divide a set of researched objects into typological groups is so called a tree which segregates them in a non-linear way [Nowak 1990, s. 69]. A process of creating a wroclaw tree in this method is a process of many stages. Apices of the graph correspond with the objects of a classification and edges (arcs and cords) – distances between the examined objects. A tree constitutes a broken line, continuous which can branch out but cannot include cycles (broken and completed) and combines all objects of the examined set, grouping them in a non-linear way [Nowak 1990, p. 70].

Building a tree consists of few stages (see Ćwiąkała-Małys, Nowak [2009]). Accumulation is understood as a coherent graph in which all the apices are connected by a continuous line of edges – a tree. There are many separate coherent graphs so creating a tree is continued and each accumulation is linked in places designated by the smallest distances. A process of creating a tree is assumed to be finished when all accumulations are linked and where is a coherent graph.

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Another issue in this research was to indicate typological groups [Ćwiąkała-Małys, 2009]. It was obtained by cutting out edges from the longest tree. There are many different ways of a tree division. To separate from a T tree some typological groups you should divide it into T parts. So from a tree you eliminate T-1, it is the longest edge. Number T may be described in various ways, for instance according to the tips suggested by authors of wroclaw taxonomy [Florek i in. 1951].

Another way of separating typological groups on the basis of a tree was suggested by Z. Hellwig [1968], by using so called limited distance d^*, which was specified according to a formula [Nowak, 1990]:

$$d^* = d_1 + u \cdot (d_2 - d_1), \tag{1}$$

Where $d_1 = \min_i \min_j \{d_{ij}\}$, $d_2 = \max_i \max_j \{d_{ij}\}$ and d_{ij} defines a distance between i and j typological object. In formula (1) an u value is determined by a researcher. It should belong to [0,1] range. The smaller the value of a parameter the more typological groups we obtain.

Another step of the research was a choice of representatives of typological groups according to a result of a classification. In this situation we choose one object from each group, a choice should be made in such a way so that representatives are as similar as possible to the remaining elements from their group (a model object represents the remaining objects from the group) and least similar to the other elements outside the group (which enables to thoroughly learn about various structures of cost from researched objects). In this work a method of the centre of gravity was used [compare Pluta 1976] to choose representatives. While choosing representatives of the typological groups it is vital to aim at choosing such an object that will possibly be the most similar to the remaining objects which were not chosen as representatives.

EMPIRICAL MATERIAL AND RESEARCHES RESULTS

A subject of the research were public universities. The researched universities were initially divided into six groups according to the following structure: academic (U), technical (T), economic (E), environmental and life sciences (R), physical education (S), and pedagogical (P). Full names of the universities from all province were included in a table 5 (Annex). Data which were taken into consideration were taken from financial reports of the researched universities from 2006 [Ćwiąkała-Małys 2010]. The data included information according to the amount of costs of amortization, materials and energy, foreign services, taxes and fees, remunerations with margins, a value of sold goods and materials, financial costs and remaining costs including operational costs.

In the first step of grouping a structure of costs was taken into consideration. The usage of taxonomic methods to classify objects requires measures of structure

similarity of classified units. In this case as a measure of distance between the units due to a structure of costs a formula for a distance was used [Nowak 1990]:

$$d_{ij} = \sqrt{\frac{1}{2}\sum_{k=1}^{K}(u_{ik} - u_{jk})^2}, \tag{2}$$

where u_{ik} is an indicator of a structure of a *k*-element for an *i*-object.

A matrix of distances calculated according to (2) constituted a basis for further analysis. Due to the usage of such a distance with a regulated value [Szymanowicz 1977], similar objects according to a structure have a value of distance close to zero and values close to unity tell about huge diversity of researched structures.

On the basis of data pairs of universities were distinguished which are the most similar as a structure of costs is concerned (Table 1) and a tree was made which shows similarity of universities due to a structure of their costs (Figure 1).

It is visible that in respect of costs groups of universities do not exchange mutually which means that universities from a given group do not always have the most similar object in respect of a structure of costs from the same group of universities. It is also noticeable that in all cases a distance of a given university from its 'closest neighbour' is small (does not exceed 0,1) which indicates that there is close similarity of researched objects.

Table 1. A specification of pairs of universities which are the most similar due to a structure of costs

University	Distance	The most similar object	University	Distance	The most similar object	University	Distance	The most similar object
U1	0.0137	U8	T3	0.0141	T6	R4	0.0336	U4
U2	0.0157	T8	T4	0.0125	T1	R5	0.0264	U14
U3	0.0137	R6	T5	0.0198	U14	R6	0.0137	U3
U4	0.0336	R4	T6	0.0136	T8	R7	0.0228	U2
U5	0.0157	U10	T7	0.0198	T3	R8	0.0126	U6
U6	0.0126	R8	T8	0.0136	T6	E1	0.0226	U7
U7	0.0159	R1	T9	0.0331	T18	E2	0.0174	T8
U8	0.0137	U10	T10	0.0391	T9	E3	0.0159	T6
U9	0.0468	S4	T11	0.0176	U2	E4	0.0329	T2
U10	0.0196	P1	T12	0.0158	T3	E5	0.0183	U12
U11	0.0149	P3	T13	0.0271	U2	S1	0.1069	S3
U12	0.0183	E5	T14	0.0188	R6	S2	0.0203	T14
U13	0.0198	U16	T15	0.0171	T6	S3	0.0521	U9
U14	0.0198	T5	T16	0.0202	T5	S4	0.0468	U9
U15	0.0830	T9	T17	0.0320	T14	S5	0.0241	T11
U16	0.0198	U13	T18	0.0266	U16	S6	0.0242	U8
U17	0.0182	T3	R1	0.0141	R6	P1	0.0196	U9
T1	0.0125	T4	R2	0.0921	T5	P2	0.0149	U6
T2	0.0268	E1	R3	0.0174	T15	P3	0.0149	U11

Source: self-study

As the following step groups of universities similar in respect of a structure of costs were indicated, by using threshold value d^*= 0,024,calculated on the basis of a formula (1) for a constant value u=0,04. (higher values made all universities to be accumulated in one place). Results of grouping with their representatives from particular groups were included in Table 2.

It is visible that one-element university groups are physical education universities which shows that there is diversification of costs in such universities and big academic centres with specific conditions following from a number of employees and students but also from a character and interdisciplinarity of the centres.

In the next step of the research the amount of costs in academic centres was taken into consideration. Costs data were standardized to make them comparable and homogenous due to changeability and location. On the basis of regulated values of features a matrix of Euklid distances between universities was distinguished. Then accumulations of first choice and pairs of universities the most similar in respect of value of costs according to groups were indicated (Table 3).

Although again there were mostly universities that did not have their similar object from the same group it is noticeable that there is common feature that appears between researched universities which should not surprise. Namely, it is visible that big academic centres have another big centre as their 'neighbour'. At the same time, it is worth noticing, that in some cases even the most similar object (so the one that has the smallest distance in respect of costs), was relatively far from its 'neighbour'. As an example we can name University of Warsaw, its distance from the closest university (Jagiellonian University in Kraków) in respect of value of costs was equal to 8,22 (which is 36 time higher than a distance of University of Bielsko-Biała from its most similar centre – University of Physical Education in Wrocław).

A tree obtained with the usage of wroclaw taxonomy method for public universities showing their diversification in respect of their value of costs was presented in Figure 2.

In another step groups of universities similar in respect of value of costs were distinguished with the usage of line value d^*= 0,7837, calculated according to a formula (1) for a constant u=0,04. The result of grouping with the representatives was included in Table 4.

Figure 1. A tree obtained with the usage of wroclaw taxonomy method for a structure of costs of public universities

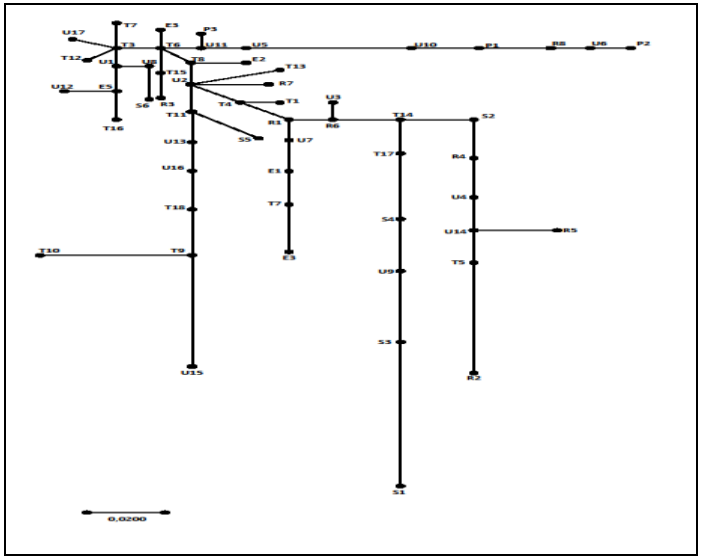
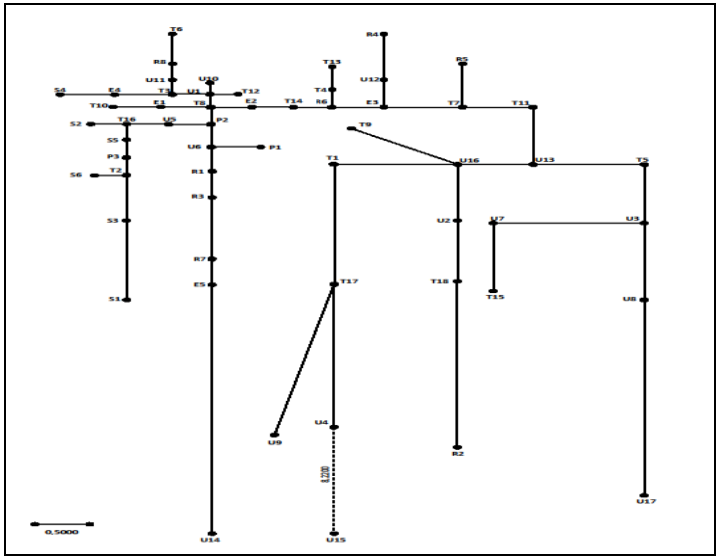


Figure 2. A tree obtained with the usage of wroclaw taxonomy method for the value of costs of public universities



Source: self-study

Table 2. A group of universities with a representatives in wroclaw taxonomy division due to a structure of costs

Group	University	Representative
group 1	U17,T12, T3, T7, U1, E5, U12, T16, U8, T6, E3, T15, R3, U11, P3, U5, T8, E2, U2, R7, T11, U13, U16, T4, T1, R1, R6, U3, T14 ,S2, U7, E1	Silesian University of Technology
group 2	E3	Warsaw School of Economics
group 3	T2	University of Bielsko-Biala
group 4	T9	Lodz University of Technology
group 5	T10	Opole University of Technology
group 6	T13	Rzeszów University of Technology
group 7	T17	Warsaw University of Technology
group 8	T18	Wrocław University of Technology
group 9	U9	Nicolaus Copernicus University of Toruń
group 10	U15	University of Warsaw
group 11	R2	Warsaw University of Life Sciences
group 12	R4	Poznań University of Life Sciences
group 13	R5	Wrocław University of Environmental and Life Sciences
group 14	U4, U14, T5	University of Warmia i Mazury in Olsztyn
group 15	U10, P1, R8, U6, P2	Kazimierz Wielki University in Bydgoszcz
group 16	S1	University of Physical Education in Cracow
group 17	S3	The Jerzy Kukuczka University of Physical Education in Katowice
group 18	S4	Józef Piłsudski University of Physical Education in Warsaw
group 19	S5	Gdansk University of Physical Education and Sport
group 20	S6	University of Physical Education in Wrocław

Source: self-study

Table 3. A specification of pairs of universities which are the most similar due to the amount of costs

University	Distance	The most similar object	University	Distance	The most similar object	University	Distance	The most similar object
U1	0.2444	U10	T3	0.3170	U1	R4	1.0553	U12
U2	1.3507	U16	T4	0.3544	T6	R5	0.8721	T7
U3	1.1080	T5	T5	0.9029	U13	R6	0.3236	T14
U4	3.9358	T17	T6	0.6884	R8	R7	0.4214	R5
U5	0.3297	T16	T7	0.6265	T11	R8	0.3919	U11
U6	0.4114	P1	T8	0.2520	U1	E1	0.3963	T8
U7	1.2463	T15	T9	1.4724	U16	E2	0.3640	T8
U8	1.5124	U3	T10	0.4398	E1	E3	0.4789	R6
U9	4.1577	T17	T11	0.6265	T7	E4	0.5423	S4
U10	0.2444	U1	T12	0.2660	U1	E5	0.4214	R7

University	Distance	The most similar object	University	Distance	The most similar object	University	Distance	The most similar object
U11	0.3204	T3	T13	0.4724	T4	S1	1.2712	S3
U12	0.5899	E3	T14	0.3236	R6	S2	0.2824	T16
U13	0.7805	U16	T15	1.2463	U7	S3	0.7326	T2
U14	6.4074	E5	T16	0.2693	S5	S4	0.5423	E4
U15	8.2200	U4	T17	3.5354	T1	S5	0.2486	P3
U16	0.7805	U13	T18	1.5160	U2	S6	0.2271	T2
U17	4.2856	U8	R1	0.4569	U6	P1	0.4114	U6
T1	1.4736	U16	R2	3.1891	T18	P2	0.3105	T8
T2	0.2271	S6	R3	0.5862	R1	P3	0.2486	S5

Source: self-study

Table 4. Groups of universities with representatives in wroclaw taxonomy division due to the amount of costs

Group	University	Representative
group 1	S4, E4, T3, U11, R8, T6, U1, U10, T12, T8, E1, T10, P2, U5, T16, S2, S5, P3, T2, S6, S3, U6, P1, R1, R3, E2, T14, R6, T4, T13, E3, U12, T7, T11	University of Bielsko-Biala
group 2	S1	University of Physical Education in Cracow
group 3	R7, E5	Wrocław University of Economics
group 4	R2	Warsaw University of Life Sciences
group 5	R4	Poznań University of Life Sciences
group 6	R5	Wrocław University of Environmental and Life Sciences
group 7	T1	University of Science and Technology in Cracow
group 8	T5	Gdańsk University of Technology
group 9	T9	Lodz University of Technology
group 10	T15	Silesian University of Technology
group 11	T17	Warsaw University of Technology
group 12	T18	Wrocław University of Technology
group 13	U2	Adam Mickiewicz University in Poznań
group 14	U3	University of Gdańsk
group 15	U4	Jagiellonian University in Kraków
group 16	U7	University of Łódź
group 17	U8	Maria Curie-Skłodowska University in Lublin
group 18	U9	Nicolaus Copernicus University of Toruń
group 19	U13	Uniwersytet Ślaski w Katowicach
group 20	U14	University of Warmia i Mazury in Olsztyn
group 21	U15	University of Warsaw
group 22	U16	University of Wrocław
group 23	U17	University of Zielona Góra

Source: self-study

It is noticeable that most of academic universities constituted a separate one-element group. Contrary to groups connected with a division according to a structure of costs, physical education universities had similar level of costs so they were allocated to groups of similar objects.

SUMMARY

The usage of graph methods in an analysis of researched objects is justified because it enables to classify them in a very detailed way. At the same time similar objects with the same characteristics are separated.

These methods should be used by Ministry of Science and Higher Education in an analysis of spending public sources by each university. Determining which universities are similar in the scope of a level and a structure of generated costs should constitute a basis for grants allocation.

ANNEX

Table 5. A list of public universities

Symbol	University
U1	University of Białystok
U2	Adam Mickiewicz University in Poznań
U3	University of Gdańsk
U4	Jagiellonian University in Kraków
U5	University of Warsaw
U6	Kazimierz Wielki University in Bydgoszcz
U7	University of Łódź
U8	Maria Curie-Skłodowska University in Lublin
U9	Nicolaus Copernicus University of Toruń
U10	University of Opole
U11	University of Rzeszów
U12	University of Szczecin
U13	Uniwersytet Śląski w Katowicach
U14	University of Warmia i Mazury in Olsztyn
U15	University of Warsaw
U16	University of Wrocław
U17	University of Zielona Góra
T1	University of Science and Technology in Cracow
T2	University of Bielsko-Biala
T3	Białystok University of Technology
T4	Częstochowa University of Technology
T5	Gdańsk University of Technology
T6	Koszalin University of Technology
T7	Tadeusz Kościuszko Cracow University of Technology
T8	Lublin University of Technology

Symbol	University
T9	Lodz University of Technology
T10	Opole University of Technology
T11	Poznań University of Technology
T12	Kazimierz Pulaski University of Technology and Humanities in Radom
T13	Rzeszów University of Technology
T14	West Pomeranian University of Technology in Szczecin
T15	Silesian University of Technology
T16	Kielce University of Technology
T17	Warsaw University of Technology
T18	Wrocław University of Technology
E1	University of Economics in Katowice
E2	Poznań University of Economics
E3	Warsaw School of Economics
E4	Cracow University of Economics
E5	Wrocław University of Economics
R1	Szczecin University of Life Sciences
R2	Warsaw University of Life Sciences
R3	University of Life Sciences in Lublin
R4	Poznań University of Life Sciences
R5	Wrocław University of Environmental and Life Sciences
R6	University of Agriculture in Cracow
R7	UTP University of Science and Technology in Bydoszcz
R8	The Jan Kochanowski University in Kielce
S1	University of Physical Education in Cracow
S2	The Eugeniusz Piasecki University of Physical Education in Poznan
S3	The Jerzy Kukuczka University of Physical Education in Katowice
S4	Józef Piłsudski University of Physical Education in Warsaw
S5	Gdansk University of Physical Education and Sport
S6	University of Physical Education in Wrocław
P1	The Jan Długosz University in Częstochowa
P2	Pedagogical University of Cracow
P3	Akademia Pedagogiki Specjalnej im. Marii Grzegorzewskiej w Warszawie

Source: self-study

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ISSUES IN MODELLING THE FINANCIAL DISTRESS AND BANKRUPTCY OF COMPANIES

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Abstract: Paper presents selected issues in designing and implementing models for financial distress and/or bankruptcy of companies. Major topics include: (1) definition of financial distress, (2) approaches to specify the distress variable, (3) aims and extent of the research on distress, (4) quantitative approaches in the field, including the attempt by Campbell, Hilscher and Szilagyi [2008], (5) survey of recent papers on the topic published in Poland.

Keywords: financial distress, bankruptcy, corporate failure, default risk

INTRODUCTION

"Stop using Altman Z-score!" Such title of a short note on a financial advisory page caught recently my attention. Advocate of a new innovation in predicting bankruptcy? Someone not satisfied with performance of Z-score? Yes and no. The note referred to a paper by Campbell, Hilsher and Szilagyi [2008] who, inspired by the developments stemming directly from the works of Altman and others, propose a new philosophy of investing on distressed stock.

Professor Edward Altman is the honorary doctor of my Alma Mater[1]. Therefore I have decided to investigate the issue. Also due to my commitment to collecting news on modelling and predicting the financial distress and bankruptcy of companies. The experience of such modelling in Poland dates back to late nineties and has been more or less alive since then.

Paper is organized as follows. Section FINANCIAL DISTRESS describes the notion of financial distress, specifying question of its vague designation and its

[1] In May 2015 professor Edward Altman has been awarded Honoraty Doctor of SGH Warsaw School of Economics.

importance to a number of stakeholders. Section OBSERVING AND MEASURING FINANCIAL DISTRESS presents a short survey of approaches to specify the distress variable in typical models. Section RESEARCH ON FINANCIAL DISTRESS points on broader subject of research on financial distress, asking about the goals of modelling and indicating the dangers in using plethora of financial ratios as distress predictors. Section METHODS discusses briefly the multitude of methods used in the research on distress. Then, in the section's second part, results of Campbell et al. [2008, 2011] are presented in short. The last section comments on the research on financial distress in Poland.

This short contribution is aimed at reviving the interest in modelling the financial distress in Poland, more than one decade after last major contributions in the field.

FINANCIAL DISTRESS

What is financial distress of a company?

Corporate finance and accounting define financial distress in various ways. Everybody understands what is the financial distress or corporate insolvency. When it comes to precise definition, the outcome is less than satisfactory. Platt and Platt [2006] state that "definition of financial distress is less precise than the legal actions that define proceedings such as bankruptcy or liquidation; despite this uncertainty, it is clear that the condition of being financially distressed deviates from corporate normality in a manner similar to bankruptcy".

Some of the issues accompanying the understanding and explaining the financial distress are as follows:

- Distress is somehow spanned between bankruptcy and good financial health of a company.
- Usually distress precedes bankruptcy, although it is not clear that the same factors are causes of both.
- Financially distressed company may possibly have unclear future, with significant probability of discontinuation. The bankrupt company terminates the activity under given legal form but sometimes it may continue with good perspectives.
- Objective measure of distress is challenging to agree upon: a company may feel like financially sound while some of its stakeholders may be already on alert.
- Category of financial distress is fuzzy and dynamic. Data on company's financial stand are usually delayed, of little use for investors. Prevailing studies concentrate on cross-sectional view. Time series analysis of distress might be better suited for practical purposes.

Parties involved/interested

Information on the financial health of a company is crucial for many stakeholders. These are:

1. Owners of equity
- The signs of distress are expressed e.g. in the possible going concern audit opinion. The opinion shall be publicly disclosed and understandably is of great importance to company owners.
- As the result, the company valuation on the market may be affected.

2. Creditors
- Financial distress of companies in banks' credit portfolios is the key to evaluation of risk in the banks. The issue has been systematically recognized by Basel Committee of Banking Supervision, specifically by recommendations of Basel II in 2004 and Basel III in 2010. Modelling probability of default, calculating loss given default etc. are fundamental for internal rating systems which are supposed to be installed in the banks. Some of those scoring systems are described by Altman and Hotchkiss [2006, Polish edition 2007].
- Evaluation of risk by internal rating systems in banks and in other crediting institutions is more and more important since the last crisis when rating agencies did not supply correct predictions on risk.

3. Investors in equity
- Investors tracking the financial performance of companies have evident interest in all news on possible distress. The likely strategy would be reducing positions on such stocks. See more in Section METHODS.

OBSERVING AND MEASURING FINANCIAL DISTRESS: MANY SHADES OF GREY?

Dichotomy or more?

Various studies indicate the necessity to strictly distinguish between Yes-No bankruptcy modelling and possibly more than two states of distress. In fact, there exists a kind of differentiation between bankruptcy and distress modelling. The first consider only two possible states, the latter – more states, although their features, number etc. are not satisfactorily established. It should be noted that conventional studies on bankruptcy give finally the probability of failing, i.e. some measure of distress. Therefore, also the two-state explanatory variable should be considered as measuring distress.

Consequently, typical studies consider two and more states of financial distress. This is usually the characteristic of explained variable in distress models. What follows is a selection of variables expressing distress in various studies.

- Cheng, Su and Li [2006], Lau [1987] – five states of increasing severity of financial distress:
  0: financial stability,
  1: omitting or reducing dividend payments,
  2: technical default and default on loan payments,
  3: protection under Chapter 10/ 11 of the Bankruptcy Act,
  4: bankruptcy and liquidation.
- Campbell, Hilscher and Szilagyi [2008, 2011], Shumway [2001], Chava and Jarrow [2004] – two states:
  0: non-failed firms,
  1: filing for bankruptcy (Ch.7, Ch.11), delisting for performance related reasons, receiving D rating from rating agency.
- Dahiya, Saunders and Srinivasan [2003] – two states:
  0: non-failed firms,
  1: firm is financially distressed if it has insufficient cash flows to meet the payments on its debt; two types of FD announcement – (1) default on a firm's public debt, and (2) filing by a firm for bankruptcy protection under Ch. 11.
- Platt and Platt [2006] – two states:
  0: non-failed firms,
  1: financially distressed firm – meets all of the following criteria: negative EBITDA interest coverage, negative EBIT, negative net income before special items.
- Hensher and Jones [2008] – three states:
  0: non-failed firms,
  1: insolvent firms: (i) failure to pay Australian Stock Exchange annual listing fees; (ii) a capital raising specifically to generate sufficient working capital to finance continuing operations; (iii) loan default, (iv) a debt/total equity restructure due to a diminished capacity to make loan repayments
  2: firms which filed for bankruptcy followed by the appointment of liquidators, insolvency administrators or receivers.

RESEARCH ON FINANCIAL DISTRESS

Scientific research on financial distress is widespread in the disciplines of corporate finance and accounting. Worldwide apparent feature of such studies is application of quantitative-statistical methods. Therefore, statistical and econometrics journals rarely allocate space to findings in the area of distress. Major outcomes are published in journals on accounting and on corporate finance. Let me mention one or two: “Accounting and Finance”, “Accounting Review”, “Advances

in International Accounting", "Financial Analysts Journal", "Journal of Accounting and Economics", "Journal of Accounting Research", Journal of Business, Finance and Accounting", "Journal of Corporate Finance", "Journal of Empirical Finance", "Journal of Finance", "Journal of Financial Economics", "Journal of Financial and Quantitative Analysis". Also important are repositories, like SSRN e-journals. Among them it is worth to indicate "Econometric Modeling: Microeconometric Models of Firm Behavior eJournal".

On another note, it is essential to mention that distress/ bankruptcy is researched in many disciplines: finance, accounting, economics, management, law. The latter may be evidenced, for example, in the recent paper by LoPucki and Doherty [2015].

Major research aims

One can argue that there are three main goals of research on financial distress, the first two being prevalent:

1. Search for determinants of financial distress. As explained before, determinants of distress might differ from those pertaining to bankruptcy. In fact, this is rarely distinguished, especially when the data-mining approach is employed.
2. Prediction of the financial distress state for a company. Such studies are close to typical credit-scoring models and are more operational than the first kind.
3. Composition of the investment portfolios. This novel target is explained in Section METHODS.

Lack of theory

Majority of researchers on financial distress do not explore the underlying theory. This stems largely from theory of enterprise and the agency theory; see Hotchkiss, John, Mooradian and Thorburn [2008].

Typical research on financial distress concentrates on verifying the hypotheses based on intuition and/or results of other researchers. This is understandable because the results are usually specific to place, extent and time period. The outcomes which are common to many markets and countries are quite scarce.

Issues with financial ratios as predictors of financial distress

There is typically a problem with determining "appropriate" set of predictors of financial distress. Commonly, predictors are chosen from the extensive set of financial ratios calculated on the basis of company's financial statements. Data miners obviously choose predictors which optimize some goodness-of-fit or forecast-error measure. Economics and finance research should try to avoid mechanical approach to selecting predictors. What follows are warning issues to consider while researching distress with the use of financial ratios.

- Classic model should include 1-2 ratios from a typical area, e.g. profitability, liquidity etc.
- Market ratios are completely unconnected with accounting ratios and often do not represent the same time periods.
- Incremental ratios like percent increase of sales introduce dynamics, frequently not utilized from methodological point of view.
- Ratios are useful for comparing firms of various sizes in "numerator" and "denominator"; they are sometimes over-used, e.g. by comparing to "industry average" etc.
- Prior classification of ratios as "good" or "bad" for explaining financial distress should be used with great caution. Distress can be well defined, although vaguely. Therefore, prior assumption that some variables shall be considered as "stimulants" or "destimulants" might be not valid. The result is always sample-specific.

METHODS

Variability of methods for assessing and explaining financial distress

Methodology employed in financial distress research covers today almost all techniques of data analysis, specifically methods of statistics, econometrics, survival analysis and data mining. It is beyond the scope of this paper to survey all the methods. A selection is presented e.g. in Gruszczyński [2012].

Common feature of most approaches is their probability-wise nature. For example, it is customary to express financial distress in terms of probability of corporate failure.

Some new interesting methods originate from modelling fraud detection, methodology of text mining or from modelling companies' churn for loans.

In such context one may ask if the popular classic methods of predicting failure/ distress such as logistic regression or discriminant analysis are still valid. They are still in wide use in research, like the one described below. However, the new approaches such as those originated from big-data methodology might soon form a significant alternative.

Nonetheless, it is worth to acknowledge the major results of classic stream in the field. These are:

- Altman's Z-Score [1968],
- Ohlson's O-Score [1980],
- Moody's KMV[2] model based on Merton [1974],
- Shumway hazard model [2001].

[2] Moody's [2000].

First two are accounting-based models, while the third and fourth are market-based models[3]. All those models have been followed by numerous researchers around the world, were subjected to many modifications by original authors and others, and are still regarded as most popular approaches in distress modelling.

"In Search of Distress Risk"

As an example of new research in the area of distress we present here the "classic" approach by Campbell, Hilscher and Szilagyi [2008, 2011]. Their attempt is based on previous studies by Shumway [2001] and Chava and Jarrow [2004].

Model is aimed at predicting "failure events" which are defined as: filing for bankruptcy (Ch.7, Ch.11) or delisting for performance related reasons or receiving D rating from rating agency. The variable Y_{it} representing failure event equals to 1, otherwise it is equal to 0. Thus, we have simple binomial model. In this case it is a kind of dynamic: failure events are observed monthly. These monthly US data cover period of 1963-2008 composing ca. 1.7 million firm-months among which there are ca. 1600 failure events.

The dynamic logit model employed here explains probability of failure event in month t by means of lagged (by 1 month) explanatory variables. The original expression from the paper is as follows:

$$P_{t-1}(Y_{it} = 1) = \frac{1}{1 + \exp(-\alpha - \beta x_{i,t-1})}$$

where Y_{it} equals one if the firm goes bankrupt or fails in month t, and $x_{i,t-1}$ represents a vector of explanatory variables in month t–1 (i.e. at the end of month t–1).

The set of explanatory (predictive) variables is composed of the following accounting and market-based predictors:

- net income to market valued total assets (NIMTA):
$$\frac{Net\ Income_{it}}{(Firm\ Market\ Equity_{it} + Total\ Liabilities_{it})}$$
- net income to total assets (NITA):
$$\frac{Net\ Income_{it}}{Total\ Assets\ (adjusted)_{it}}$$
- total liabilities divided by the sum of market equity and book liabilities (TLMTA):
$$\frac{Total\ Liabilities_{it}}{(Firm\ Market\ Equity_{it} + Total\ Liabilities_{it})}$$
- total liabilities relative to total assets (TLTA):

[3] The distinction by Outecheva [2007].

$$\frac{Total\ Liabilities_{it}}{Total\ Assets\ (adjusted)_{it}}$$

- book to market equity ratio,
- ratio of cash and short-term assets to the market value of assets (CASHMTA):

$$\frac{Cash\ and\ Short\ Term\ Investments_{it}}{(Firm\ Market\ Equity_{it} + Total\ Liabilities_{it})}$$

- monthly log excess return on equity relative to the S&P 500 index (EXRET):

$$\log(1 + R_{it}) - \log(1 + R_{S\&P500,t})$$

- standard deviation of daily stock return over the past 3 months (SIGMA),
- relative size measured as the log ratio of its market capitalization to that of the S&P 500 index (RSIZE):

$$log\frac{Firm\ Market\ Equity_{it}}{Total\ S\&P500\ Market\ Value_t}$$

- log price per share, truncated above at $15 (PRICE).

The novelty of this approach lies mostly in fine-tuning of explanatory variables. Those are principally the same as in previous studies by Shumway [2001] and Chava and Jarrow [2004]. However, the profitability (NIMTA) and excess stock returns (EXRET) variables have been also introduced in a distributed lag form – which has helped to increase model performance.

Model outperforms other approaches in prediction: as forecast horizon increases, market-based variables are more important than accounting variables. Model outperforms O-score and Z-score, doubles the accuracy of Moody's KMV distance-to-default. Safest 5% stocks have the average failure probability of 0.01 while riskiest 5% have 0.34.

Second exercise described in the paper uses the stocks' failure probabilities from the 12-month ahead model (reestimated each January) to form 10 portfolios of stock falling in different regions of the failure risk distribution. The portfolios contain stocks in percentiles 0 to 5, 5 to 10, 10 to 20, 20 to 40, 40 to 60, 60 to 80, 80 to 90, 90 to 95, 95 to 99, and 99 to 100 of the failure risk distribution. Stocks with high risk of failure (above 60 percentile) have anomalously low average returns. As a whole, the distressed portfolios have low average returns (with high standard deviations and market betas).

However, theoretically, the bearers of the risk of owning financially distressed stock shall charge a premium for that. The paper shows the opposite: distressed stock underperform safe stock for decades. As authors state, their measure of financial distress generates underperformance among distressed stocks in all quintiles of the size and value distributions. In discussing this result, authors offer several explanations for the anomalously low returns on distressed stocks such as: unexpected developments, valuation errors by investors, private benefits of control by majority investors (like buying assets at bargain prices), expensive low turnover stock (when traded in large quantities by institutional investors).

So, what did we learn from the paper? That one may construct financial distress models with better and better predictive performance – using only "classic" market and accounting variables. And – that investing in distressed stock is not profitable (even less than one might expect).

FINANCIAL DISTRESS RESEARCH IN POLAND

Studies on bankruptcy and financial distress of companies in Poland have the origin at the very beginning of political and economic transformation of economic system 25 years ago. It took some time to adjust and recognize the symptoms of "proper" distress/ bankruptcy under new system.

The extensive research resulted in quite a number of publications which emerge until now, although the current pace is a bit slower. Research was pioneered in the nineties by academics in University of Lodz and at Polish Academy of Sciences. Until now, all academic centres in Poland propose some sort of research involvement in modelling bankruptcy and distress. However, there are not many with rigorous up-to-date research with quantitative edge. In my opinion, the accounting and corporate finance research in Poland is not yet ready to diverge into these areas, with few notable exceptions.

In order to mark publications on distress in Poland, at first let me mention some books, most of them with quantitative approach. These are books by Mączyńska [2001, 2009, 2010], Prusak [2006], Lasek [2007], Kisielińska [2008], Korol [2010], Gruszczyński [2012]. The list is not comprehensive and represents a choice of books published since 2000.

Secondly, the flow of papers on bankruptcy and distress in Poland which has been witnessed at turn of the century is someway reduced in recent years. Few new contributions since 2012 are shown below, along with the list of journals which were searched for presence of contributions on distress and bankruptcy research.

- *Bank i Kredyt* (National Bank of Poland) – no contributions (nc),
- *Finanse* (Polish Academy of Sciences) – nc,
- *Przegląd Statystyczny* (Polish Academy of Sciences) – nc,
- *FindEcon* (University of Lodz) – nc,
- *Rachunkowość* – nc,
- *Copernican Journal of Finance and Accounting* (Nicolaus Copernicus University Toruń) – nc,
- *Quantitative Methods in Economics* (Warsaw University of Life Sciences – SGGW) – Ptak-Chmielewska [2013], Zielińska-Sitkiewicz [2013],
- *Journal of Management and Financial Sciences* (SGH Warsaw School of Economics) – Altman and Rijken [2012], Tomczak [2014],
- *Finanse, Rynki Finansowe, Ubezpieczenia* (University of Szczecin) – Balina [2012], Góralski, Pietrzak i Jędralski [2012], Sukiennik [2013], Wasylkowska and Szopik-Depczyńska [2014], Bolibok [2014],

- *Folia Oeconomica Stetinensia* (University of Szczecin) – Markowicz [2014],
- *Journal of Management and Finance* (University of Gdańsk), Śmiglak-Krajewska and Just [2013], Waszkowski [2013],
- *Zeszyty Teoretyczne Rachunkowości* – Wędzki [2012].

The list above is obviously a selection. Yet, it shows that recent contributions on the topic of financial distress in Poland are scarce. Most mainstream journals in economics, finance and accounting did not publish articles on distress since 2012. On the other hand, it would be interesting to survey international contributions on this topic by the authors from Poland. Just to show one: paper by Siedlecki [2014] in "Procedia Economics and Finance".

Research on financial distress and bankruptcy in Poland is potentially destined for success. With narrowing of the gap between mutual competencies of academics in finance and accounting on one side and in quantitative methods on the other, more and more contributions with internationally appreciated quality should be foreseen.

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APPLICATION OF MIXED MODELS AND FAMILIES OF CLASSIFIERS TO ESTIMATION OF FINANCIAL RISK PARAMETERS

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Abstract: The essential role in credit risk modeling is Loss Given Default (LGD) estimation. LGD is treated as a random variable with bimodal distribution. For LGD estimation advanced statistical models such as beta regression can be applied. Unfortunately, the parametric methods require amendments of the "inflation" type that lead to mixed modeling approach. Contrary to classical statistical methods based on probability distribution, the families of classifiers such as gradient boosting or random forests operate with information and allow for more flexible model adjustment. The problem encountered is comparison of obtained results. The aim of the paper is to present and compare results of LGD modeling using statistical methods and data mining approach. Calculations were done on real life data sourced from one of Polish large banks.

Keywords: LGD, mixed models, random forests, gradient boosting

INTRODUCTION

New Basel Accords introduced a possibility of applying IRB systems in banks for the need of risk parameters estimation. Within that approach three key risk parameters should be estimated: PD (Probability of Default), LGD (Loss Given Default) i. e., the percentage of total exposure at the time of default that cannot be recovered and EAD (Exposure at Default). Contrary to PD, LGD estimation has received much less attention so far. If fact it has been a subject of more intense scientific research for hardly five years now. The reasons are, among others, lack of unified definitions of default or economic loss as well as the scarcity of LGD data. Moreover, LGD can exhibit difficult behavior. Its values are fractions, often with

high concentration at 0 (full recovery) and/or at 1 (total loss). LGD is treated as a random variable, frequently with a bimodal distribution.

In the first section of the paper we briefly describe models utilized so far in LGD modeling. Our idea was to apply families of classifiers to modeling LGD. While parametric models are easier to explain, ensemble methods are able to better cope with often bimodal or highly skewed distribution of LGD. Therefore in the next section we present two ensemble models: gradient boosting and random forests that have to our knowledge not been applied yet in LGD modeling. In the final section we present the results of our research. We compare classical methods applied in LGD estimation with ensemble methods. We compare both approaches using graphical methods, among others the REC curve.

PARAMETRIC MODELS FOR LGD ESTIMATION

Industrial approach to LGD modeling

Capital loss in credit risk is represented by a random variable L. Its expected value can be calculated as:

$$E(L) = EAD \cdot LGD \cdot PD \tag{1}$$

In the paper we consider only LGD for individual transactions.

Models of LGD

LDG is expressed in percentage therefore it is a fractional target variable. LGD is usually modeled by regression methods, like fractional regression or beta regression [Loterman et al. 2012]. Fractional regression was introduced by Papke and Wooldridge in 1996 [Papke et al. 1996]. Fractional regression is a Generalized Linear Model (GLM) with logit link function.

$$G(\beta x) = \frac{1}{1+\exp(-\beta x)} \tag{2}$$

Beta regression was first applied to model proportions in 2004 by Ferrari and Cribari-Neto [Frerrari et al. 2004]. The distribution function for $p > 0,\ q > 0$ and $y\epsilon(0,1)$ is given by:

$$f(y; p, q) = \frac{y^{p-1}(1-y)^{q-1}}{B(p,q)} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1} \tag{3}$$

where $B(\cdot,\cdot)$ is a Beta function and $\Gamma(\cdot)$ denotes Gamma function. Ferrari and Cribari-Neto proposed a transformation of p and q into a location/mean parameter μ and dispersion/precision parameter ϕ. We have:

$$E(Y) = \frac{p}{p+q} = \mu \tag{4}$$

$$Var(Y) = \frac{pq}{(p+q)^2(p+q+1)} = \frac{\mu(1-\mu)}{\phi+1}, \text{ where } \phi = p + q \tag{5}$$

Ferrari and Cribari-Neto estimated parameters using maximum likelihood function and their approach was similar to the maximum likelihood method applied

in GLM. Beta function is well suited to describe highly skewed data [Karwański et al. 2015]. Unfortunately, the model proposed by Ferrari and Cribari-Neto is restricted to the open interval (0,1). The observations on boundaries are neglected. Real data indicate that observations at boundaries appear with high frequencies. Only in the recent three years extensions of beta regression models covering the whole range [0,1] have been discussed. The first generalization of beta regression for the boundaries of the (0,1) interval was proposed by Ospina and Ferrari [Ospina et al. 2011]. The authors proposed a zero or one inflated beta regression for modeling fractional outcomes. In 2012 Calabrese proposed a mixed model

$$B_{inf01}(y,\pi_0,\pi_1,\mu,\phi)=\begin{cases}\pi_0 & for\ y=0\\ \pi_1 & for\ y=1\\ \pi_0+[1-\pi_0-\pi_1]f(y;\mu,\phi) & for\ y\epsilon(0,1)\end{cases} \quad (6)$$

where $f(y;\mu,\phi)$ is a beta distribution [Calabrese 2012]. The model is a mixture of Bernoulli distribution and Beta distribution.

A zero and one beta inflated model was first introduced in 2014 by Xiao Yao [Xiao Yao et al. 2014]. It can be written in the following way:

$$B_{inf01}(y,\pi,\psi,\mu,\phi)=\begin{cases}\pi(1-\psi) & for\ y=0\\ \pi\psi & for\ y=1\\ (1-\pi)f(y;\mu,\phi) & for\ y\epsilon(0,1)\end{cases} \quad (7)$$

$$E(y)=\pi\psi+(1-\pi)\mu \quad (8)$$

In our calculations we have applied three different models based on beta distribution. Two of them belong to the family of mixed models. The parameters of our models were estimated by maximum likelihood method.

FAMILIES OF CLASSIFIERS

Random forests and gradient boosting are extensions of regression trees, that is simply the partition of the space X, which consists of predictors of target variable y, into disjoint regions R_j. We will describe briefly both methods.

Random forests

Random forests were introduced in 2001 by L. Breiman as a method of classification [Breiman 2011]. In this approach a large number of simple trees is constructed with a random sample of predictors taken before each node is split. The object is classified based on an average vote of the set of de-correlated trees [Berk 2008]. The random forest algorithm can be described as follows [Hastie et al. 2009]:

Create N bootstrap samples {S1,…,SN} out of a data set S as follows:
Si: random drawings of |N| elements from S with replacement
For each i=1,…,N select random set of attributes {X*} h*i=Learn(Si ; X*)

Output H=[{h*1,…,h*N}, majority Vote].

Gradient boosting

Gradient boosting was introduced by J. Fridman in 1999. In gradient boosting, similarly as in random forests, a family of trees is grown. Each tree is constructed based on a previous one in such a way that one minimizes a given loss function in the gradient direction. Gradient boosting can be described in the following way [Berk 2008]:

Let Y=learn(h(X)) and $b_m(x)$ be a set of predictors of Y

$$h(x) = \sum_{m=1}^{M} b_m(x) = \sum_{m=1}^{M} \beta_m b(x;\theta_m) \tag{8}$$

Put: $h_0(x) \leftarrow 0$. For m=1,…,M and the loss function L( ):

$$r(x,y) \leftarrow -\frac{d}{dh} L(h_{m-1}(x), y) \tag{9}$$

$$b_m \leftarrow \arg\min_b \sum_{(x,y)} \big(b(x) - r(x,y)\big)^2 \tag{10}$$

$$h_m(x) \leftarrow h_{m-1}(x) + v \cdot b_m(x) \tag{11}$$

In gradient boosting one applies a property that for small v:

$$\sum_{(x,y)} L(h_{m-1}(x) + b_m(x), y) \approx \sum_{(x,y)} L(h_{m-1}(x), y) + \sum_{(x,y)} \frac{d}{dh} L(h_{m-1}(x), y)\, b_m(x) \tag{12}$$

DATA

Our calculations were made based on data covering small and medium enterprises sourced from one of large Polish banks. Data was collected in a few operational data bases built for the needs of various bank departments. It comprised both information about the client as well as about products offered. The data was collected over the period of 3 years with the defaults registered until October 2007 and the recovery process observed till October 2008. In the calculations 12000 observations and 12 variables were used. The data was normalized to make the comparison possible. The selected variables used in calculations are described in Table 1.

Table 1. Explanatory variables used in the analysis

X1	Average monthly withdrawal in the last 3 months
X2	Average overdraft balance in the last 6 months
X3	Average credit balance in the last 3 months
X4	Average balance in the last 3 months
X5	Trend for average balances in the last 12 months
X6	Ratio of total balance to average balance in the last 3 months
X7	Total interest delinquency ratios at the time of analysis (PIT* approach)
X8	Total capital delinquency ratios at the time of analysis (PIT approach)
X9	Average increas in capital arrears in the last 6 months
X10	Total amount of monthly payments done by the client
X11	Coefficient of debt/loan repayment
X12	Status of the first loan account

Source: own preparation

RESULTS AND CONCLUSIONS

In our research we have applied three regression models. The models were selected in accordance with GLM model selection. The first model was a beta regression model proposed by Ferrari and Cribari-Neto (we denote it by beta model). The second one was a zero-one inflated beta regression with constant parameters π, ψ and ϕ (we denote it by inflated beta model 1) and the last one was a zero-one inflated beta regression with parameters following logistic distribution (we denote it by inflated beta regression 2). The parameters of our regression models were estimated by maximum likelihood method. The calculations were done in SAS 9.4.

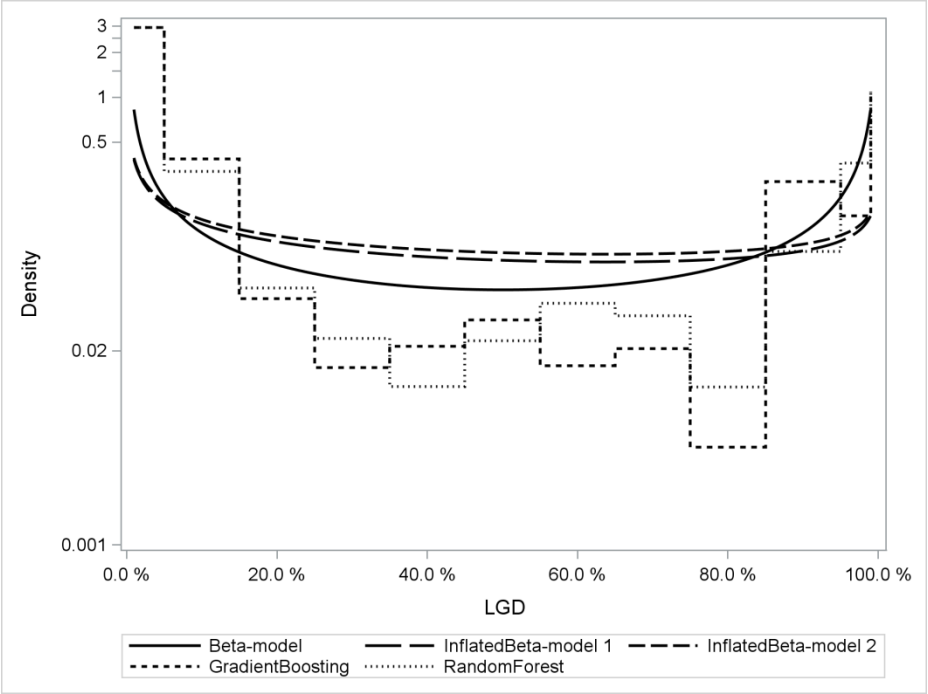
We have also applied two ensemble methods: random forest and gradient boosting. The calculations were done in SAS Enterprise Miner ver. 13.2. To make the model comparison possible, the target variable LGD in ensemble modeling was categorized into ten classes.

The research that is based on two different approaches encounters the problem of comparison between obtained results. No sufficient theory of comparing statistical and data mining models has been developed so far. Namely, regression models are based on minimization of residuals, while data mining methods are based on information maximization. In the latter Gini coefficient or entropy measure are commonly used. In fact, there are no popular measures that compare simultaneously both approaches.

* PIT (point in time), a methodology of evaluating risk parameters opposite to TTC

In order to compare the results of our research we have used an underrated measure called REC (Regression Error Characteristic) [Bi et al. 2003]. REC curve is a powerful tool for visualizing and comparing model results. REC curves plot the error tolerance (loss function) versus the percentage of points predicted within the tolerance (cumulative distribution). The REC curve visually presents commonly-used statistics. The area-over-the-curve (AOC) is a biased estimate of the expected error. The R^2 statistic can be estimated using the ratio of the AOC for a given model to the AOC for the null model. Moreover, the shapes of these curves give some additional information about model goodness-of-fit.

Figure 1. Density functions $f(LGD|x's\ equal\ their\ averages)$ estimated by various models

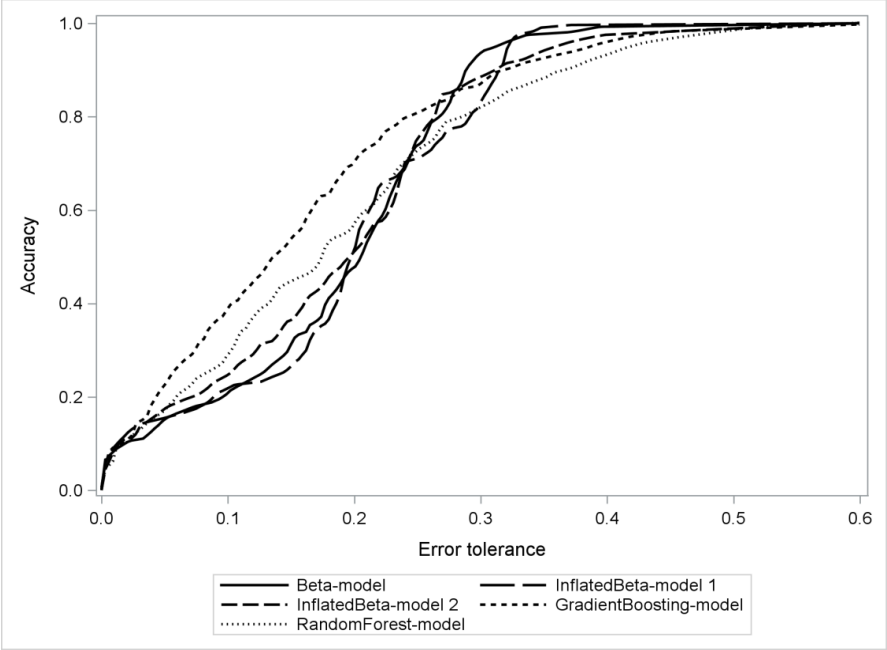


Source: own calculations

The plots of density functions for LGD shown on Figure 1 indicate that all regression models and families of classifiers give similar results. The figure was plotted for average values of all covariates.

The REC curves in Figure 2 show that gradient boosting outperforms other methods for the majority of residual values. Beta regression models exhibit similar behavior.

Figure 2. Comparison of REC curves for all beta regression models, random forests and gradient boosting



Source: own calculations

Table 2. Area Over the REC Curve as the measure of goodness of fit

	Area Over the REC Curve (AOC)
Beta model	0.1831
Inflated beta model 1	0.1868
Inflated beta model 2	0.1812
Gradient boosting	0.1525
Random forest	0.1813

Source: own preparation

The gradient boosting model clearly dominates the others over the greater part of range of possible errors. On the contrary, the performances of beta model and both beta inflated regression models are harder to compare. For smaller errors inflated beta model 2 dominates, but for larger errors beta model overcomes. The decision which of these two models is preferable may be domain dependent. Random forests behave similarly as the area over the curve (i.e., the expected error) is almost equal to that of inflated beta regression 2.

The aim of our research was to show that ensemble methods can be applied in LGD estimation. The results revealed that families of classifiers not only can successfully be applied in LGD modeling but also gradient boosting outperforms all considered beta regression models.

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ARTIFICIAL NEURAL NETWORK SUPPORTING THE PROCESS OF INVESTING ON THE FOREIGN STOCK EXCHANGES

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Abstract: The publication presents the use of artificial neural networks as a tool expert that supports the process of decision-making for the quarterly period to invest in selected stock exchanges. It proposes a set of 10 features of exchanges, which is of enough universal character that the approach presented in the publication may be useful for any chosen stock exchange. The conducted study was based on actual data.

Keywords: artificial intelligence, process of portfolio management, stock exchanges features, investments

INTRODUCTION

Globalization, mostly associated with the economic sphere, significantly affects the process of managing a portfolio of financial instruments. It follows clearly and unequivocally that this process must take into account the international diversification. It should be noted that in the globalization conditions, financial integration does not always have a positive effect on economies, as strong shock to the financial system interfere with the flow of capital [Kalemli-Ozcan et al. 2013], which in turn reduces the profitability of investments in the stock markets. As a result, investors who wish to use the international diversification of the portfolio, should take into account the fact that inadequate time to invest in foreign stock exchange, whose index at the time of the investment can increase their volatility, can be considered as a major factor influencing negatively the achieved rate of return.

All this suggests that international diversification, despite the numerous advantages presented in the literature, cannot be regarded as a part of the portfolio

management process, which in all conditions increases the return on investment. As it is widely known, the investments on foreign exchanges are also connected with the purchase of foreign currencies, however detailed analytical studies show that the currency risk is an important component of the overall investment risk [Menkhoff et al. 2012]. The purpose of this article is to present the artificial neural networks (ANN) as a tool to support international diversification of the portfolio of financial instruments during the period for a medium-term investing. Artificial neural networks in this area, according to the authors, were not yet used. It should be added that the article is based on actual data and empirical literature as well as surveys and considerations of their own.

GENERAL PRESENTATION OF THE SELECTED STOCK EXCHANGES (ON 06.30.2015)

For research purposes, we selected seven foreign stock exchanges, namely: NYSE Euronext, NASDAQ OMX, Japan Exchange Group - Tokyo, Hong Kong Exchanges Deutsche Börse, NYSE Euronext (Europe), SIX Swiss Exchange. Stock exchanges were selected for this reason that it is they who are most often subject to analyzes, published by the most reputable financial portals[1], suggesting the most interest investors in the world. It is for this reason that we carry out a study of the use of designated stock exchanges. The reason is that investors around the world tend to be interested in investing on them. While the exchanges are so large (due to the transactions made on them and their capitalization, especially in comparison with the stock exchanges of countries with developing economies) that studies based on them may become a base for such a configuration tool to support (artificial network neural), which will be useful for analyzes other than those selected exchanges, including Polish, thereby stimulating international portfolio diversification. The data characterizing the selected stock exchange, relevant to the analysis, are presented in a Table 1.

Table 1. The basic data relating to the selected stock exchanges (end of June 2014)

Stock exchange	Capitalization (USD millions)	The share of the stock market capitalization of all stock exchanges (members of WFE)
NYSE Euronext	19 237 429.6	28.03%
NASDAQ OMX	7 243 276.1	10.55%
Japan Exchange Group-Tokyo	4 944 150.2	7.2%
Hong Kong Exchanges	3 751 454.3	5.47%
Deutsche Börse	1 752 563.7	2.55%
NYSE Euronext (Europe)	3 414 827.8	4.98%
SIX Swiss Exchange	1 545 786.2	2.25%

Source: own studies based on WFE - World Federation of Exchanges , (http://www.world-exchanges.org/)

[1] That follows from the observation of the authors.

ARTIFICIAL NEURAL NETWORKS IN THE CONTEXT OF SUPPORTING THE DECISION TO INVEST IN THE SELECTED STOCK EXCHANGES

Considerations conducted in this publication were restricted to shares due to investor interest, as these financial instruments are most purchased among all listed on the stock exchanges that are members of WFE[2] [WFE 2015]. Despite the fact that carried out research, devoted to the analysis of artificial neural networks in the context of promoting investment decisions in the stock market, should take into account the fact that there are many other tools that perform this function[3], but they do not support the process of international portfolio diversification. What's more, they do not include the definition of an optimal period of investment from the perspective of medium and long term investments. Technical analysis can be included to some of them, which is a form of investor support, especially for weekly investment horizon [Menkhoff 2010]. In the long-term investment horizon, investors use more greately the fundamental analysis, also for forecasting [Lui, Mole 1998].

Another tool that allows even to determine the shares of selected assets, is to analyze the portfolio, which is largely based on the values of rates of return and risk levels measured by standard deviation. Such an assumption can be regarded as hindering the decision support. Firstly, in practice of creating portfolios, investors pay attention not only to their return and risk [Lynch 2001], which can be calculated by different methods. Secondly, the standard deviation is not a monotone measure, mainly because of that, it cannot be considered to be appropriate for risk measurement [Foster, Hart 2009]. As part of portfolio analysis there are most frequently used techniques for constructing portfolios: the Markowitz method [Markowitz 1952], the method takes into account borrowing and lending at the risk-free rate [Elton, Gruber 1995] and the method based on Lagrange function [Francis 2000]. Given that the portfolio analysis is useful for weighting of assets in the portfolio, it is a tool willingly used by institutional and individual investors. Econometric models that have not received attention, also play an important role in portfolio management, but their deep analysis is beyond the scope of the publication. As it was mentioned, these methods do not help greatly in determining optimal investment period and do not support the process of international diversification, so as a conclusion of this part of the work there comes out the necessity to present the use of artificial neural networks in these areas.

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[2] World Federation of Exchanges is an organization bringing together the most advanced exchanges in the world (http://www.world-exchanges.org/).

[3] Giving an overview of the characteristics abstracted tools to support investors in the publication of efficient market hypothesis.

The design of artificial neural networks, which are formed by neurons, as well as their ability to learn, shows conclusively the usefulness of international portfolio diversification and to determine the optimal time to invest. As you know, the main problem of investment funds is the lack of reliable information on future prices and trends of investment assets. For this reason, neural networks are used for many years for the purpose of forecasting in the financial markets [Azoff 1994]. In addition, it is also used successfully to generate a transaction strategies [Morajda Domaradzki 2005].

Returning to the substance of the discussion, it is worth noting that in this publication its purpose can also be formulated as the attempt to use artificial neural network in the context of the support of international portfolio diversification from the perspective of determining optimal investment horizon with a view to maximizing the rate of return on long-term investment in equities. Because of the possibility of determining the learning network to a type of research study defined as teaching "supervised" (Supervised) with the teacher. It is often used when configuring the network in order to make predictions [Pokharel, Deardon 2014]. All these considerations can be defined in such a way that the themes and issues of international diversification investment period is not treated separately, but as one area for the simplification of research. This approach is mainly caused by the fact that in the analysis of investment period on selected stock exchanges, the object becomes concrete exchange during the period. This allows you to assign the appropriate characteristics, because as the desired end result, the artificial neural networks should recognize different situations in stock exchange, which will provide the basis for a decision to assign them to the appropriate class. With this in mind, a set of features in the presented concept plays an important role, but literature does not specify the team. Therefore, restricting the study to the segment of shares proposed 10 universal characteristics of these exchanges given periods. Their values will generate 3 classes decisions. The first would cover the period during which you should not invest in stocks on the selected stock exchange (based on expert suggestions, the value pattern in this case is "-1"). The second class of the pattern equal to "1" would be ascribed to the period in which you should invest, while the third and final would determine a decision regarding the period during which you can continue an already made investment, but rather, without enlarging the portfolio value (value pattern is in this case " 0 "). It can also be added that the division is compiled on the basis of proposals of their own, however it comes as a result of the stems of cut-tested to assess their attractiveness from the perspective of obtaining a high return on investment in shares traded on various exchanges. The study takes into account the fact that the international short-term investments could become unviable if only because of transaction costs and the costs of buying and selling foreign currencies. For this reason, the publication was quarter as the minimum holding period on stock exchanges in shares, without analysis of concrete shares. Key assumptions concerning their empirical study are as follows:

- A single period of the investment in selected markets is 3 months, however for the network learning there have been studied quarterly periods from 07.01.2003 to 12.31.2014 (the number of periods was 46, while the size of the training set - 322 as a single period falls 7 exchanges) . It assumes also that the attractiveness of each period are examined from the perspective of 3 months. On the basis of the characteristics of each period they rated their attractiveness at the suggestion of the expert by assigning the value of the standard, which is "-1", "0" and "1". Such a concept has allowed the science of artificial neural network in order to evaluate future 3-month periods based on the expected value characteristics. This means that after 3 months you can decide to continue investing or enlarging the portfolio or sell shares.
- To assess the effectiveness of network learning, there was prepared a testing set for 7 stock exchanges, which included two quarterly periods from 01.02.2015 to 06.30.2015. This means that the size of this set was 14.
- In the research process there was applied the "supervised" learning and experiments were carried out using a variety of network configurations, wherein: the number of inputs always was 10 and the multiple learning has been amended (learning network was continued to a predetermined accuracy).
- The study finally adopted the following network architecture: two hidden layers (the penultimate layer composed of two neuron, the second hidden layer - with six neurons) and the output layer - one neuron. Transfer function in the hidden layer was tansig nonlinear function, and the output layer – linear.
- The network was trained by Back Propagation, according to Levenberg-Marquardt algorithm, and the purpose of learning was to obtain the smallest value of the sum of the squares of the difference between the output of the network, and the value of the pattern constructed by an expert. Multiple of learning in studies provide established network quality.

The precise nature of the proposed universal set of 10 features is presented in tabular form (whereas 5 features are related to a stock exchange, and other 5 features are the change in the value of the former). It should be added that the higher amount of change (growth), the better is the stock exchange estimated during the period.

Table 2. Set of features quarters of investing in the selected stock exchange (related to the stock segment)

Name of the features	The indicator used to calculate the value of feature	Essence in relation to the returns on investment in shares
The increase in the number of listed companies and its change in % compared to the previous quarter.	The increase in the number of new companies listed on the stock exchanges during the quarter.	A growing number of companies shows a positive trend on the stock markets.
The quarterly growth rate of the index of the largest companies and its change in % compared to the previous quarter .	Indices of the largest companies in selected stock exchange (according to the assumptions WFE)	The index of the largest companies is a barometer of the economy.
The quarterly growth rate of the main index and its change in % compared to the previous quarter.	Main indexes of selected stock exchanges (according to the assumptions WFE).	The main index is a synthetic indicator of the stock market situation.
The growth rate of stock market capitalization in % - report quarterly to the previous quarter, and its change in %.	The sum of the stock market capitalization of all the companies creates the capitalization of selected exchanges.	The growth rate reflects a market capitalization directly increase the prices of all shares listed on the stock exchanges.
The rate of increase in the value of trading in shares-relational quarter to the same quarter of the previous year and its change in% compared to the previous quarter.	Turnover in terms of value obtained by multiplying the exchange rate and the number of shares bought and sold.	The growing trade in equities means increasing the liquidity of these financial instruments.

Source: own study

The presented set of features is aimed to reflect in a general way the situation on the stock market during the 3 month period. All of the features are calculated on the basis of well-known database WFE. Suggestion of expert became a standard, which had used artificial neural networks to learn. As you can see, all the features are expressed in "%". Sample time along with the actual data are presented in Table 3.

Table 3. Examples of the actual data values of 10 features in a fourth quarter of 2014. (that is on 12.31.2014) belonging to a learning set for the German Stock Exchange (Deutsche Börse)

Name of the features	Value of the feature	Change of feature value
Number of listed companies	-3.18%	-1.75%
Blue chip index performance	3.50%	7.14%
Broad stock index performance	4.14%	8.32%
Domestic market capitalization	1.30%	12.66%
Value of share trading	-0.05%	0.93%
Expert suggestion	Z (pattern value)	1

Source: own study

USING ARTIFICIAL NEURAL NETWORKS TO DETERMINE THE QUARTERLY INVESTMENT DECISION ON THE SELECTED STOCK EXCHANGES

Simulation studies were performed in MATLAB software environment for obtaining simulation results according to the suggestions of an expert. In its initial phase of research, the neural network did not receive satisfactory results (did not establish the correct weights), so they changed the learning process in such a way that the outcome of every learning became the beginning of the next. Number of repetitions is chosen experimentally, until the quality of the network was no worse than a predetermined (the results of specific experiments quantified with key elements of the study are shown in Table 4). In addition, in the experiments there were also used modifications of startup to improve the learning process, using multipliers for certain data input. There were used the following multipliers:

a – multiplier, which multiplied the value of first set of two features.

b – multiplier, which multiplied the value of second set of two features.

c – multiplier, which multiplied the value of third set of two features.

d – multiplier, which multiplied the value of fourth set of two features.

e - multiplier, which multiplied the value of fifth set of two features.

To improve the quality of the results, the outcome of the experiments is presented in a quantitative form (see Table 4).

Table 4. Selected information regarding experimentation of ANN (the values shown are calculated on the basis of the average of three learning outcomes of network)

Description of the experiment	Type of data	The averaged result	Comment
Standard lear-ning networks without multipliers	The training set (n=322)	F=1.12 G=0	Learning of network was not effective (big value of F).
	The test set (n=14)	F=9.65 G=0.33	

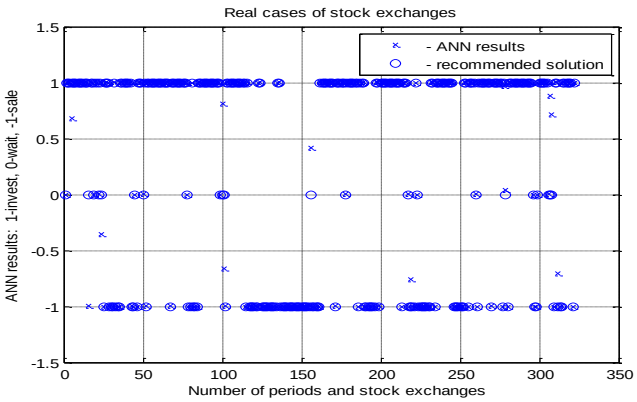
Description of the experiment	Type of data	The averaged result	Comment
Multiple learning network without the use of multipliers	The training set (n=322)	F=0.76 G=0	Network activity has become more efficient, but it cannot be considered as a tool expert.
	The test set (n=14)	F=4.52 G=0	
Multiple learn-ing network of multipliers for parameter data (a=0.9; b=1.5; c=1.8; d=0.9; e=0.25)	The training set (n=322)	F=0.58 G=0	This method proved to be the best. The network was taught, as compared to the data training set, a network of well-recogni-zed expert suggestions (Fig. 1).
	The test set (n=14)	F=4.24 G=0	

"F" is the sum of the squares of the difference between the output of ANN and the value of the pattern, "G" is the number of indications (data network) which differ from the experts' not less than 0.99, „n" is the number of the examined periods and stock exchanges.

Source: own study

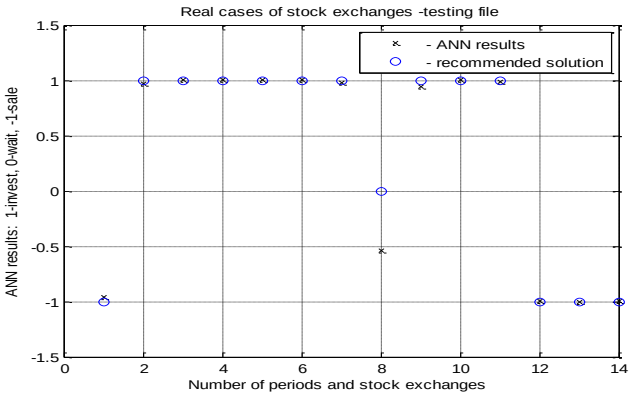
It is easy to observe that the standard learning network without the use of multipliers was not effective. Therefore a repeated learning network of multipliers chosen by trial and error, which made it possible to learn the network (this is presented in Chart 1, in which the OX axis is the number of periods studied, and the axis OY - value, whereby the network results were determined in the form of "O" and suggestions expert - in the form of an "X").

Figure 1. An example network response for multiple learning from multipliers data on actual values in the training set



Source: own preparation

Figure 2. Results of network with multipliers data on actual values in the set testing



Source: own study on the basis of the result of MATLAB

Figure 1 shows the results of simulation of option 3 in Table 4 for a training set, which visually can be considered as satisfactory (in almost all cases the position "O" and "X" is almost identical). In contrast, Figure 2 shows the results of tests carried out on the basis of a set of testing, checking utility network. A network set up under option 3 made it possible to obtain satisfactory results (this is also illustrated by the position "O" and "X").

CONCLUSIONS

To solve the problem of the research, there was proposed in the first place a universal set of 10 features, based on artificial neural network which may have undergone a process of learning by analyzing data of 7 foreign stock exchanges. The learning process was most effective when:

- network composed of two hidden layers, which included 6 and two neurons, and the output layer (one neuron), the transfer function in the hidden layer was "tansig" and in the output layer - "PURELINE",
- the network was trained by Back Propagation, according to Levenberg-Marquardt algorithm,
- multipliers used the changing values of the characteristics in the same way for each of the seven exchanges,
- network was learnt until the network performance was satisfactory.

In summary, the authors obtained satisfactory results of the research, as an artificial neural network obtained results consistent with expert suggestions based on actual data training set. In this way, the network generated the correct results also for a testing set , therefore it could be considered as an expert system, supporting making decision for the quarterly period to invest in various stock exchanges. Furthermore, it appears that the proposed set of 10 features has enough

universal character that the presented way configuration of the network together with this set of data can be useful in any chosen market. Therefore, the approach outlined in this article, supplemented with modified Prognostics (artificial neural network with data multipliers), concerning the values of the characteristics in future 3 monthly periods for various stock exchanges, can provide a comprehensive expert system for portfolio management process shares, listed on various stock exchanges.

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THE INNOVATIVE, DEMAND-BASED ORGANIZATIONAL DESIGN FOR EFFICIENT ONCOLOGICAL CARE IN POLAND: A VORONOI DIAGRAM ANALYSIS

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Abstract: In this paper, we apply a Voronoi diagram to map the existing design of oncological care in Poland. First, we correlate the actual organization of care with epidemiological data on cancer survival rate and morbidity. On this basis, we build a proxy for localized demand for oncological care. As a further step, we construct a Voronoi diagram to discuss the project of a national network for oncology in Poland based on comprehensive cancer centers and institutes, aiming at improving accessibility and equality in access to specialized oncological care.

Keywords: health economics, Voronoi diagram

INTRODUCTION

Cancers creep toward the top of the list of leading causes of morbidity and mortality worldwide. World Health Organization estimates that there were 14 million of new cases and 8.2 million cancer related deaths in 2012 [World Cancer Report 2014]. The number of new cancer diagnosis is expected to rise by about 70% over the next two decades. Only in Europe the number of cancer-related deaths borders up one million each year. In Poland, cancers are the second cause of mortality with 100 000 fatalities and this number is forecasted to increase rapidly over the next decade and cancer will become the most pestilent illness. Thus, Poland will be shortly facing a real epidemiological crisis associated with cancer incidence.

At the same time, there are significant differences in effectiveness of health care system in Europe. EUROCARE-5 study compares the cancer survival rate across European countries. Although the 5 year survival rate increased from 73 to 83% (the European mean, all cancers) during the 1999-2007 period [De Angelis et al. 2014], there are quite large divergences in regional benchmarking. For instance, the Eastern European countries show the lower survival rate for colon or lung cancer

(compared to Northern and Central European countries). Moreover, Eastern Europe performs poorly with survival rate for breast cancer (Eastern Europe 72% *versus* 82% as European mean). The good news is, however, that the distance in terms of survival outcomes between Eastern Europe and Nordic/Central Europe countries consequently shortens. Sadly, survival rates for rapidly fatal cancers (lung, pancreas, pleura and liver cancer) were impressively high for Austria, Belgium, Croatia, Germany, and Poland.

The efficiency of oncological system can also be evaluated by assessing the accessibility of care. Euro Health Consumer Index 2014 provides the measure of waiting time for oncological treatment (Time to get radiation/chemotherapy after decision). Poland classifies in the middle of the range, with 50-90% of procedures done in less than 21 days[1]. This result, however, is dimmed by the average waiting time for clinic visit. For instance, in some regions, waiting time for hematological clinic visit exceeds 120 days)[2].

There is an extensive knowledge about the risk factors associated with cancers. Since many years public health policy has been focusing on warning populations about the side effects of smoking and alcohol consumption. Information campaigns also embraced recommendations on nutrition habits or sun exposure. While prevention efforts are extremely important in containing cancer disease, experts agree on the fundamental role of screenings, early diagnosis and immediate start of therapy.

In Poland, specialists and policy makers share the opinion that low epidemiological indicators related to cancers are the consequence of late diagnosis. Therefore, the Poland's "Cancer Plan 2015-2014" sets the key-goal of improving morbidity and mortality indicators related to cancer disease, along with raising the quality of life for oncological patients.

Oncologists claim [Warzocha 2013] that achieving this goal is only possible with creating of a national network for oncology based on the comprehensive cancer centers and institutes. Thus, there is a need for redesigning the organization of oncological care based on the concept of reference centers displaying the highest level of competences, ensuring an adequate and rapid treatment. Furthermore, comprehensive oncological centers are of crucial importance for patients suffering from rare cancers and children. As an example, accurately diagnosed sarcoma can be totally cured in case of an early diagnosis using biopsy and molecular biology. However, this can be done only in referral centers[3].

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[1] Survey commissioned by HCP from Patient View 2014. Interviews with healthcare officials, feedback from national agencies. Report available at: http://www.healthpowerhouse.com/files/EHCI_2014/EHCI_2014_report.pdf (accessed on: 2015-05-03)

[2] PWC (2014) Obecny stan zwalczania nowotworów w Polsce. A real-time 'waiting times' for services are available at https://kolejki.nfz.gov.pl/

[3] http://www.rynekzdrowia.pl/Serwis-Onkologia/Nowotwory-rzadkie-brakuje-osrodkow-referencyjnych,128375,1013.html (accessed on: 2015-05-03)

Therefore, the goal of this article is two-fold. First, we aim at analyzing the actual organization of oncological care in Poland. Correlating mortality and morbidity data with demographic characteristics, we construct the proxy of localized demand for oncology care. As a following step, we apply the Voronoi procedure to depict the current design of care. Finally, we discuss the shortcomings of actual design and draw proposals. Thus, the article is structured in three sections, corresponding to the previously announced research objectives.

CURRENT ORGANIZATION OF ONCOLOGICAL CARE IN POLAND

In Poland, oncological care for adult patients with solid tumors is based on a centralized system, with the major role of the National Institute of Oncology and Hematology (with three branches: in Warsaw, Cracow and Gliwice). In most voivodships there are also regional cancer centers operating as independent units or within multidisciplinary hospitals. An important role in oncological care in Poland is ensured by the clinical hospitals. In some regions oncological care is provided by the smaller cancer centers existing within hospitals of different levels. Patients with blood cancer are treated in hematological centers, whereas patients with cancer of the lymphatic system are treated in regional oncology centers. Childhood cancer care is provided by the regional pediatric cancer centers and some pediatric wards. Surgery treatment is ensured by the Institute of Oncology, oncological surgery wards, general surgery wards and related organ surgery wards of regional cancer centers. Children are treated in pediatric cancer centers and pediatric surgery wards.

Cancer is extremely costly, both in terms of human costs and resources involved to fight cancer. In 2013, Poland total health expenditure (both public and private) accounted for 6,33% of GDP in 2012[4]. This ranks Poland in the tail of OECD countries. In 2013, the National Health Fund signed contracts with providers of oncological care services for the amount of 5,74 PLN billion. Table 1 below presents the detailed information about contracting by category:

Table 1. Contracting in oncology care by service category (2013)

	16 regional cancer centers		Other facilities	
Radiation therapy	70%	580 438 k PLN	30%	253 706 k PLN
Surgery	42%	262 236 k PLN	58%	366 746 k PLN
Chemotherapy	36%	798 718 k PLN	64%	1 428 023 k PLN
Palliative care	2%	6 765 k PLN	98%	333 162 k PLN
Diagnostics	13%	129 005 k PLN	87%	869 405 k PLN
Related treatment services	30%	211 217 k PLN	70%	501 532 k PLN

Source: Authors' calculation based on NHF data (Feb. 2014)

[4] GUS (2014) Health and health care in 2013, Warszawa, p. 127.

Analysis of this table provide us with some important insights. In 2013, 16 regional cancer centers contracted 36,5% of total NHF financial resources directed to cancer care. We can also observe an important dispersion of diagnostics. Furthermore, 70% of resources for radiation therapy was contracted by regional cancer centers, whereas the share of NHF contracting attributed to regional cancer centers was 35,8%.

Information about financial resources must be complemented by the analysis of epidemiological data and the index of resources utilization. This important data is presented in Table 2 below:

Table 2. NHF contracting, cancer morbidity rates and resource utilization index

	National Health Fund cancer contracting (per population mln, in mln PLN, 2012)	All cancer morbidity count (per 105 for males, 2012)	population/megavoltage device ratio (2012)
Dolnośląskie	98.44	269.0	364 572
Kujawsko-Pomorskie	139.26	291.8	299 767
Lubelskie	94.05	256.0	361 976
Lubuskie	97.70	268.7	341 053
Łódzkie	76.33	261.2	422 280
Małopolskie	102.88	274.3	334 680
Mazowieckie	157.87	210.5	310 918
Opolskie	89.19	267.9	337 983
Podkarpackie	79.16	274.9	425 737
Podlaskie	102.81	226.8	300 246
Pomorskie	110.64	313.3	380 583
Śląskie	134.44	254.0	257 020
Świętokrzyskie	128.87	263.5	319 529
Warmińsko-Mazurskie	110.89	275.0	484 199
Wielkopolskie	127.68	290.3	345 548
Zach-pomorskie	107.17	250.0	246 106
min	76.33	210.5	246 106
max	157.87	313.3	484 199
average	109.83	265.4	345 762.30
st. dev.	22.48	24.33	61 831.25

Source: Authors' compilation based on PWC report *Obecny stan zwalczania nowotworów w Polsce*, [Wojciechowska, Didkowska][5] and GUS [2012]

In the second column of this table, we present the values of the NHF regional contracting for the entire spectrum of oncological services (including palliative care

[5] Available at http://onkologia.org.pl/raporty/ (accessed on 2015-05-03)

and health programs). For each region, the value of contracts has been scaled to population ratio. As a further step, we have proceed with calculation of Pearson correlation coefficients between: contracting and morbidity count ($r_{c,m}$) and resource utilization index and morbidity count ($r_{r,m}$). Because of the fact that the correlation coefficient is sensitive to outliers, we have omitted Mazowieckie and Łódzkie for $r_{c,m}$ calculation and Mazowieckie, Pomorskie, Warmińsko-Mazurskie and Zachodniopomorskie for $r_{r,m}$ calculation. The value for $r_{c,m}$ (.20) suggests that increasing only spending is inefficient in reducing cancer mortality. However, this must be interpreted with caution, as we do not control for the quality of care. Similarly, the value of $r_{r,m}$ (.40) would suggest there is an inefficiency component associated with cancer screening procedure. One should remember, however, that there is likely a temporality problem associated with static approach. Moreover, analysis of the accessibility indicator (population/magevoltage device ratio) highlights quite large odds that may create additional waiting time for radiation treatment that is not associated with limited NHF contracting.

VORONOI DIAGRAM FOR THE CURRENT SYSTEM OF ONCOLOGICAL CARE IN POLAND

In the light of previous remarks on accessibility of oncology care in a current design, we have applied a Voronoi diagram in order to depict the map of existing facilities for adults and children. Before, in the subsequent sections, we present the theoretical foundation of the concept along with applied examples.

Theoretical background

The Voronoi diagram is the nearest-neighbor map for a set of points. Each region contains those points that are nearer one input site than any other input site. We start with describing the principles of calculation and elementary properties of Voronoi diagrams.

Let χ be a finite set of k points, which belongs to the Euclidean space S. The elements of the set χ we call centres. The Voronoi region (Voronoi cell) which is adherent to the element p of the set χ we call the set of points which are closer to the point p than any other element of this set [Aurenhammer 2000]

$$\chi(p) = \{x \in S | \forall q \in S, d(x,p) \leq d(x, q)\} \quad (1)$$

where d is the distance between two points described by:

$$d(a,b) = d(b,a) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$
$$= \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2} \quad (2)$$

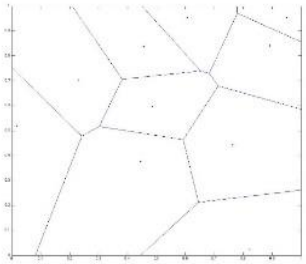
For any pair of points which belong to the space χ $(a, b \in \chi)$ in the 2-dimension space, there is a set $\Pi(a, b)$ of points which are in the same distance from both points a, b. This set is called the perpendicular bisector. This ideal line is also the limit between the set of points which are less remote from b than a.

To describe the Voronoi cell, we need to describe first the set of points half-plane which is limited by the ideal line *Π(a, b)* in which the point *a* is included. This half-plane is an area which contains every point that is closer to the point *a* than *b*.

$$H(a, b) = \{x \in S | d(x,a) \leq d(x, b)\} \quad (3)$$

The Voronoi cell, Voronoi region which is assigned to the point *a* is the part of the common of every half planes where *b* may replace each point from the set $S-\{a\}$. The Voronoi cells are the convex polygon. The set of those polygons separates the 2-dimensional Euclidean space and represents the Voronoi diagram, which corresponds to the set of points *S*.
An example of a Voronoi diagram is presented in Figure 1.

Figure 1. The example of Voronoi diagram based on the set of random points



Source: Authors' computation in Matlab

Based on the Voronoi diagram, we have also made calculation/computation of triangulation. We have used the Delaunay triangulation. The Delaunay triangulation is based on the Voronoi diagram through the principle of duality.[6]

Let *P* be a finite set of points in a sub-domain Ω^n of the n-dimensional space R^n. Two points p_i and p_j are connected by a Delaunay edge *e* if and only if there exists a location $\chi \in \Omega^n$ which is equally close to p_i and p_j and closer to p_i, p_j than to any other $p_k \in P$. The location *x* is the center of an n-dimensional sphere which passes through the points p_i, p_j and which contains no other points p_k of *P* [Fleishmann 1999].

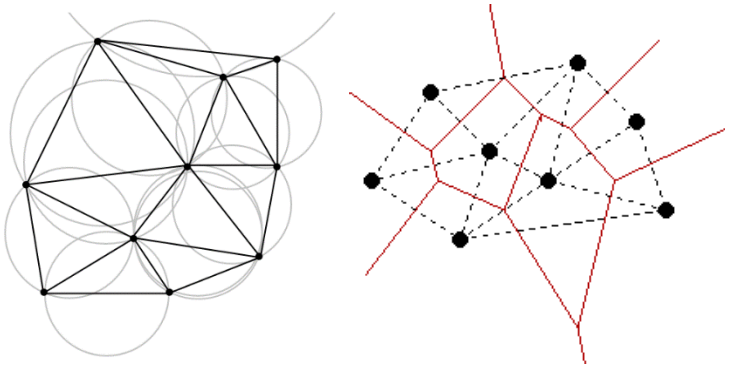
$$\begin{gathered} e_{Delaunay}(p_i, p_j) \Leftrightarrow \exists x \\ x \in \Omega^n \wedge \\ \| \| x - p_i \| = \| x - p_j \| \wedge \\ \forall k \neq i, j ; \quad \| x - p_i \| < \| x - p_k \| \end{gathered} \quad (4)$$

[6]

[6] The equation and definition from: Fleischmann P. (1999) Dissertation: Mesh Generation for Technology CAD in Three Dimensions, Vienna.

Combining this criterion for the three edges of a triangle and furthermore for the four triangles of a tetrahedron leads to the following criteria for Delaunay simplices. A Delaunay triangle is thereby the dual of a Voronoi edge.

Figure 2. Dealunay triangulation and Voronoi diagram



Source: compgeom.com, geom.uiuc.edu

On the left we present the Delaunay triangulation based on the circumscribed triangles, whereas the right picture shows the combination of Delaunay triangulation (black dotted line) and the Voronoi diagram (solid line).
So defined diagram have many applications in robotics (path planning in the presence of obstacles)[Drysdale 1993], biology (area potentially available to a tree), [Okabe 2009, Drysdale 1993, astrophysics (identifying clusters of stars and galaxies)[Qiang et al.1999] or in business retail stores market area) [Kalcsics et al. 1999, Clarkson 1985].

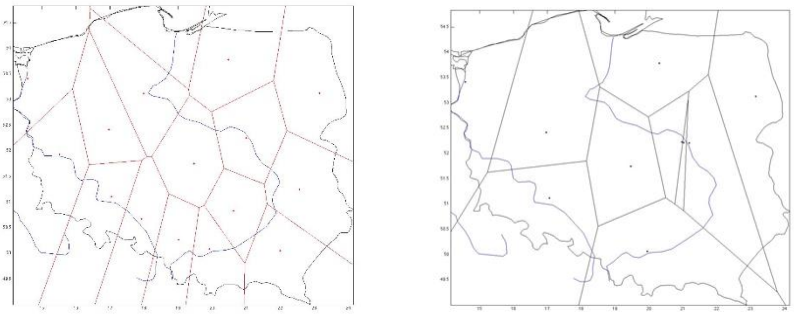
Data and procedure

Following the publicly available information about the location of cancer centers in Poland, we have constructed a related database, distinguishing adults and children facilities. That database enabled us to create maps of Poland with marked centers on it.
As a further step, we have proceeded with creating a Voronoi diagram, which divide the map for 16 regions in case of reference centers for adults and into 10 regions in case of reference centers for children (see Figure 3).

Results

Our map (depicted on Figure 3) shows the division of territory into zones that belong to each reference center. In the ideal situation, each cell should have the same area and should have the reference center in the middle of the Voronoi region.

Figure 3. The Voronoi Diagram of reference centers for adults (left) and for Children (right)

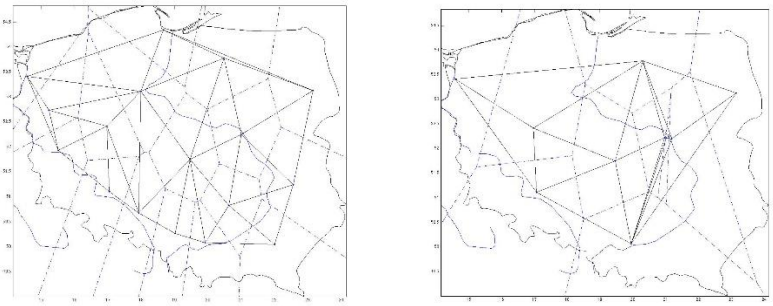


Source: Authors' computation in Matlab

As can be seen on both maps, for adults and for children, the cells have different areas and, in most cases, the reference center is not even close the center of a region. In case of the map, which represents the location of centers for adults, we see that there are centers in each region's capital city, but when we take a look at the map representing the locations of facilities for children the outcome is significantly worse. There is a substantial regional gap that is supposed to create inequalities in access to care.

In the ideal situation we should see that the map divided into a similar sized areas, which cover the entire surface of the country. If in addition to the map we add a triangulation based on the points corresponding to the reference centers (see Figure 4), we see that only the central area of Poland is covered by the graph. Once again, this can be interpreted in terms of limited accessibility of care and, partly elucidate the unsatisfactory health outcomes related to oncological care

Figure 4. Delaunay Triangulation based on the position of reference centers in Poland, for adults (left) and children (right)



Source: Authors' computation in Matlab

DISCUSSION

Our analysis confirms the partial inefficiency of current organization of oncology care in Poland. We have proved that there is an important, territorial barrier in accessibility of care. Moreover, limited resources (rationing of NHF contracting, limited medical devices) may partly explain unsatisfactory cancer survival rates in Poland. Polish *Cancer Plan* aims at containing the spread of cancer by implementing, among others, early detection and diagnosis for cervical, breast, prostate, bladder, stomach, esophagus, colorectal, and melanoma cancers, i.e. those diseases which recognition in Poland is much delayed compared to opportunities arising from the current state of practical knowledge. The plan also targets the improvement in the quality of diagnostic tests and the availability of effective treatments for malignant tumors. Furthermore, Poland intends to widen cancer screening programs (based on quality of epidemiological data), improve the accessibility of coordinated treatment, as well as raise the quality and scope of palliative cancer care. Fulfillment of these goals is only possible with the implementation of the latest science achievements. In turn, this is conditional of clinical and research excellence.

It appears, however, that this can be accomplished by rethinking the existing networks of cancer centers, following the idea of efficient and patient-oriented *Comprehensive Cancer Centers* (CCCs).

Towards Comprehensive Cancer Centers

The success factors in countries having the best outcomes in cancer diagnosis and treatment embrace, among others, the organization of care system based on the concept of reference centers under the label of *Comprehensive Cancer Centers* (CCCs). These centers share a series of distinctive features. First, they should be patient-centered and based on a coordinated mechanism of multidisciplinary treatment. Moreover, the Comprehensive Cancer Centers are supposed to be efficient in terms of resources and health outcomes. The centers' activity should rely on reducing waiting time for procedures, monitoring the effects of treatments and applying therapeutic guidelines based on best practices. Among others, there is also a competence requirement for the medical staff (continuous training, research activities). Finally, the network of comprehensive cancer centers should initiate and oversee the screening programs nationwide. The successful networks of CCCs exist in the U.S. (with 41 facilities nationwide), the Netherlands (7 facilities) and France (20 facilities)[7]. It is worth mentioning that the activities of American, Dutch and French CCCs are coordinated and overseen by the National Cancer Institutes.

In order to evaluate the potential to become CCCs for cancer centers in Poland, we apply the assessment criteria developed by the Organization of European Cancer

7 http://www.unicancer.fr/en/healthcare-professionals/innovative-oncology-model (accessed on: 2015-06-15)

Institutes (OECI)[8]. In 2008, the OECI launched its Accreditation and Designation Program aiming at certifying the quality of oncology care and designating the various types of cancer structures, including CCCs. We have chosen to evaluate Centrum Onkologii-Instytut Marii Skłodowskiej-Curie (COI) in Warsaw, Gliwice and Cracow The certification criteria along with their assessment are presented in Table 3 below.

Table 3. Assets assessment for Centrum Onkologii – Instytut Marii Skłodowskiej-Curie

Designation criteria for Comprehensive Cancer Center (OECI)	Warsaw, Gliwice and Cracow (situation as of 2013, except clinical trials)
budget for care: > 5000 k€	Yes, 863 680.70k PLN (205 638.26k €) [source: annual report]
budget for research: > 3000 k€	Yes, 64 237.40k PLN (15 294.62k €) [source: annual report]
no. of beds and ambulatory day care beds: > 100	Yes, 1.443 [source: annual report]
active clinical trials: > 50	Yes, 76 [source: ClinicalTrials.gov]
no. of scientific publications: >30	Yes, 407 [source: annual report]
no. of scientific publications with IF over 10: > 17	Yes, 23 [source: annual report and Web of Science™]

Source: Author's simulation based on Saghatchian M. (2014) and COI (2014) annual report

Our simulation confirms that COI meets the CCC's accreditation criteria. Therefore, it already constitutes an important nexus and example to follow for transforming remaining regional cancer centers into CCCs. Of course, this organizational design should be complemented by the study of effectiveness of administrated care. However, such a study remains beyond the competences of economists. We can only claim its necessity. Finally, we think that Polish cancer centers should undergo the independent accreditation procedure and the general paths of their activities and development should be initiated overseen by the Ministry of Health.

Developing economically efficient quality and inclusive oncology care is a quite challenging task. Nevertheless, it is possible once we allow to be innovative in financing (e.g. new models of financing with fair remuneration of providers with value-based pricing), organization (e.g. Comprehensive Cancer Centers) and treatment (e.g. proton therapy, personalized medicines).

[8] The OECI is a non-government, non-profit Organisations founded in Vienna in 1979 with the primary goal of strengthening cooperation between oncological institutions in the EU. The OECI promotes the concept of Comprehensiveness in treatment, as well as encourages improvement in the quality of cancer care, research and education. For more information see http://www.oeci.eu/About_OECI.aspx

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TRADE DURATION AND MARKET IMPACT

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Abstract: In this article the problem of the algorithm of the transaction execution as the factor in market impact modelling is studied. The current state of research in this area is presented and discussed. The paper adds new arguments to the discussion on this topic. Moreover, the solution to the problem of the trade execution's duration in practical application of [Almgren et al. 2005] market impact model is proposed.

Keywords: market impact, square root impact law, trade duration, transaction cost

INTRODUCTION

Market impact (also called price impact) can be defined as some sort of a change in the asset price with respect to adequate reference price, caused by trading. This change, if occurs, is against the trade initiator, that is the price grows when buying and drops when selling, and thus the price impact is a source of transaction costs. It in intuitive and a standard in economic theory that a demand increase should result in growth of the price and a supply increase should result in the price drop. The concept of market impact is closely related to the notion of bid-ask spread, which is the difference between the best available in the market bid and ask prices (called just bid and ask prices, respectively) and often is expressed as a fraction of a so called mid-price which is defined as the average between best bid and ask prices and represents the market value of an asset. If, as some authors do, the mid-price is the reference price, then bid-ask spread is a part of market impact. However, it is often assumed that the reference price is a bid price in case of a seller-initiated trade and an ask price in case of a buyer-initiated trade. In such approach the spread and price impact are treated separately.

Market impact is the main source of liquidity risk. It is the reason of not being able to execute a transaction at the current quoted price because execution moves the price in an unfavourable manner. Spectacular examples showing how important is market impact are: the fiasco of Metallgesellschaft in 1993, the LTCM crisis in 1998 and the cancelling of the portfolio of Jérôme Kerviel by Societé Générale in 2008 [Schied and Slynko 2011]. Since market impact moves adversely the prices at which transactions are made, it can, sometimes significantly, reduce profits and turn theoretically profitable strategy into a financial failure.

Therefore, it is no surprise that modelling, estimation and analysis of market impact interests many asset managers and scholars. In fact, research on price impact has become one of the most popular activities in quantitative finance since the mid-1990s [Tóth et al. 2011].

The aim of this article is to analyse the problem of importance of the algorithm of the trade execution as the variable in modelling of market impact. The paper contributes to the literature on market impact by adding new arguments confirming that the execution speed is of minor significance in price impact modelling. Next contribution is a new view on the model of [Almgren et al. 2005] which allows for better practical use of this model. This article contains also the example of the calculation of market impact in Warsaw Stock Exchange, with use of order book.

MARKET IMPACT MODELS AND TRADE DURATION

Market impact modelling and estimation has been very important to scholars interested in market microstructure and practitioners. A well-calibrated price impact model is an important part of quantitative investment management. It is a useful tool in predicting transaction costs and price changes due to trading activity. Such expectations allow to forecast the consequences of implementing portfolio strategies. Today, any decent pre-trade analytic software takes into account the price impact of a proposed transactions as a function of trade-based parameters and characteristics of the traded security [Gatheral 2010].

The simple and popular approach to modelling price impact suggested in the literature is to consider it as one of the components of transaction costs. Then the formula for market impact as a relative fraction of the price of the traded security at the beginning of the trade, is given as follows:

$$MI = c\sigma\left(\frac{V_{trade}}{\overline{V}}\right)^{\delta} \tag{1}$$

where σ is the daily volatility, V_{trade} is the volume of the executed trade, $\overline{V}$ is the average daily volume, c is the numerical constant of order unity that can be estimated from the representative sample of transactions and the exponent δ does not exceed 1 and its estimation has often the range between 0.4 and 0.7, however

an important practically and theoretically case is linear function of market impact with $\delta = 1$.

A particular variant of the formula (1) is the so-called square root impact law which is widely used in academia and financial service industry:

$$MI = c\sigma\sqrt{\frac{V_{trade}}{\bar{V}}} \tag{2}$$

Equation (2) is strongly supported by the empirical data, reasonable arguments given in [Grinold and Kahn 2000] and it is consistent with a trading rule of thumb according to which the transaction cost of the volume equal to the average one day's volume, costs roughly one day's volatility of the price. [Grinold and Kahn 2000], [Gatheral 2010].

Formula (1) suggests that the only trade-based variable which is necessary to calculate the market impact is the transaction size V_{trade}, it does not take into account the execution algorithm used by the trader. In this context it should be noticed that there is a great variety of execution strategies – apart from static (determined in advance of trading) there are also dynamic which are conditioned on movement of the security price during execution of transaction. The trade execution is roughly characterized by duration which describes how long the executing lasts. The duration is determined by the trading rate (the speed of execution) and the transaction volume. Low importance of the execution characteristics is more emphasised by some authors [Tóth et al. 2011], [Zarinelli et al. 2014] by using, for the volume of the executed trade, in the market impact equation the name "metaorder", which denotes the sequence of trading decisions. A metaorder is usually fragmented and traded incrementally by single orders which are, in this context, called child orders.

Such approach is however contrary to the widespread opinion that market impact can be reduced by dividing intended transaction into smaller orders and placing them in separate time intervals. In short, the popular view is that slower trade execution lowers price impact. It is also empirically confirmed that the trading rate can, in some circumstances, significantly affect the market impact. There is an extensive theoretical research and practical solutions on the problem of optimal counteracting market impact while executing transaction. The question arises, therefore, about the explanation of this conceptual contradiction. In order to answer this question, it is worth pointing out that the observations where the duration was important pertain to the cases of very large trading rates were trading sizes were large relative to the volume of trade offers in the order book. For reasonable trading rates (about 1% to 25% of average daily volume per day), it seems that the market impact is roughly independent of trade duration [Gatheral 2010]. It is even presented in [Gatheral and Schied 2013] as the empirical rule of thumb that market impact is roughly proportional to the size of the transaction and not very dependent on the trading rate. There is also a heuristic argument that

duration can or maybe even should be omitted as variable in formula (1). Namely, [Grinold and Kahn 2000] claimed that in a framework of inventory risk model, for a proposed trade of size V_{trade}, the estimated time before a sufficient number of opposing trades appears in the market to execute the transaction (time to clear the transaction) is given by the formula

$$\tau_{clear} \propto \frac{V_{trade}}{\overline{V}} \tag{3}$$

Formula (3) establishes a strictly linear relationship between the size of the transaction and the time of execution. Thus, since it is natural that the durations can be measured by time τ_{clear}, it is, according to (3), fully characterized by the trading volume. Consequently, since the size of the transaction is a variable in (1), duration does not have to appear there.

Weak dependence of market impact on trade duration is also confirmed by the empirical data in [Engle et al. 2008].

In order to provide new arguments for discussion on the meaning of execution algorithm in market impact modelling I would like to notice that it is not uncommon to consider the problem of the optimal portfolio selection in multiperiod setting where neither the total transaction volume nor the investment horizon has to bounded in advance. Then, it is appropriate to ask how long lasts the market impact effect of trading in one period, on the asset's price dynamics. It is clear that the value of market impact in next time period strongly depends on the answer to this question. Most practitioners in execution models use the decomposition of the market impact into permanent and temporary market impact [Guéant 2014]. Temporary price impact affects a single transaction and may be considered as the cost of providing enough liquidity to absorb the trade. The permanent price impact component is an information-based effect and measures the change of the market value before and after trade. This is due to the fact that there is no easy, method to distinguish not informed traders from informed traders and therefore each transaction is considered as a source of information on the market value of the traded asset. Thus, a buyer-initiated transaction tells the market participants that an asset may be underpiced and a seller-initiated transaction is a signal that an asset is overvalued. As a result, the transaction causes the change in the theoretical value of the asset which is unfavourable to the initiator of the trade.

It seems that the speed execution has different effects on the levels of the considered components of market impact. The higher trading rate results in larger temporary impact and lower permanent impact, in case of lower trading rate it is the other way round. Therefore the coexistence of the two components of market impact which differently react on the speed of trading I find as one of the arguments for low significance of the execution style in modelling market impact.

In case of informed traders there is also another factor that counteracts the effect of reducing market impact of the strategy of slower execution. It is opportunity cost. This notion assumes that, if the motivation of the trade is information on the future value of the traded assets, then quick execution is necessary because such information can be used only for a limited time. Rapid execution enables to benefit from the underpricing in case of buying and from overpricing in case of selling.

Physical time is not the only method of measuring the duration of the execution. The duration is sometimes quantified in so called volume time [Almgren et al. 2005], [Zarinelli et al. 2014] which is calculated for time periods shorter than trading day, as the fraction of an average daily volume that has been executed up to physical time t. Speaking formally let $V(t)$ be the total volume traded in the market from the trading day's open up to physical time t. Volume time (also called volume duration) is defined as $v = \frac{V(t)}{V(t_c)}$ or $v = \frac{V(t)}{\bar{V}}$, where t_c is the market close. It is easily seen that independently of the total daily volume, the volume time defined that way equals 0 at the market opening and 1 at market closing time [Zarinelli et al. 2014].

The duration measured in volume time is an input variable in the elaborately worked out and seemingly ready for use model used by [Almgren et al. 2005]. In its estimation [Almgren et al. 2005] used the data set of almost 700,000 trade orders from the US market, executed by Citigroup equity trading desk from December 2001 to June 2003, in which a direction of the trade (buyer or seller initiated) is known. The market impact defined as the execution cost in the model of [Almgren et al. 2005], assuming that price impact is positive for buy as well as for sell orders (in the original version of this model the execution cost can be negative), is given by the formula:

$$MI = \frac{1}{2}\gamma\sigma\frac{V_{trade}}{\bar{V}}\left(\frac{\Theta}{\bar{V}}\right)^{1/4} + \eta\sigma\sqrt{\frac{V_{trade}}{\bar{V}T}}^{3/5} \tag{4}$$

where Θ is the total number of shares outstanding, T is volume duration of active trading, γ, η are the constants.

The estimated values of γ and η were calculated by linear regression [Almgren et al. 2005] and they calculated that $\gamma = 0.314 \pm 0.041$ and $\eta = 0.142 \pm 0.0062$.

An example of the application of this model is presented in [Kociński 2014] where the duration was assumed to be an arbitrary value. The variable T I find the most problematic in the model given in [Almgren et al. 2005]. It seems that the trader is rather not able to control the duration of execution to the extent which is necessary to produce reliable estimator of the volume duration. However, by reasonable assumptions, applying of low-frequency estimator which uses only the

daily volumes in estimating the time to clear the trade and application of linear approximation, it is possible to eliminate T from the formula (3).

Namely, to justify the use of low frequency estimation, it seems sound to assume that average time to fulfil an order does not depend on whether it is a buy or sell order. By this assumption, the formula (3) and the fact that the coefficient of proportionality in (3) which is estimated by using daily volumes, equals 1, it is possible to write the following formula for physical time of transaction execution:

$$\tau_{clear} = \frac{V_{trade}}{\overline{V}} \tag{5}$$

The volume time given in [Almgren et al. 2005] is defined as $v = \frac{V(t)}{\overline{V}}$. From the assumption that $V(t)$ is a linear function of t, it follows that physical time is identical with volume time and consequently from (6) it follows that the value of T is given by the formula:

$$T = \frac{V_{trade}}{\overline{V}} \tag{6}$$

The equations (5) and (6) imply the following, simplified expression for the market impact:

$$MI = \frac{1}{2}\gamma\sigma\frac{V_{trade}}{\overline{V}}\left(\frac{\Theta}{\overline{V}}\right)^{1/4} + \eta\sigma \tag{7}$$

According to (7) the price impact is a linear function of the traded volume, what is an interesting in view of the popularity of the square root model. However the assumption that market impact is linear in the traded volume one can meet in the literature (see for example [DeMiguel et al. 2014]). Such approach can be partly justified in market microstructure theory by the Kyle model [Kyle 1985].

The model given in [Almgren et al. 2005] treats the bid-ask spread as a part of the market impact and this allows to interpret the second component of the sum in (7) as the bid-ask spread, and in this approach it is a linear function of volatility. Moreover, an interesting observation is that in view of (7) the average volume is not a significant determinant of the spread. The negligibility of the market volume in case of the bid-ask spread in the option market was found in [Cho and Engle 1999].

EMPIRICAL RESEARCH

To verify the conclusions on the role volatility and volume in determining bid-ask spread following from formula (7) the empirical research was carried out on a random sample of 300 stocks quoted on the Warsaw Stock Exchange (WSE) in 2014. The annual volatility was computed from the formula:

$$volatility = \ln\left(\frac{P_{\max}}{P_{\min}}\right) \tag{8}$$

where $P_{\max}$ and $P_{\min}$ denote the maximal and minimal price of the stock in year 2014, respectively.

Using data 2014 WSE Statistic Bulletin I calculated the Pearson correlation coefficients between the average spread and volatility ($r_{volatility}$) and the average spread and the average daily volume (r_{volume}). Then, I verified their significance by the standard significance test with test statistics $t = r\sqrt{\frac{n-2}{1-r^2}}$ where r is the correlation coefficient and n is a number of observations. The following results were obtained (p is the p-value):

$$r_{volatility} = 0{,}256;\ p = 0{,}0000068$$

$$r_{volume} = 0{,}037,\ p = 0{,}5256843.$$

The correlation between volatility and spread is highly statistically significant, in contrary to the correlation between the spread and volume. I also estimated the regression coefficients of the spread on volatility:

$$spread = \underset{(2{,}95*10^{-25})}{142{,}48} + \underset{(6{,}94*10^{-6})}{63{,}18} * volatility \tag{9}$$

The results concerning correlation coefficients presented above are consistent with equation (7). However, the non-zero constant in regression equation suggests that formula (7) should be supplemented by an additive constant.

PRICE IMPACT IN MARKET MICROSTRUCTURE: AN EXAMPLE FROM WARSAW STOCK EXCHANGE

One of the most important characteristics of a market is the type of its execution system. In this respect there are three major types of markets: quote driven markets, order driven markets and brokered markets. Warsaw Stock Exchange (WSE) is classified as order driven market [Doman 2011]. Table 1 shows the first five rows of an order book for the stock of the company Stalexport Autostrady S.A. (denoted as STALEXPORT), from the WSE at some point in time during the trading session on 09 September 2015.

Table 1. The first five rows of the order book for the shares of STALEXPORT at some moment during the trading day in WSE on 09 Sep 2015

Bid size	Bid price (PLN)	Ask price (PLN)	Ask size
2000	3.21	3.24	2700
1600	3.18	3.25	3582
2375	3.17	3.27	1500
1900	3.16	3.28	5821
4433	3.15	3.29	2550

Source: http://biznes.onet.pl/gielda/notowania/gpw-rynek-glowny/akcje-wszystkie,101,notowania-gpw-ciagle-szczegolowe.html

The second row of Table 1 represents the highest available bid price (3.21), the number of stocks to buy at this price (2000), the lowest available ask price (3.24) and the number of stocks to sell at this price (2700). The theoretical price of the stock STALEXPORT is the average of the best bid and ask prices and equals 3.225. The spread is computed as $\frac{3.24-3.21}{3.225}$ and equals approximately 0.009.

Consider an investor who wants to buy some shares and places market buy order. For the sake of transparency of this example I assume that commission here is negligible, since this sort of cost is just an additive constant. On the frictionless market the cost of his or her transaction would be 3.225 PLN per share. The transaction cost here is calculated as the relative increase in average price per share with respect to the theoretical price. On real market if the number of shares does not exceed 2700 then the required by the market price per share is 3.24 PLN and transaction cost is just half of the spread. However, if the trading volume increases then the average price per share grows. It is clear that the average price per share can be computed from the formula:

$$\overline{S} = \frac{\sum_{i=1}^{n} x_i S_i}{\sum_{i=1}^{n} x_i} \tag{10}$$

where n is the number of price levels in the order book, S_i is the i-th price level and x_i is the number of shares for which the ask price is S_i, respectively. Then the average cost per share equals $\frac{\overline{S}-S_0}{S_0}$ where S_0 is the theoretical price. Market impact is here the difference between transaction cost and the half of the spread. The calculation of the average costs and market impact for trading volumes corresponding to the cumulated values of ask sizes from the order book are shown in Table 2.

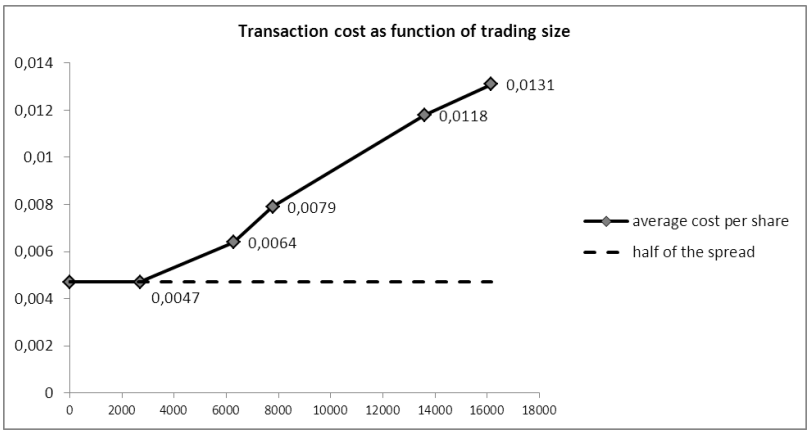
Table 2. The average cost per share for purchase transactions of the stock

Trading size	Average cost	Market impact
2700	0.0047	0.0000
3582	0.0064	0.0018
1500	0.0079	0.0032
5821	0.0118	0.0071
2550	0.0131	0.0085

Source: data from Table 1 and own elaboration

The graph of the transaction costs function is presented in Figure 1.

Figure 1. The function of transaction cost



Source: data from Table 2 and own elaboration

The dashed line corresponds to the case of proportional transactions costs, where the only source of payments for exchange of shares is the bid-ask spread. The market model with proportional cost is much more realistic then the assumption of frictionless market where there are no costs associated with trading. Figure 1, however, shows, that in order to precisely estimate the risk in asset management, one should take the price impact into account.

CONCLUSIONS

The paper presents the problem of the influence of the method of transaction execution on the magnitude of market impact. Neglecting the style of trading as factor affecting price impact is popular among both theoreticians and practitioners. Taking into account the literature on this subject and the arguments presented in this article, It appears reasonable to assume that when the transaction volume is not extremely large and the investment horizon is not very long the execution style is not very important. The effect of lowering market impact with execution method is usually of minor order than the value of price impact, and therefore the execution algorithm need not be taken into account when estimating market impact.

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PENSION FUND MARKETS IN EUROPE. COMPARATIVE ANALYSIS APPLYING SYNTHETIC MEASURE OF DEVELOPMENT

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Abstract. The pension systems in the majority of European states have been reformed because of serious changes in the demographic structure of the populations. Therefore funded pillars have been added to the pension systems. This additional (to pay as you go system) pillar, created by pension funds, is mandatory in some countries. The investments made by the pension funds influence the development of financial markets and affect the situation in national economies. The aim of the paper is comparative analysis of the pension fund markets in selected European states in the years 2000-2013, using the synthetic measure of development.

Keywords: pension funds market, comparative analysis, synthetic measure of development

Population ageing in the majority of European countries is leading to the significant increase of the old-age dependency ratios because it causes the increase in the number of people in retirement relative to the size of the working-age population, and the increase in the number of years that people spend on retirement**.** Therefore essential transformations of the pension systems have been introducing in many European states to make pension systems more financially sustainable. The main ideas of changes in the retirement system consist in heightening the pension age and introducing funded system instead of pay as you go system (PAYG). There are six major pension reform key objectives [OECD 2013, p. 18]. (1) Pension system coverage in both mandatory and voluntary schemes. (2) The financial

sustainability and affordability of pension promises to taxpayers and contributors. (3) Incentives that encourage people to work for longer parts of their lifetimes and to save more while in employment. (4) Adequacy of retirement benefits. (5) Administrative efficiency to minimize pension system running costs. (6) The diversification of retirement income sources across providers (public and private), the three pillars (public, industry-wide and personal), and financing forms (PAYG and funded).

The economic crisis caused reduction in government revenues to finance PAYG public pensions leaving the space for the private pension system development. Founded pillar of the pension systems is created by the private pension funds, which operate in similar way as mutual funds. The most frequent reason given in the public policy debate for a funded system is the apparently superior performance of the capital market in terms of the rate of return on investment it can offer. Indeed, many studies have shown how poor the rate of return on PAYG pension contributions really is (see [Sinn 2000], [Feldstein 1997]).

Pension funds play an important role in financial market and affect the development of national economies because they are one of the main institutional investors. In 2013 assets accumulated by pension funds totaled USD 24.7 trillion (i.e. 26.7% of total assets held by all institutional investors) while assets of public pension reserve funds were USD 5.1. trillion (- 5.5%) [OECD 2014, p. 7].

The aim of the research[1] is comparison of the pension fund markets in selected OECD European states in the years 2001-2013. Analysis is provided in terms of the pension funds performance together with investment dynamics and assets accumulated relative to the size of the national economies, using synthetic measure of development.

PENSION FUNDS IN EUROPE

Ensuring coverage of employees through one or more pension plans is fundamental in fighting income poverty in old age. All OECD countries have set up mandatory or quasi-mandatory pension plans, either public or private, to achieve quasi-universal coverage. However, mostly in low-income countries, there is still a significant share of society not covered by public or national schemes. Policies to diversify and secure savings have taken four main forms [OECD 2013, p. 25]. (a) Voluntary pension plans to improve investment options for workers and increase competition among funds. Canada, the Czech and Slovak, Poland and the United Kingdom have introduced such schemes. (b) Regulations that allow individuals greater choice over the way their retirement savings are invested in private plans. Canada, Estonia, Hungary, Israel, Mexico and Poland, for example, have adopted

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this policy, supported by measures to move people automatically into less risky investments as they get closer to retirement, a policy recommended in earlier OECD analysis. (c) The relaxing of restrictions on investment options to foster greater diversification of pension funds' portfolios. Chile, Finland, Switzerland and Turkey have followed this path, with Chile and the Slovakia allowing pension funds to take larger shares in foreign investments in order to hedge the risk of national default. (d) Action to improve pension funds' solvency rates. Canada, Chile, Estonia and Ireland have introduced stricter rules on investment in risky assets in order to protect pension plans' members more effectively. In Canada and Ireland, state direct intervention has helped financially insolvent funds to recoup losses in their asset values caused by the financial crisis. Finally, Finland and the Netherlands temporarily relaxed solvency rules to allow funds a longer time to recover.

Table 1. Percentage real average net annual and 5-year rate of investment returns generated by pension funds

Country	2008	2009	2010	2011	2012	Average 5-year	Average annual	Year
Austria	-14.4	7.3	3.7	-6	5.5	-1.13	0.62	2002
Belgium	-22.3	13.4	4.4	-4.6	9.3	-0.83	2.04	2002
Czech Rep.	-1.5	-0.6	0.7	0.5	0.2	-0.14	0.65	2002
Denmark	5.1	1.2	7.1	12.1	5.4	6.12	4.79	2002
Estonia	-32.4	14.8	2.1	-8.0	5.2	-5.17	-1.63	2003
Finland	-19.7	14	7.1	-5.2	6.6	-0.19	2.23	2002
Germany	0.5	3.9	3.4	1.0	3.3	2.41	2.52	2002
Greece	2.3	0.3	-7.8	-5.6	5.0	-1.28	-1.28	2008
Hungary	-21.7	12.8	4.2	-0.5	6.8	-0.44	0.93	2002
Italy	-5.3	5.3	1.2	-2.8	4.0	0.40	1.35	2002
Luxembourg	-11.3	6.5	0.7	-2.3	5.0	-0.49	3.21	2005
Netherlands	-17.3	11.5	8.8	4.3	13.5	3.50	3.70	2002
Poland	-17.3	8.9	7.2	-9.1	1.6	-2.27	3.95	2002
Portugal	-13.2	11.6	-3	-7.3	5.8	-1.62	1.59	2002
Slovakia	-8.9	1.0	0.0	-3.8	0.4	-2.33	-1.96	2007
Slovenia	-5.4	4.2	1.8	-1.8	4.5	0.59	0.32	2007
Spain	-9.9	6.9	-2.2	-2.3	3.6	-0.95	-0.95	2008
Switzerland	-13.8	9.9	2.8	0.6	7.5	1.04	1.78	2002
U.K.	-0.9	-0.9	-2.1	-2.5	-1.2	-1.52	-0.53	2002

Source: own calculations on the basis of OECD Global Pension Statistics

In our investigation we employ data from OECD Global Pension Statistics concerning pension funds operating in selected European countries[2]. In fact the in-

[2] Analysis, basing on data from OECD Global Pension Statistics, provided for the pension fund markets in the USA and selected European countries is presented in the paper Foo J., Witkowska D. (2015) Pension Fund Efficiency Performance between US and Europe, discussion paper, presented at International Atlantic Economic Conference, Milan.

vestigation covers 19 countries[3] in the years 2001-2013 however in some analysis it is necessary to shorten the sample because of lack of data. Table 1 contains observations concerning percentage real average net annual rate of investment returns generated by pension funds in the years 2008-2012 since only for these years the observations are available for all considered countries (Table 1). Therefore average annual returns are calculated on the basis of available data and the first year of analysis is presented in the last columns in Tables 1, 3 and 4.

As one can notice, only six transitional countries are represented in the analysis. Let us remind that pension funds in these states started to operate latter than in Western Europe. It is visible that global financial crisis influences the pension funds performance since 5-year real average rates of return evaluated for years 2008-2012 are negative for majority of states while negative average annual rates of returns are observed only in Estonia, Greece, Slovak, Spain and the UK. It worth mentioning that pension funds in United Kingdom were the only ones that did not recovered from the crisis in 2012. While pension funds in Denmark, Germany and Switzerland generated nonnegative returns in the years 2008-2012.

Table 2. Annual real returns from mandatory pension funds in Central and Eastern Europe

States	2002-2007	2002-2012	States	2002-2007	2002-2012
Bulgaria	4.0	0.5	Poland	10.8	6.4
Croatia	5.0	3.2	Russia	-3.1	-2.7
Estonia	3.1	0.1	Romania	n.a.	5.1
Lithuania	3.2	0.8	Slovakia	0.8	-12.0
Latvia	-2.0	1.3	Hungary	4.2	n.a.
Macedonia	2.6	2.4			

Source: [Lewicka-Banaszak 2014]

Analyzing the performance of mandatory pension funds in Central and Eastern Europe one can notice that in Poland the efficiency of pension funds is the highest among European transitional states where this pillar is mandatory (Tab. 2).

The market value of assets accumulated relative to the size of economy is described by pension funds' investment as percentage of GDP. The OECD weighted average asset-to-GDP ratio for pension funds increased from 77.1% of GDP in 2012 to 84.2% of GDP in 2013. The Netherlands reached the highest ratio at 166.3%. Table 3 contains average and variability measures of asset-to-GDP ratio calculated for all years of observations, together with measures of dynamics, evaluated for all available data from the period 2001-2013. There are two countries the Netherlands and Switzerland where pension fund investments exceed the value

[3] There is no available data concerning pension funds performance for France in OECD pension data. We also remove Ireland and Sweden from this analysis because for these states there were only data concerning years 2007, 2008, and 2011, 2012, respectively.

of their GDP[4]. In UK the value of investments is bigger than 70% of GDP, while in Finland it equals 60% of GDP. The smallest share of pension funds' investments in GDP is observed in Greece, Luxemburg and Slovenia. Due to variation coefficient Greece with Slovakia are characterized by the biggest, while Switzerland – the smallest variability among the analyzed countries. In Estonia and Slovakia investments raised the most dynamically (according to annual average increase), while in Belgium and Portugal investments decreased (annually).

Table 3. Pension funds investments in the years 2001-2013 as percentage of GDP

Country	Average	Standard deviation	Variation coefficient	Changes in time - annual average	Year
Austria	4.67	0.71	0.15	5.82%	2001
Belgium	4.33	0.57	0.13	-0.52%	2001
Czech Republic	4.78	1.68	0.35	11.04%	2001
Denmark	37.92	8.85	0.23	3.85%	2001
Estonia	4.41	3.11	0.71	61.29%	2001
Finland	60.68	12.09	0.20	0.19%	2001
Germany	4.7	0.97	0.21	5.03%	2001
Greece	0.03	0.02	0.66	30.37%	2007
Hungary	7.66	3.69	0.48	0.32%	2001
Italy	3.63	1.26	0.35	8.80%	2001
Luxembourg	1.51	0.65	0.43	22.48%	2004
Netherlands	122.36	20.52	0.17	3.81%	2001
Poland	10.86	4.97	0.46	18.46%	2001
Portugal	11.09	1.66	0.15	-1.54%	2001
Slovakia	5.31	3.43	0.65	45.90%	2005
Slovenia	2.24	1.12	0.50	19.66%	2003
Spain	7.38	1.02	0.14	4.61%	2001
Switzerland	106.77	7.74	0.07	1.53%	2001
United Kingdom	78.18	14.04	0.18	3.08%	2001

Source: own calculations on the basis of OECD Global Pension Statistics

Taking into consideration value of investments made by pension funds we analyze only investment dynamics because of lack of the data for some years and countries. Investment dynamic is measured by geometric mean, which is calculated individually for each country due to availability of data (Table 4). This measure describes annual average changes of investment values in all considered states. The biggest dynamics are observed in Slovakia (116% annually) and Estonia (81%). The second group is created by Czech Republic (16%), Poland (27.5%), Greece (28%), Slovenia (31%) and Luxemburg (33%), while the slight decrease of investments is observed in Portugal.

[4] Similar situation is in Iceland, which is not considered in our study.

Table 4. Dynamics of total investment of pension funds [millions of national currency]

Country	2007	2008	2009	2010	2011	2012	Average	Year
Austria	13.15	12.55	14.06	15.22	14.76	16.31	8.97%	2001
Belgium	14.79	11.41	13.80	13.31	15.63	17.24	1.74%	2001
Czech Rep.	167.20	191.71	215.87	232.42	247.51	273.20	16.00%	2001
Denmark	548.98	824.24	718.05	867.88	887.90	913.14	8.74%	2001
Estonia	0.71	0.74	0.95	1.07	1.13	1.48	81.01%	2001
Finland	127.00	112.74	133.07	148.06	143.66	152.75	7.46%	2001
Germany	112.76	117.88	126.36	134.85	149.09	167.57	7.88%	2001
Greece	0.02	0.03	0.05	0.05	0.07	0.09	28.44%	2007
Hungary	2766.27	2567.25	3412.00	3964.53	1060.48	919.05	4.06%	2001
Italy	50.14	53.69	62.51	70.81	76.85	87.64	10.92%	2001
Luxembourg	0.37	0.39	0.84	0.80	0.83	0.90	32.78%	2004
Netherlands	772.45	670.24	679.86	760.12	815.87	960.22	6.93%	2001
Poland	141.35	139.61	181.35	223.01	229.02	274.20	27.50%	2001
Portugal	22.36	20.28	21.92	19.72	13.24	14.47	-0.22%	2001
Slovakia	2.29	3.17	3.97	4.88	5.80	6.82	116.16%	2005
Slovenia	0.63	0.71	0.91	1.09	1.20	1.31	30.73%	2003
Spain	86.48	78.13	85.07	83.99	83.66	86.59	7.48%	2001
Switzerland	605.46	538.52	598.93	621.23	625.29	672.50	3.91%	2001
U.K.	1092.67	927.72	1124.26	1289.07	1444.02	1474.50	6.70%	2001

Source: own calculations on the basis of OECD Global Pension Statistics

METHODOLOGY AND EMPIRICAL RESULTS

Our investigation consists in comparative analysis of the pension funds operating in selected European states taking into account their performance together with investment dynamics and assets accumulated relative to the size of the national economies. We apply the synthetic measure of development [Hellwig 1968]. Let us define the taxonomic measure SMR_i for the *i*-th country:

$$SMR_i = 1 - \frac{q_i}{\overline{q} + 2 \cdot S_q} \tag{1}$$

where q_i is the distance of the *i*-th object (state) from the benchmark:

$$q_i = \sqrt{\sum_{j=1}^{k} \left(z_j^i - z_j^0 \right)^2} \tag{2}$$

evaluated for standardized variables z_j^0, z_j^i that describe the benchmark and the *i*-th investigated state, respectively. The benchmark is defined as the hypothetical object that is characterized by maximal values of stimulants and minimal values of destimulants:

$$z_j^0 = \begin{cases} \min\limits_{i=1,2,\ldots,n} \left\{ z_j^i \right\} & for \quad x_i \in D \\ \max\limits_{i=1,2,\ldots,n} \left\{ z_j^i \right\} & for \quad x_i \in S \end{cases} \tag{3}$$

where for each *j*-th variable: z_j^i - standardized variables, x_j^i, $\overline{x}_j$, S_j^x - observations of for the *i*-th country, average and standard deviation, respectively. *D* and *S* are sets of destimulants and stimulants, respectively. Other symbols denote $\overline{q}$, S_q - the average and the standard deviation of distances q_i, respectively.

The synthetic taxonomic measure *SMR* is evaluated for each country on the basis of the diagnostic variables, and it can be treated as a measure of the pension fund market development in analyzed countries. To compare considered states we construct clusters for selected countries as following:

A. very well developed - efficient pension fund market if $SMR_i \geq SMR + S_{SMR}$,
B. well developed pension fund market if $SMR + S_{SMR} > SMR_i \geq SMR$,
C. developed pension fund market if $SMR > SMR_i \geq SMR - S_{SMR}$,
D. undeveloped - inefficient pension fund market if $SMR_i < SMR - S_{SMR}$,

where *SMR* and S_{SMR} are mean and standard deviation of the measures SMR_i, respectively.

The key question concerns the selection of the diagnostic variables, which are used to evaluate taxonomic measures SMR since changes of the set of these variables may influence the position of states (objects) in the ranking and the country belonging to the certain cluster. Therefore if several sets of diagnostic variables are used it is convenient to generalize the results of investigation either

- making a ranking of states taking into consideration all measures:

$$SMGR_i = \sum_{k=1}^{K} R_{ik} \tag{4}$$

  where R_{ik} is the position of the *i*-th state due to measure SMR_{ik}, where SMR_{ik} is evaluated for the *k*-th set of diagnostic variables or

- evaluating the cluster for states taking into consideration the frequency of being the member of the certain cluster:

$$SMGC_i = \sum_{k=1}^{K} f_{ik} \cdot G_{ik} \tag{5}$$

  where G_{ik} is the scoring for the cluster where the *i*-th state belongs due to measure SMR_{ik}, (we assumed: A=1, B=2, C=3 and D=4), f_{ik} describes how many times the analyzed *i*-th object (state) belongs to certain cluster A-D. Using average and standard deviation of $SMGC_i$ we construct generalized clusters based on all applied sets of variables (as it was made before for SMR_i).

To construct the synthetic measure we use seven diagnostic variables: (1) five-year rates of return (OECD data), (2) annual rates of return (geometric mean evaluated from yearly OECD data), (3) annual average share of the pension funds

investments in GDP (arithmetic mean), (4) standard deviation coefficient of the share of the pension funds investments in GDP, (5) variation coefficient of the pension funds investments in GDP, (6) average annual increase of the share of the pension funds investments in GDP, (geometric mean), (7) average annual increase of the pension funds investments in local currency (geometric mean).

Table 5. Ranking of states due to SMR_i evaluated for different set of diagnostic variables

SMR(Ia) – 7 variables		SMR(Ib) – 4 variables		SMR(IIa) – 6 variables		SMR(IIb) – 5 variables	
Denmark	0.248	Netherlands	0.436	Denmark	0.464	Denmark	0.544
Luxembourg	0.224	Denmark	0.383	Switzerland	0.378	Switzerland	0.466
Switzerland	0.213	Switzerland	0.329	Germany	0.356	Germany	0.391
Germany	0.186	Finland	0.278	Slovenia	0.355	Netherlands	0.388
Slovenia	0.173	Luxembourg	0.257	Italy	0.342	Italy	0.323
Netherlands	0.165	Germany	0.251	Netherlands	0.338	Slovenia	0.321
Poland	0.158	Slovenia	0.221	Czech Rep.	0.322	Czech Rep.	0.312
Italy	0.139	Poland	0.206	Luxembourg	0.320	Luxembourg	0.285
Finland	0.139	Italy	0.190	Slovakia	0.281	U.K.	0.275
Czech Rep.	0.135	Czech Rep.	0.176	Poland	0.269	Poland	0.259
Slovakia	0.133	U.K.	0.171	U.K.	0.264	Austria	0.253
Austria	0.095	Slovakia	0.156	Austria	0.238	Belgium	0.251
Belgium	0.085	Hungary	0.144	Belgium	0.231	Finland	0.248
Greece	0.066	Belgium	0.143	Greece	0.222	Spain	0.239
U.K.	0.060	Austria	0.125	Spain	0.215	Slovakia	0.218
Portugal	0.059	Portugal	0.117	Finland	0.169	Portugal	0.142
Spain	0.057	Greece	0.102	Estonia	0.154	Greece	0.128
Hungary	0.048	Spain	0.083	Portugal	0.085	Estonia	0.088
Estonia	0.028	Estonia	0.038	Hungary	-0.142	Hungary	-0.066

Source: own calculations

Due to availability of data for selected countries the synthetic measures are calculated using the measurement of diagnostic variables, which was made in two ways. The first one employs all available observations i.e. dynamics, means or dispersion for each state are measured for different periods (as in Tables 1, 3 and 4), and the aggregated measure is evaluated using diagnostic variables listed above – this set of variables we denote (I). The second measurement assures the same length of dynamic samples for all states although this length differs for selected variables. Data concerning variables (3)-(6) cover the period 2007-2013, while for variables (2) and (7) observations are available for all states in the period 2008-2012, thus the variable (1) is excluded and this set of variables contains six variables denoted by (II).

Table 6. Ranking of states due to generalized measures

Ranking of states			Frequency of the cluster belonging		Clustering of states		
No.	States	$SMGR_i$			No.	States	$SMGC_i$
1	Denmark	5	Austria	C(4)	A	Denmark	4
2	Switzerland	10	Belgium	C(4)		Switzerland	5
3	Germany	16	Czech Rep.	B(3); C(1)	B	Luxembourg	7
4	Netherlands	17	Denmark	A(4)		Netherlands	7
5	Slovenia	22	Estonia	C(1); D(3)		Germany	8
6	Luxembourg	23	Finland	B(2); C(2)		Slovenia	8
7	Italy	27	Germany	B(4)		Czech Rep.	9
8	Czech Rep.	34	Greece	C(3); D(1)		Italy	9
9	Poland	35	Hungary	C(1); D(3)		Poland	9
10	Finland	42	Italy	B(3); C(1)		Finland	10
11	U.K.	46	Luxembourg	A(1); B(3)		Slovakia	10
12	Slovakia	47	Netherlands	A(1); B(3)	C	U.K.	11
13	Austria	50	Poland	B(3); C(1)		Austria	12
14	Belgium	52	Portugal	C(2); D(2)		Belgium	12
15	Greece	62	Slovakia	B(2); C(2)		Greece	13
16	Spain	64	Slovenia	B(4)	D	Portugal	14
17	Portugal	66	Spain	C(2); D(2)		Spain	14
18	Hungary	69	Switzerland	A(3); B(1)		Estonia	15
19	Estonia	73	U.K.	B(2); C(1); D(1)		Hungary	15

Note: Numbers in parenthesis denote how many times the *i*-th state is classified to the certain class A-D on the basis of taxonomic measures SMR_{ik}.

Source: own calculations

In addition, after correlation analysis of variables for both samples we created sets of uncorrelated variables i.e. variables for which Pearson coefficient is smaller than 0.8. In other words we apply two sets of variables denoted as (a) and (b). The sets containing all variables (i.e. 7 for the sample I and six for the sample II) we denote as (a) and the sets, which contain only uncorrelated variables is denoted as (b). Standard deviation (4) and variation coefficient (5) together with annual changes in the share of pension funds investments in GDP (6) are strongly correlated for the sample (I) thus they are excluded from set of variables (a). While from the sample (II) only variable (6) is excluded from set of variables (a), creating the set (b). Therefore synthetic measures SMR are calculated on the basis of seven variables – SMR(Ia), four variables SMR(Ib), six variables SMR(IIa) and five variables SMR(IIb). According to evaluated taxonomic measures we rank all countries and classified them to the four clusters describing the level of development of the pension markets (Table 5).

The efficient pension systems are in Denmark and Switzerland, while inefficient ones - in Estonia and Hungary. Poland belongs the second cluster i.e. the pension system containing mandatory funded pillar was well developed since in all

four rankings the lowest position of Poland is the 10-th. Among other countries in transition which were taken into account only Slovenia keeps better position in all rankings, and Czech Rep. in rankings provided on the basis of the sample (II), and Slovakia for SMR(IIa). The mentioned relations are even better visible using measures that summarize ranking and clustering for all defined set of variables (Table 6).

CONCUSION

Our paper compares the development of the pension fund markets in selected European states. In our analysis we applied seven diagnostic variables describing the pension fund markets to construct the synthetic measure of development *SMR* for each analyzed country. Diagnostic variables are evaluated using OECD data from the years 2001-2013.

According to our investigation pension fund markets are the most developed in rich countries i.e. Denmark, Switzerland Netherlands, Luxemburg and Germany. However some new European Union member states, namely Slovenia, Poland, Czech Republic and Slovakia keeps high positions in the ranking while the pension fund markets are inefficient in Estonia and Hungary. It is worth noticing that United Kingdom, Austria, Belgium and Spain are classified to the third and fourth classes i.e. pension fund markets are not well developed in these countries.

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MODEL-ORIENTED DECISION SUPPORT SYSTEM AND FUZZY INFORMATION PROCESSING FOR INCREASING EFFICIENCY OF UNIVERSITY – IT-COMPANY CONSORTIA

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Abstract: In the paper the existing approaches for improving abovementioned collaboration processes, technologies and methodologies based on efficient methods of optimization and decision making, modern computer-based systems and Internet opportunities are been discussed. The main contribution of the authors in this paper is a structure of computerized decision making system (CDMS) which can help partners from education and industry to find the best model of university – IT-company consortia from proposed set of efficient models A1, A2, B, C and their rational combinations.

Keywords: decision support system, fuzzy logic, linguistic model, membership function, linguistic term, rule base, fuzzyfication, defuzzyfication, university – IT-company consortium

INTRODUCTION

The essential influence on the general development and integration level of informational technologies into Ukrainian or any national economy and into world market's segments is done by results of high-efficiency and mutually profitable cooperation of universities and IT-companies. Herein implementation of the new models of cooperation requires consideration and preliminary

processing of large amount of input data, in particular, based on analysis of preliminary cooperation experience of involved parties, their main achievements, competitiveness, advantages and directions for the development, scientific and educational levels of participants of future academic-industrial consortium, employment level of students, university professors and IT-companies, etc. Incorrectly chosen model of cooperation as well as non observation of relevant conditions of collaboration within the consortium such as "University – IT-companies" can lead to undesired and unexpected consequences, including the loss of significant amount of intellectual and/or material resources, lowering educational-qualification level of specialists, appearing of limitation in education and development of ability to creative thinking [Kharchenko and Sklyar 2014, Kondratenko and Kharchenko 2014, Kondratenko 2011].

THE STATEMENT OF RESEARCHED PROBLEM

Increase of cooperation efficiency can be influenced by model-oriented decision support systems (DSS), which is developed on the basis of the latest methods, technologies, and approaches of system analysis, forecasting, fuzzy logics, neural networks, artificial intelligence, etc. [Kazymyr et al. 2015, Lytvynov et al. 2015]. Usage of the above mentioned methods when designing modern DSS allows to process the essential amount of different-type information on a new level of intellectual cooperation of a decision maker (DM) and computer system [Chrzanowska and Chudzian 2014, Gaspars-Wieloch 2014, Rotshtein 1999, Zadeh 1965]. Nowadays there is still an unsolved question of selecting partnership models based on developing the system of multicriterion assessment of possible level of cooperation between universities and IT-companies. Usage of such class DSS in some specific practical cases makes it possible to select the best variant of the model of consortia development such as "University – IT-company" [Kondratenko and Kharchenko 2014, Starov et al. 2014].

The aim of this work is development and research model-oriented DSS based on fuzzy logic to increase the efficiency of multicriterion decision making processes for choosing a model of cooperation within consortia such as "University – IT-company".

Preliminary researches and analysis of successful cooperation experience within different-type consortia prove that nowadays solving the task of estimating the level of cooperation between universities and IT-companies involves the selection of one of the four formed alternative models [Kharchenko and Sklyar 2014] as alternative decision variants $E_i,(i=1...4)$, where decision variant E_1 corresponds to the model A1 (cooperation between university and IT-company in the sphere of education and study organization, knowledge sharing, targeted personnel training for IT-industry); variant E_2 – model A2 (organization and

support of certification processes of cooperation results); variant E_3 – model B (creating collective center of scientific researches, developing collective scientific projects); variant E_4 – model C (creation of student research groups with business orientation and realization of startups). Herein the efficiency of process of selecting cooperation model essentially depends on chosen criterion x_j, (j=1,2,…,n), which characterize each partner of the relevant future consortium such as "University – IT-company". Usage of fuzzy logics and hierarchical structure of input data (coordinates) when developing model-oriented DSS of such type allows to increase efficiency of multicriterion selection of cooperation model between universities and IT-companies, which is achieved by simplifying the process of formation and processing knowledge, taking into account significant amount of quality indicators and selection of optimal solution for a large amount of input expert information [Kondratenko et al. 2011, Rotshtein 1999, Zadeh 1965].

THE ANALYSIS OF RESEARCHING INTELLECTUAL DSS WITH HIERARCHICALLY-ORGANIZED STRUCTURE

In this study there is considered the developed by Authors model-oriented DSS for selecting model ($m=4$) of cooperation between universities and IT-companies according to preliminary proposed and defined criterion ($n=27$). The experience of professionals in the sphere of designing specialized fuzzy system of different purpose shows that with one-level structure of DSS in cases of large dimension of input coordinates vector $X=\{x_j\}, j=1...n$ sensitivity of their fuzzy rule bases to changes of input coordinates (criterion) values reduces $x_j,(j=1,2,...,n)$ [Kondratenko and Kondratenko 2014, Kondratenko and Sidenko 2014]. This is primarily due to complexity of creating relevant fuzzy rules to realize all possible dependences between input and output parameters of the system $y_k=f(x_1,x_2,...,x_{27}),k=1...K$.

Let's describe all linguistic variables in the model structure (Figure 1): x_1 – level of scientific novelty of projects; x_2 – practical significance of projects; x_3 – accordance to study direction; x_4 – work experience in IT-sphere; x_5 – participation in international programs of students exchange; x_6 – level of students cooperation with IT-companies; x_7 – success in study; x_8 – level of innovative projects; x_9 – number of patents; x_{10} – number of grants; x_{11} – level of scientific publications in university department; x_{12} – number of scientific publications in university department; x_{13} – university category; x_{14} – IT-certification of department teachers; x_{15} – number of business courses; x_{16} – experience in organizing student companies; x_{17} – experience in organizing mixed creative teams

for execution and realization of IT-projects; x_{18} – level of knowledge transfer of IT staff considering their employment; x_{19} – experience level of IT company personnel; x_{20} – educational qualification level of IT company personnel; x_{21} – experience of IT company in supporting the development of innovative researches; x_{22} – experience of cooperation with universities; x_{23} – age of IT-company; x_{24} – potential of university department for cooperation; x_{25} – potential of IT-company for cooperation; x_{26} – expected results for the university department, x_{27} – expected results for the IT company; y_1 – evaluation level of diploma/master's work; y_2 – level of professional students orientation; y_3 – level of scientific activity of the university department; y_4 – level of business orientation of the university department; y_5 – assessment of a possible knowledge exchange of IT-company staff; y_6 – assessment of a possible level of scientific and educational support from the IT-company; y_7 – possibility of cooperation; y_8 – general educational level of students; y_9 – general activity level of the university department; y_{10} – general activity level of IT-company; y – cooperation level under consortia such as "University – IT-company".

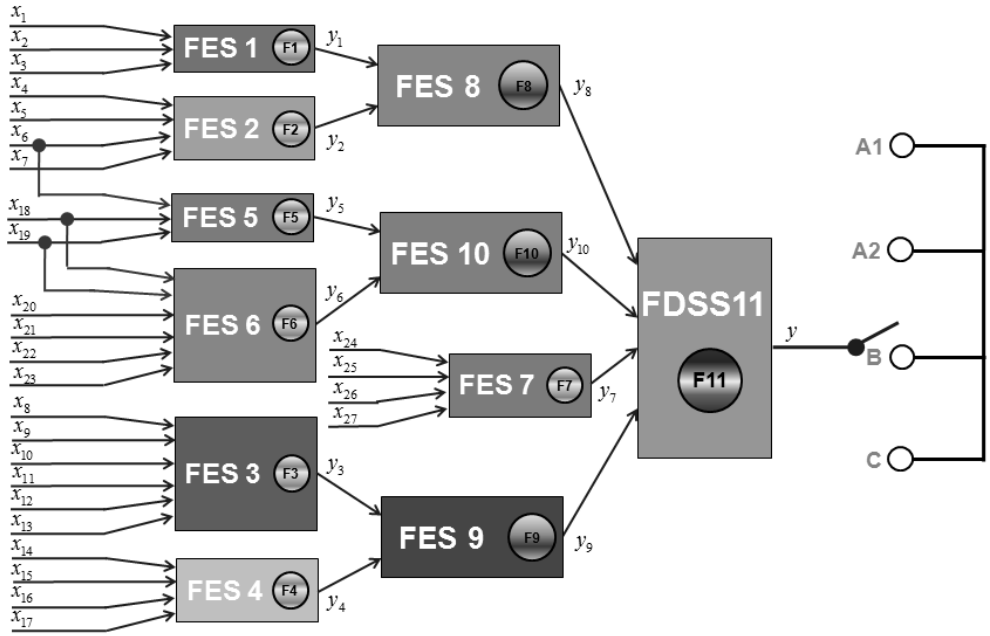
In Figure 1 there is shown the variant of proposed by Authors hierarchically-organized DSS St_s to select the best model $E^*, \left(E^* \in E, E = \{E_1, E_2, E_3, E_4\}\right)$ of cooperation between universities and IT-companies, which is created on the basis of input coordinates vector decomposition $X = \{x_j\}, j = 1...27$ with their association in the next s -group combination (1):

$$X_s = \begin{Bmatrix} \{x_1, x_2, x_3\}, \{x_4, x_5, x_6, x_7\}, \{x_8, x_9, ..., x_{13}\}, \{x_{14}, x_{15}, x_{16}, x_{17}\}, \\ \{x_6, x_{18}, x_{19}\}, \{x_{18}, x_{19}, ..., x_{23}\}, \{x_{24}, x_{25}, x_{26}, x_{27}\} \end{Bmatrix} \quad (1)$$

Herein, corresponding subsystems of DSS (Figure 1), among them $\{FES_1, FES_2, ..., FES_{10}, FES_{11}\}$, realize next functional dependencies for s alternative structure $St_s = \{y_1, y_2, ..., y_{10}, y\}$ of DSS (2):

$$St_s = \begin{Bmatrix} y_1 = f_1(x_1, x_2, x_3), y_2 = f_2(x_4, x_5, x_6, x_7), y_3 = f_3(x_8, x_9, ..., x_{13}), \\ y_4 = f_4(x_{14}, x_{15}, x_{16}, x_{17}), y_5 = f_5(x_6, x_{18}, x_{19}), y_6 = f_6(x_{18}, x_{19}, ..., x_{23}), \\ y_7 = f_7(x_{24}, x_{25}, x_{26}, x_{27}), y_8 = f_8(y_1, y_2), y_9 = f_9(y_3, y_4), \\ y_{10} = f_{10}(y_5, y_6), y = f_{11}(y_7, y_8, y_9, y_{10}) \end{Bmatrix} \quad (2)$$

Figure 1. The structure of hierarchical DSS based on fuzzy logics for selecting model of cooperation within consortia such as "University – IT-company"



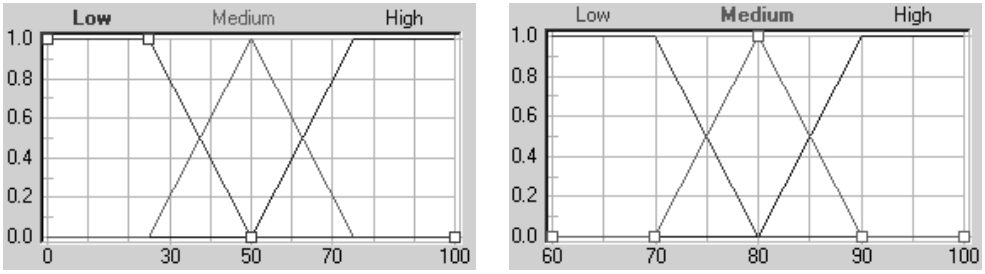
Source: own elaboration

Fuzzy DSS with the corresponding alternative structure St_s (Figure 1) evaluates the possible level of cooperation by 4 models (A1, A2, B, C). To change (increase or decrease) the number of evaluation models it necessary to change the number of linguistic terms of output variable in the eleventh subsystem $y = f_{11}(y_7, y_8, y_9, y_{10})$, and accordingly to correct the rule base of the eleventh subsystem. For example, for evaluation of the level of cooperation by 7 models (A1, A1-A2, A2, A2-B, B, B-C, C), where A1-A2, A2-B, B-C are combined models, it necessary to increase the number of linguistic terms of output variable to 7 ("Very Low" – VL , "Low" – L, "Lower than Medium" – L, "Medium" – M, "Higher than Medium" – M, "High" – H, "Very High" – VH).

So, for example, the second subsystem $y_2 = f_2(x_4, x_5, x_6, x_7)$ for assessment of level of professional orientation of students in relevant university is being created (Figure 1) on the basis of four input coordinates (x_4 – experience of work in IT-sphere: the range of change – [0 100], x_5 – participation in international programs of students exchange: the range of change – [0 100], x_6 – level of students cooperation with IT-companies: the range of change – [0 100], x_7 – success in study: the range of change – [60 100]), which are combined according to

common abilities, and one output coordinate (y_2 – level of professional students orientation: the range of change – [0 100]) with realization of relevant knowledge base, which includes 81 fuzzy rules of productional type (Table 1). To design fuzzy rule bases for developed structure of DSS (Figure 1) there are used linguistic terms $\{Low, Medium, High\}$ with triangular shape of membership function [Kondratenko and Sidenko 2011], [Zadeh 1965]. Graphical representation of possible values of variables x_4, x_5, x_6, x_7, y_2 and their relation to linguistic terms are represented in Figure 2.

Figure 2. Linguistic terms for variables x_4, x_5, x_6, y_2 (a) and x_7 (b)



a) b)

Source: own elaboration

The function structure $f_2(x_4, x_5, x_6, x_7)$, which is realized through the corresponding fuzzy inference engine, consists of fuzzyfication of input variables x_4, x_5, x_6, x_7, activation of rules, aggregation of antecedent components of corresponding rules based on *MIN* operation, accumulation of consequents and formation of the resulting fuzzy set. Defuzzyfication procedure using the center of area method (Fast CoA) applicable only to the output variable y in the eleventh subsystem $y = f_{11}(y_7, y_8, y_9, y_{10})$.

Double-sided fuzzyfication/defuzzyfication procedure is performed mainly in single-level (non-hierarchical) fuzzy systems. In a hierarchically-organized systems fuzzyfication procedure is performed for input variables in all subsystems, and defuzzyfication procedure – only for output variable in the subsystem of the last hierarchical level (in our case, for the eleventh subsystem $y = f_{11}(y_7, y_8, y_9, y_{10})$). At the intermediate levels defuzzyfication procedure is not performed. Result of the fuzzy logic inference engine in the form of resulting fuzzy set directly transmitted to the fuzzy logic inference engine of the next hierarchical level. This allows decreasing the calculation time of output variable.

Table 1. Selective ruleset of knowledge base of the second subsystem

Number of rule	x_4	x_5	x_6	x_7	y_2
1	Low	Low	Low	Low	Low
2	Low	Low	Low	Medium	Low
3	Low	Low	Low	High	Low
… … …					
37	Medium	Medium	Low	Low	Low
38	Medium	Medium	Low	Medium	Medium
39	Medium	Medium	Low	High	Medium
… … …					
76	High	High	Medium	Low	Medium
77	High	High	Medium	Medium	Medium
78	High	High	Medium	High	High
79	High	High	High	Low	Medium
80	High	High	High	Medium	High
81	High	High	High	High	High

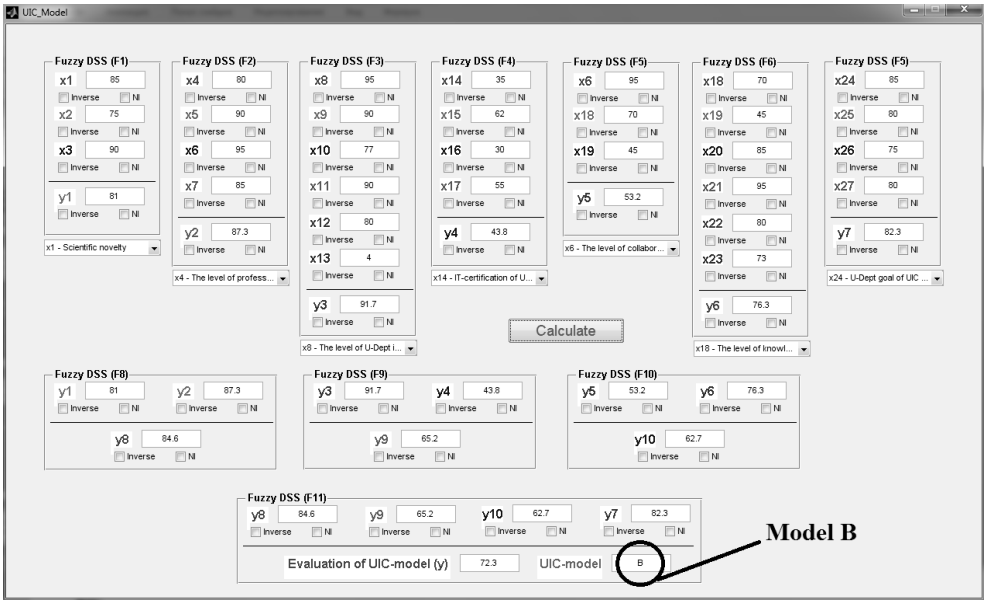
Source: own elaboration

Human-computer interface of CDMS, program realization and results of DSS work for selecting the model of cooperation within consortia such as "University – IT-company" are shown on Figure 3. For the presented on Figure 3 set of input data $X = \{x_j\}, j = 1...27$ developed DSS on fuzzy logics (Figure 1) creates on its output consolidated signal, which recommends corresponding future partners for cooperation (specific University and specific IT-company) to choose as optimal model E^* the model of cooperation B: $E^* = B, \left(E^* \in E, E = \{E_1 = A1, A2, B, C\}\right)$.

As input data for modular DSS different-type input variables $X = \{x_j\}, j = 1...27$ are used, which characterize performance indicators of university (of relevant IT-department) and IT-companies, which are part of academic-industrial consortia. Some of input data are quantitative, and some – qualitative. Quantitative input indicators can be created on the basis of results of statistical information processing [Binderman, Borkowski and Szczesny 2011], and qualitative – on the basis of results of expert evaluations (using individual and group assessments) [Gil-Aluja 1999, Gil-Lafuente and Merigo 2010, Kondratenko and Kondratenko 2014, Lodwick and Kacprzhyk 2010].

The working efficiency of the developed hierarchically-organized DSS based on fuzzy logic for selecting the model of cooperation within consortia such as "University – IT-company" tested on the different types of sets of the input data, that were received from specific universities and IT-companies, that corresponding to the possible models of cooperation.

Figure 3. The interface of developed model-oriented DSS for selecting the model of cooperation within consortia such as "University – IT-company"



Source: own calculations

Practical usage of the developed fuzzy DSS to for selecting the model of cooperation within consortia such as "University – IT-company" tested in successful cooperation of Petro Mohyla Black Sea State University (PMBSSU), Ukraine and following IT-companies:

- Global Logic Ukraine, cooperation since 2010 (model A1);
- Template Monster, cooperation during 2008 - 2010 (Model C);
- D-Link, cooperation since 2010 (model A2);
- Camo IT, cooperation since 2015 (model B).

CONCLUSIONS

In this paper there are shown results of developing hierarchically-organized DSS, which is synthesized on the basis of using fuzzy logics, to increase efficiency of decision-making processes for selecting optimal model E^* of partner cooperation under consortia such as "University – IT-company". Made by authors analysis of samples of successful innovative cooperation of academic institutions and IT-companies [Kharchenko and Sklyar 2014], [Kondratenko and Kharchenko 2014], [Kondratenko 2011] proves that creation of different groups, consortia, assosiations and alliances such as "University – IT-company" to solve current and future problems in higher education sphere based on mutual working experience in computer science area and internet-communications is a perspective direction in

the area of improving efficiency of higher education system. In particular, the National Aerocosmic University "Kharkiv Aviation Institute" named after M. E. Zhukovskiy, Odessa National Polytechnic University, Yuriy Fedkovych Chernivtsi National University, Chernihiv State University, Petro Mohyla Black Sea State University, Institute of Cybernetics of National Academy of Sciences of Ukraine and others are members of such international academic-industrial consortia, which includes universities and IT-companies from Great Britain, Spain, Italy, Portugal, Ukraine and Sweden [Kharchenko and Sklyar 2014]. This consortium is created to develop and implement models of cooperation between universities and industry (IT-companies) such as A1, A2, B and C within the project TEMPUS-CABRIOLET 544497-TEMPUS-1-2013-1-UK-TEMPUS-JPHES "Model-oriented approach and Intelligent Knowledge–Based System for Evolvable Academia-Industry Cooperation in Electronics and Computer Engineering" (2013-2016).

Aprobation of the developed model-oriented DSS proves its high efficiency, that is confirmed by authors as in solving practical tasks of selecting a model of cooperation within consortia such as "University – IT-company", and in solving different-type tasks of transport logistics [Kondratenko et al. 2006], [Kondratenko and Sidenko 2014], in particular when selecting the best transport company from the set of existing alternative variants, etc.

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